Randomized Dual Coordinate Ascent with Arbitrary Sampling

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Empirical Risk Minimization

ERM:

$$\min_{w\in\mathbb{R}^d}\left[P(w)\stackrel{\text{def}}{=}\frac{1}{n}\sum_{i=1}^{n}\phi_i(A_i^\top w)+\lambda g(w)\right]$$

- supervised learning;
- train a linear predictor $w\in\mathbb{R}^d$;
- n training samples $A_1,\ldots,A_n\in\mathbb{R}^d$;
- convex and $1/\gamma$-smooth loss function $\phi_i:\mathbb{R}\to\mathbb{R}$;
  - ex.: Squared loss $(\phi_i(a)=\frac{1}{2\gamma}(a-b_i)^2)$, Logistic loss $(\phi_i(a)=\frac{4}{\gamma}\log(1+e^a))$, ...
- 1-strongly convex regularizer $g:\mathbb{R}^d\to\mathbb{R}\cup\{+\infty\}$;
  - ex.: L_2 regularization $(g(w)=\frac{1}{2}\|w\|_2^2)$, ...

Primal Dual Formulation

- ERM:

$$\min_{w\in\mathbb{R}^d}\left[P(w)\overset{\text{def}}{=}\frac{1}{n}\sum_{i=1}^{n}\phi_i(A_i^\top w)+\lambda g(w)\right]$$

- Dual problem of ERM:

$$\max_{\alpha\in\mathbb{R}^n}\ D(\alpha)\overset{\text{def}}{=}\underbrace{-\lambda g^\star\left(\frac{1}{\lambda n}\sum_{i=1}^{n}A_i\alpha_i\right)}_{\textit{smooth}}\underbrace{-\frac{1}{n}\sum_{i=1}^{n}\phi_i^\star(-\alpha_i)}_{\textit{strongly convex and separable}}$$

- Optimality conditions:

$$\textbf{OPT1}:\ w^*=\nabla g^\star\left(\frac{1}{\lambda n}A\alpha^*\right)$$

$$\textbf{OPT2}:\ \alpha_i^*=-\nabla\phi_i\left(A_i^\top w^*\right),\ \ \forall i=1,\ldots,n.$$

Stochastic Dual Coordinate Ascent

Primal solution

For $t \geq 0$:

1. $w^t = \nabla g^\star(\frac{1}{\lambda n} A\alpha^t)$

Dual solution

For $t \geq 0$:

1. $\alpha^{t+1} = \alpha^t$;
2. Randomly pick $i_t \in \{1, \ldots, n\}$ according to a distribution p;
3. Update $\alpha_{i_t}^{t+1}$:

$$\alpha_{i_t}^{t+1} = \arg\max_{\beta \in \mathbb{R}} \left\{ -\phi_{i_t}^\star(-\beta) - (A_{i_t}^\top w^t)\beta - \frac{\|A_{i_t}\|^2}{2\lambda n} |\beta - \alpha_{i_t}^t|^2 \right\}$$

Uniform sampling (SDCA: [Shalev-Shwartz & Zhang '13])

$$p_i = \textbf{Prob}(i_t = i) = \frac{1}{n}, \quad \forall i \in [n].$$

Analysis of SDCA and Iprox-SDCA

$$\mathbb{E}_t \underbrace{\left[D(\alpha^{t+1}) - D(\alpha^t)\right]}_{\text{dual value increase at iteration } t} \geq \theta \underbrace{\left(P(w^t) - D(\alpha^t)\right)}_{\text{primal dual gap at iteration } t}, \quad \forall t \geq 0.$$

$$\underbrace{\mathbb{E}[P(w^t) - D(\alpha^t)]}_{\text{expected primal dual gap at iteration } t} \leq \frac{1}{\theta}(1-\theta)^t \underbrace{\left(D(\alpha^*) - D(\alpha^0)\right)}_{\text{initial dual optimality gap}}, \quad \forall t \geq 0.$$

where

$$\theta = \begin{cases} \min_i \dfrac{\lambda\gamma}{\|A_i\|^2 + \lambda\gamma n} & \text{if } p_i \sim \frac{1}{n} \text{ (SDCA)} \\ \dfrac{\lambda\gamma n}{\sum_{i=1}^n (\|A_i\|^2 + \lambda\gamma n)} & \text{if } p_i \sim \|A_i\|^2 + \lambda\gamma n \text{ (Iprox SDCA)} \end{cases}$$

