

# A new max-plus based randomized algorithm for solving a class of HJB PDEs

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The work was conducted when I was a Ph.D. student at CMAP, École Polytechnique, France.

# Infinite horizon linear quadratic optimal control

$$V(x) = \sup_{\mathbf{u} \in L_2^{\text{loc}}([0, \infty); \mathbb{R}^k)} \int_0^{\infty} \frac{1}{2} \mathbf{x}(t)' D \mathbf{x}(t) - \frac{\gamma^2}{2} \|\mathbf{u}(t)\|^2 dt$$

where the state dynamics are given by

$$\begin{aligned} \dot{\mathbf{x}}(s) &= A\mathbf{x}(s) + \sigma \mathbf{u}(s) \\ \mathbf{x}(0) &= x \in \mathbb{R}^d. \end{aligned} \tag{1}$$

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## Assumption (Existence assumption)

The matrices  $D$  and  $-A - A'$  are positive definite and

$$\gamma^2/c_{\sigma}^2 > c_D/c_A^2$$

where  $c_D = \lambda_{\max}(D)$ ,  $c_A = \lambda_{\min}(-A - A')/2$  and  $c_{\sigma} = \sqrt{\lambda_{\max}(\sigma\sigma')}$ .

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• Under the existence assumption, the solution  $V : \mathbb{R}^d \rightarrow \mathbb{R}$  is quadratic:

$$V(x) = \frac{1}{2} x' P x, \quad \forall x \in \mathbb{R}^d,$$

where  $P \in \mathbb{R}^{d \times d}$  is the solution of an Algebraic Riccati equation:

$$A'P + PA + P\Sigma P + D = 0$$

where  $\Sigma = \sigma\sigma' / \gamma^2$ .

# Infinite horizon switched problem [McEneaney 07]

$$V(x) = \sup_{\substack{\mathbf{u} \in L_2^{loc}([0, \infty); \mathbb{R}^k) \\ \mu: [0, \infty) \rightarrow [M] := \{1, \dots, M\}}} \int_0^\infty \frac{1}{2} \mathbf{x}(t)' D^{\mu(t)} \mathbf{x}(t) - \frac{\gamma^2}{2} \|\mathbf{u}(t)\|^2 dt$$

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where  $c_D = \max_m \lambda_{\max}(D^m)$ ,  $c_A = \min_m \lambda_{\min}(-A^m - (A^m)')/2$  and  $c_\sigma = \max_m \sqrt{\lambda_{\max}(\sigma^m (\sigma^m)')}$ .

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- nonconvex problem
- nonlinear robust  $H_\infty$  control ( $V$  is the available storage function)

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Under the existence assumption, the solution  $V : \mathbb{R}^d \rightarrow \mathbb{R}$  is the (unique) viscosity solution of the following stationary HJB PDE:

$$H(x, \nabla V(x)) = 0, \quad \forall x \in \mathbb{R}^d,$$

where  $H(\cdot, \cdot) : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$  is the associated Hamiltonian:

$$H(x, p) = \sup_{m \in [M]} H^m(x, p) := \sup_{m \in [M]} (A^m x)' p + \frac{1}{2} x' D^m x + \frac{1}{2} p' \Sigma^m p$$



# Lax-Oleinik semigroup [McEneaney 07]

The semigroup  $\{S_t\}_{t \geq 0}$  associated with the switched problem:

$$S_t[\phi](x) := \sup_{\substack{\mathbf{u} \in \dot{L}_2^{loc}([0, \infty); \mathbb{R}^k) \\ \mu: [0, \infty) \rightarrow [M] := \{1, \dots, M\}}} \int_0^t \frac{1}{2} \mathbf{x}(t)' D^{\mu(t)} \mathbf{x}(t) - \frac{\gamma^2}{2} \|\mathbf{u}(t)\|^2 dt + \phi(\mathbf{x}(t)).$$

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- order-preserving property:  $\phi_1 \leq \phi_2 \Rightarrow S_t[\phi_1] \leq S_t[\phi_2]$
- group property:  $S_{t_1+t_2}[\phi] = S_{t_1}[S_{t_2}[\phi]]$
- max-plus linearity:  
 $S_t[\sup(\phi_1, \phi_2) + \lambda] = \sup(S_t[\phi_1], S_t[\phi_2]) + \lambda$

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where

$$\dot{\mathbf{x}}(s) = A^{\mu(s)} \mathbf{x}(s) + \sigma^{\mu(s)} \mathbf{u}(s), \quad \mathbf{x}(0) = x \in \mathbb{R}^d .$$

- For all  $t \geq 0$ ,  $S_t[V] = V$
- For all  $V_0 \leq V$ ,  $V = \lim_{T \rightarrow +\infty} S_T[V_0]$  .
- $V \simeq S_T[V_0] = \{S_\tau\}^N[V_0] \simeq \{\tilde{S}_\tau\}^N[V_0]$  .