Iteration Complexity Results

Uniform sampling (SDCA: [Shalev-Shwartz & Zhang '13])

$$p_i = \textbf{Prob}(i_t = i) \sim \frac{1}{n},$$

Iteration complexity:

$$T \geq \left(n + \frac{\max_i \|A_i\|^2}{\lambda\gamma}\right) \log\left(\left(n + \frac{\max_i \|A_i\|^2}{\lambda\gamma}\right)\left(\frac{D(\alpha^*) - D(\alpha^0)}{\epsilon}\right)\right)$$

Importance sampling (Iprox-SDCA: [Zhao & Zhang '15])

$$p_i = \textbf{Prob}(i_t = i) \sim \|A_i\|^2 + \lambda\gamma n,$$

Iteration complexity:

$$T \geq \left(n + \frac{\sum_{i=1}^n \|A_i\|^2}{n\lambda\gamma}\right) \log\left(\left(n + \frac{\sum_{i=1}^n \|A_i\|^2}{n\lambda\gamma}\right)\left(\frac{D(\alpha^*) - D(\alpha^0)}{\epsilon}\right)\right).$$

Main Contributions

Quartz (Randomized dual coordinate ascent methd with arbitrary sampling)

- arbitrary sampling:
  - at each iteration, update a random subset $S_t \subset [n]$ of dual variables
- direct primal-dual analysis:

$$\mathbb{E}_t \left[P(w^{t+1}) - D(\alpha^{t+1}) \right] \leq (1-\theta)(P(w^t) - D(\alpha^t)), \quad \forall t \geq 0.$$

$$T \geq \frac{1}{\theta} \log\left(\frac{P(w^0) - D(\alpha^0)}{\epsilon}\right)$$

- additional data-driven speedup

Algorithm (Quartz)

Primal solution

For $t \geq 0$:

1. $w^{t+1} = (1-\theta)w^t + \theta \nabla g^\star(\frac{1}{\lambda n} A\alpha^t)$

Dual solution

For $t \geq 0$:

1. $\alpha^{t+1} = \alpha^t$;
2. Generate a random subset $S_t \subset [n]$ according to $\hat{S}$;
3. For each $i \in S_t$ do:

$$\alpha_i^{t+1} = (1 - \theta p_i^{-1})\alpha_i^t + \theta p_i^{-1}(-\nabla \phi_i(A_i^\top w^{t+1}))$$

$$p_i \stackrel{\text{def}}{=} \textbf{Prob}(i \in \hat{S})$$

Interpretation through Fenchel's Duality

Primal dual gap ($\bar{\alpha} = \frac{1}{\lambda n} A\alpha$):

$$P(w) - D(\alpha)$$

$$= \lambda \underbrace{\left(g(w) + g^*(\bar{\alpha}) - \langle w, \bar{\alpha} \rangle\right)}_{GAP_g(w,\alpha)} + \frac{1}{n} \sum_{i=1}^{n} \underbrace{\phi_i(A_i^\top w) + \phi_i^*(-\alpha_i) + \langle A_i^\top w, \alpha_i \rangle}_{GAP_{\phi_i}(w,\alpha_i)}$$

$$\geq 0.$$

Optimality conditions:

$$\textbf{OPT1}: \; w^* = \nabla g^* \left(\frac{1}{\lambda n} A\alpha^* \right)$$

$$\textbf{OPT2}: \; \alpha_i^* = -\nabla \phi_i \left(A_i^\top w^* \right), \quad \forall i = 1, \ldots, n.$$

Quartz as a Randomized SOR Method

$$0 = \lambda(\underbrace{g(w) + g^*(\bar{\alpha}) - \langle w, \bar{\alpha}\rangle}_{GAP_g(w,\alpha)}) + \frac{1}{n}\sum_{i=1}^{n} \underbrace{\phi_i(A_i^\top w) + \phi_i^*(-\alpha_i) + \langle A_i^\top w, \alpha_i\rangle}_{GAP_{\phi_i}(w,\alpha_i)}$$

$$\textbf{OPT1}: \; w^* = \nabla g^\star\left(\frac{1}{\lambda n} A\alpha^*\right)$$

$$\textbf{OPT2}: \; \alpha_i^* = -\nabla\phi_i\left(A_i^\top w^*\right), \quad \forall i = 1, \ldots, n.$$

Quartz:

$$w^t = (1-\theta)w^{t-1} + \theta\nabla g^*(\bar{\alpha}^{t-1})$$

$$\alpha_i^t = \begin{cases} \left(1 - \theta p_i^{-1}\right)\alpha_i^{t-1} + \theta p_i^{-1}\left(-\nabla\phi_i(A_i^\top w^t)\right) & , \; i \in S_t \sim \hat{S} \\ \alpha_i^{t-1} & , \; i \notin S_t \end{cases}$$

Flexibility of Quartz

Arbitrary proper sampling:

$$p_i \stackrel{\text{def}}{=} \mathbf{Prob}(i \in \hat{S}) > 0, \qquad i \in [n].$$

- serial sampling

$$\mathbf{Prob}(|\hat{S}| = 1) = 1$$

- standard mini-batching
  - τ-nice sampling:

$$\mathbf{Prob}(\hat{S} = S) = \frac{1}{C_n^\tau}, \quad \forall S \subset [n], |S| = \tau.$$

- distributed sampling
  - c nodes/machines:

$$\hat{S} = \hat{S}_1 \cup \hat{S}_2 \cdots \cup \hat{S}_c$$

- ...

Formula of θ

$$w^t = (1-\theta)w^{t-1} + \theta \nabla g^*(\bar{\alpha}^{t-1})$$

$$\alpha_i^t = \begin{cases} \left(1 - \theta p_i^{-1}\right)\alpha_i^{t-1} + \theta p_i^{-1}\left(-\nabla \phi_i(A_i^\top w^t)\right) & , \ i \in S_t \sim \hat{S} \\ \alpha_i^{t-1} & , \ i \notin S_t \end{cases}$$

Theorem (Q., Richtarik & Zhang '14)

Let $\hat{S}$ be a proper sampling. If

$$\theta = \min_i \frac{p_i \lambda \gamma n}{v_i + \lambda \gamma n}$$

where $v_1, \ldots, v_n$ satisfy:

$$\mathbb{E}\left\| \sum_{i \in \hat{S}} A_i \alpha_i \right\|^2 \leq \sum_{i=1}^n p_i v_i |\alpha_i|^2, \ \ \forall \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n,$$

then

$$\mathbb{E}[P(w^t) - D(\alpha^t)] \leq (1-\theta)^t (P(w^0) - D(\alpha^0)), \ \ \forall t \geq 0.$$

Quartz specialized to serial sampling

Lemma

If $\hat{S}$ is a serial proper sampling, then

$$\mathbb{E}\left\|\sum_{i\in\hat{S}} A_i\alpha_i\right\|^2 \leq \sum_{i=1}^{n} p_i v_i |\alpha_i|^2, \quad \forall \alpha = (\alpha_1,\ldots,\alpha_n)\in\mathbb{R}^n,$$

holds for

$$v_i = \|A_i\|^2, \quad \forall i \in [n].$$

Corollary (Q., Richtarik & Zhang '14)