# Semigroup approximation

- Semigroup approximation:

$$S_\tau \simeq \tilde{S}_\tau := \sup_m S_\tau^m$$

- $\{S_t^m\}_{t \geq 0}$  is the semigroup associated to the  $m$ th linear quadratic control problem:

$$S_t^m[\phi](x) = \sup_{\mathbf{u}} \int_0^t \frac{1}{2} \mathbf{x}(t)' D^m \mathbf{x}(t) - \frac{\gamma^2}{2} \|\mathbf{u}(t)\|^2 dt + \phi(\mathbf{x}(t)).$$

$$\dot{\mathbf{x}}(s) = A^m \mathbf{x}(s) + \sigma^m \mathbf{u}(s); \quad \mathbf{x}(0) = x \in \mathbb{R}^d .$$

- $S_t^m[\phi]$  is a quadratic function if  $\phi$  is. (Riccati)

$$V \simeq \{S_\tau\}^N[V_0] \simeq \{\tilde{S}_\tau\}^N[V_0] = \sup_{i_N, \dots, i_1} S_\tau^{i_N} \circ \dots \circ S_\tau^{i_1}[V_0] .$$

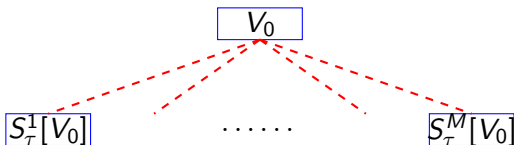
# Arborescent propagation

$$V \simeq V_T = \{S_\tau\}^N[V_0] \simeq \{\tilde{S}_\tau\}^N[V_0] = \sup_{i_N, \dots, i_1} S_\tau^{i_N} \circ \dots \circ S_\tau^{i_1}[V_0] .$$

$V_0$

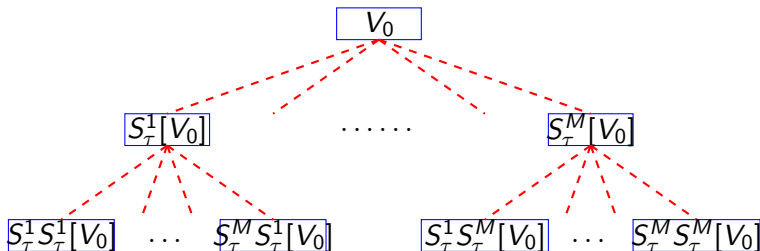
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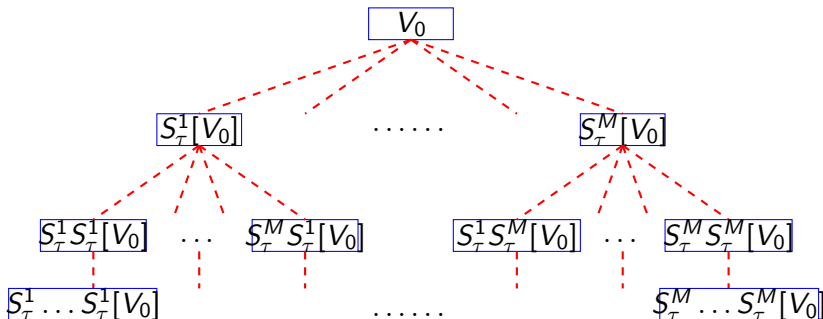
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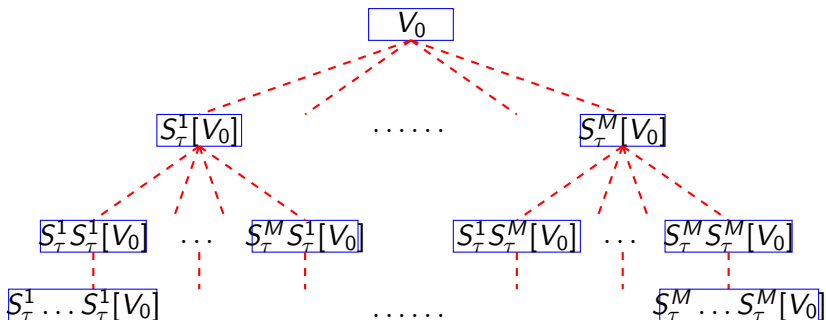
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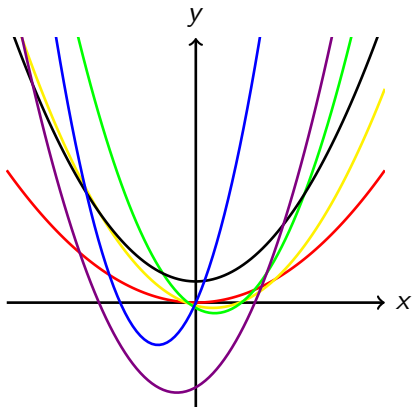
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Computational complexity:  $O(M^N d^3) \Rightarrow$  **curse of dimensionality free**

# Pruning operation:

$$V \simeq \sup(\phi_{green}, \phi_{red}, \phi_{violet}, \\ \phi_{yellow}, \phi_{black}, \phi_{blue})$$



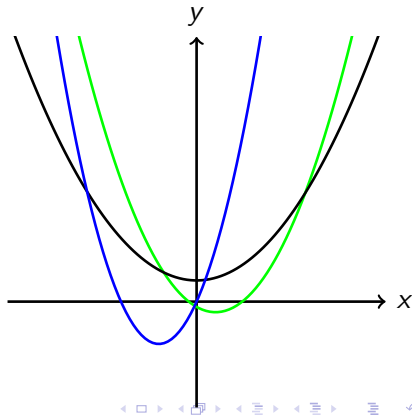
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# SDP based pruning algorithms

Suppose that the approximation at current step is:

$$\sup_{i \in I} \phi_i(x) = \sup_{i \in I} (x' \ 1) Q_i \begin{pmatrix} x \\ 1 \end{pmatrix}.$$

SDP approach ( [McEneaney, Deshpande, Gaubert 08] ): for every quadratic function  $\phi_i$ , compute the maximum lost caused by removing it from the basis set, keep those functions with largest maximum lost.

## Importance metric of $\phi_j$

$$\nu_j = \max \nu$$

$$\nu \leq z'(Q_j - Q_i)z, \forall i \neq j$$

$$z^T z = 1.$$

## SDP relaxation

$$\bar{\nu}_j = \max \nu$$

$$\nu \leq \langle (Q_j - Q_i), Z \rangle, \forall i \neq j$$

$$Z \geq 0, \text{ trace}(Z) = 1.$$

# Motivation

- SDP based COD free method:

$$V \simeq \sup_{i_1, \dots, i_N} S_{i_N}^T \cdots S_{i_1}^T [V_0] .$$

add all the possible basis functions + prune

- SDP based COD free algorithm works but is time-consuming:

$\tau=0.2, N=25$	Total time	Propagation	SDP	Pruning
<i>sort lower</i>	1.04h	1.85%	98.15%	0.00%
<i>sort upper</i>	1.34h	1.52%	98.43%	0.05%
<i>J-V p-d</i>	1.38h	1.45%	89.47%	9.08%
<i>greedy</i>	1.43h	1.63%	97.84%	0.53%

- New algorithm:

add "useful" basis functions (no need to prune)

# Idea

Suppose that at current step we have a **sub-approximation** of  $V$  by:

$$V \simeq \sup_{i \in \{1, \dots, k\}} \phi_i \leq V.$$

Instead of computing  $S_\tau^m[\phi_i]$  for every  $i \in \{1, \dots, k\}$  and every  $m \in \{1, \dots, M\}$  and then prune by solving  $kM$  SDP programs:

$$V \simeq \sup_{i \in \{1, \dots, k\}} \phi_i$$

$$S_\tau^1[\phi_1] \quad \dots \quad S_\tau^1[\phi_k] \quad \dots \quad S_\tau^M[\phi_1] \quad \dots \quad S_\tau^M[\phi_k]$$

we will find out efficiently a propagation time  $t_0 > 0$ , an index  $i_0 \in \{1, \dots, k\}$  and an index  $m_0 \in \{1, \dots, M\}$  such that

$$\sup_{i \in \{1, \dots, k\}} \phi_i(x_0) < \sup_{i \in \{1, \dots, k\}} \phi_i(x_0), S_{t_0}^{m_0}[\phi_{i_0}](x_0)$$

# Key observation

Recall that:

$$H^m(x, p) = (A^m x)' p + \frac{1}{2} x' D^m x + \frac{1}{2} p' \Sigma^m p$$

Let  $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$  be a quadratic function,  $x_0 \in \mathbb{R}^d$  and  $m_0 \in \{1, \dots, M\}$ . If

$$H^{m_0}(x_0, D\phi(x_0)) > 0 ,$$

then for all sufficiently small  $t_0 > 0$ ,

$$S_{t_0}^{m_0}[\phi](x_0) > \phi(x_0) .$$



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# Key observation

Let  $\phi_1, \dots, \phi_k : \mathbb{R}^d \rightarrow \mathbb{R}$  be quadratic functions such that:

$$\sup_{i \in \{1, \dots, k\}} \phi_i(x) \leq V(x), \quad \forall x \in \mathbb{R}^d .$$

If  $\phi_{i_0}(x_0) = \sup_{i \in \{1, \dots, k\}} \phi_i(x_0)$  ,  
 $H(x_0, D\phi_{i_0}(x_0)) > 0$  ,

then there is  $m_0 \in \{1, \dots, M\}$   
such that for all sufficiently small  
 $t_0 > 0$ ,

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We find proper  $i_0$  and  $m_0$  in  $O(k + M)$ .

# Randomized max-plus algorithm

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## Algorithm 1 Randomized max-plus algorithm

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1: INPUT:  $V^0 = \sup_i \phi_i^0 \leq V$ , compact  $X \ni 0$ , threshold  $\vartheta > 0$

2: **for**  $k = 0, 1, 2, \dots$  **do**

3:   Randomly choose a point  $x_0 \in X$ ;

4:   **if**  $H(x_0, DV^k(x_0)) > \vartheta$ , **then**

5:

$$V^{k+1} = \sup(V^k, \Psi(V^k, x_0));$$

6:   **else**

7:

$$V^{k+1} = V^k.$$

8:   **end if**

9: **end for**

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# Convergence of the algorithm

## Theorem (Convergence result)

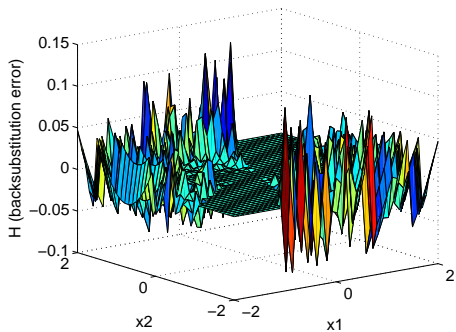
Under the *existence assumption*, for every threshold  $\vartheta > 0$ , the randomized algorithm stops surely after a finite number of iterations. There is a constant  $L > 0$  such that for all  $\vartheta > 0$ , almost surely we have:

$$\lim_{k \rightarrow +\infty} \sup_x |V(x) - V^k(x)|/|x|^2 \leq L\vartheta$$

# Experimental results

$M=6, d=6$

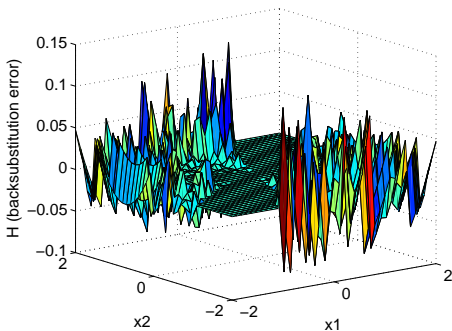
SDP based pruning (greedy),  $\tau=0.1, N=25, \text{time}>1\text{h}$



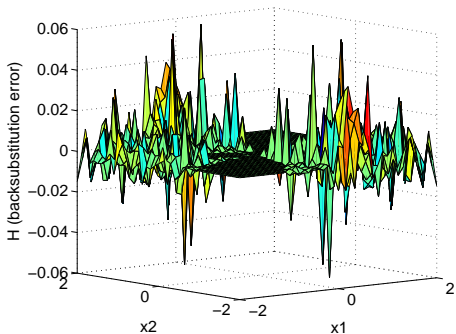
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$M=6, d=6$

SDP based pruning (greedy),  $\tau=0.1, N=25$ , time  $>1h$

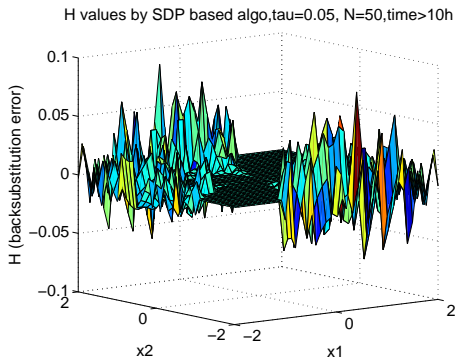


Randomized algorithm, time =103s



# Experimental results

$M=6, d=6$

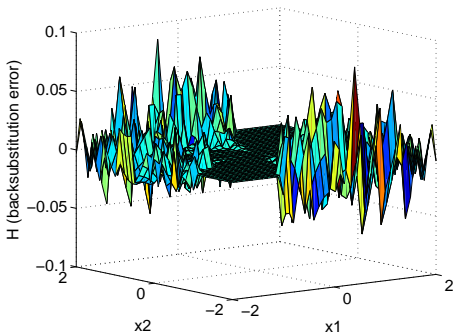




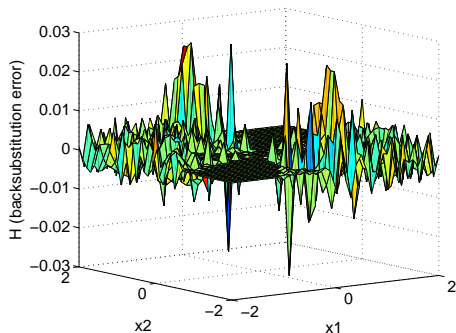
# Experimental results

$M=6, d=6$

H values by SDP based algo,  $\tau=0.05, N=50, \text{time}>10\text{h}$

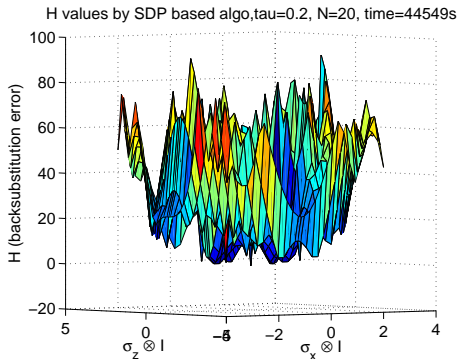


H values by randomized algo,  $\text{time}=1217\text{s}$



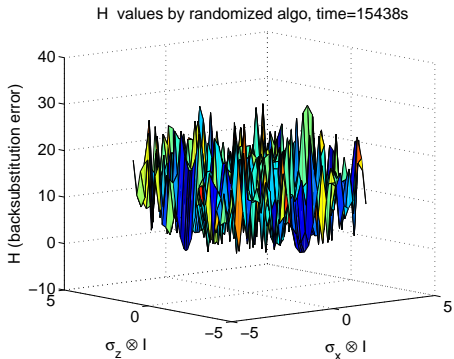
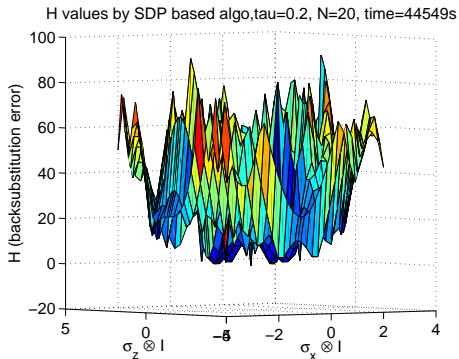
# Experimental results

$M=6$ ,  $d=15$  (an optimal control problem on  $SU(4)$  –quantum optimal gate synthesis)



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Instance nb.	d	M	initial	backsub.	SDP		randomized algo	
			backsub. error	error	based cpu time	basis nb.	cpu time	basis nb.
1	4	10	1.2	0.8	280s	100	10s	345
1	4	10	1.2	0.007	-	-	527s	26605
2	4	20	4.5	0.14	-	-	184s	9925
2	4	20	4.5	0.024	-	-	2900s	75831
3	4	50	3.8	1.5	5923s	80	41s	120
3	4	50	3.8	0.1	-	-	155s	5938
3	4	50	3.8	0.034	-	-	3650s	99848
4	6	6	2.3	0.21	188s	170	27s	1231
4	6	6	2.3	0.12	1.5h	170	37s	1636
4	6	6	2.3	0.08	>10h	320	103s	3281
4	6	6	2.3	0.0257	-	-	1217s	10227
5	8	20	0.13	0.1	163s	50	16s	56
5	8	20	0.13	0.01	-	-	90s	8108
5	8	20	0.13	0.0045	-	-	1887s	32860
6	8	30	0.16	0.012	-	-	151s	13698
6	8	30	0.16	0.012	-	-	2994s	76393
7	8	50	0.21	0.01	-	-	120s	10011
7	8	50	0.21	0.0044	-	-	3124s	67515
8	12	20	0.7	0.01	-	-	1485s	44894
8	12	20	0.7	0.0052	-	-	5368s	93292
9	12	30	0.83	0.012	-	-	1322s	39987

# End

Thank you!



Akian, Gaubert, Lakhoua 06.

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