Let $\hat{S}$ be a serial proper sampling. If

$$\theta = \min_i \frac{p_i \lambda\gamma n}{\|A_i\|^2 + \lambda\gamma n},$$

then

$$\mathbb{E}[P(w^t) - D(\alpha^t)] \leq (1-\theta)^t (P(w^0) - D(\alpha^0)), \quad \forall t \geq 0.$$

Comparison with SDCA

QUARTZ+ uniform sampling

$$T \geq \left(n + \frac{\max_i \|A_i\|^2}{\lambda\gamma}\right) \log\left(\frac{P(w^0)-D(\alpha^0)}{\epsilon}\right)$$

SDCA [S-Shwartz & Zhang '15]

$$T \geq \left(n + \frac{\max_i \|A_i\|^2}{\lambda\gamma}\right) \log\left(\left(n + \frac{\max_i \|A_i\|^2}{\lambda\gamma}\right)\left(\frac{D(\alpha^*)-D(\alpha^0)}{\epsilon}\right)\right)$$

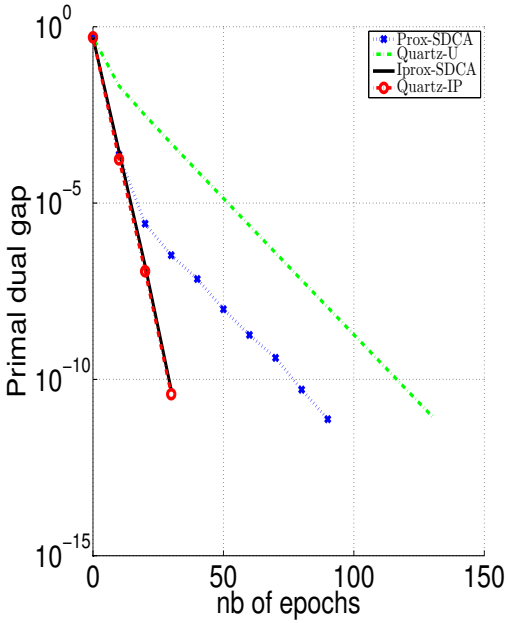
QUARTZ+ importance sampling

$$T \geq \left(n + \frac{\sum_i \|A_i\|^2}{n\lambda\gamma}\right) \log\left(\frac{P(w^0)-D(\alpha^0)}{\epsilon}\right)$$

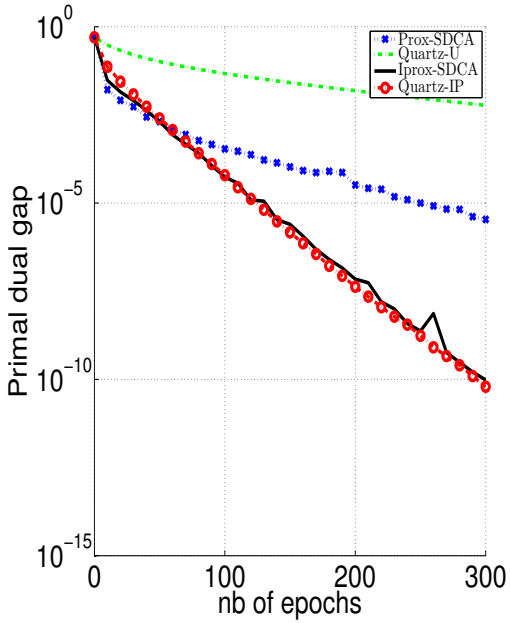
Iprox-SDCA [Zhao & Zhang '15]

$$T \geq \left(n + \frac{\sum_i \|A_i\|^2}{n\lambda\gamma}\right) \log\left(\left(n + \frac{\sum_i \|A_i\|^2}{n\lambda\gamma}\right)\left(\frac{D(\alpha^*)-D(\alpha^0)}{\epsilon}\right)\right)$$

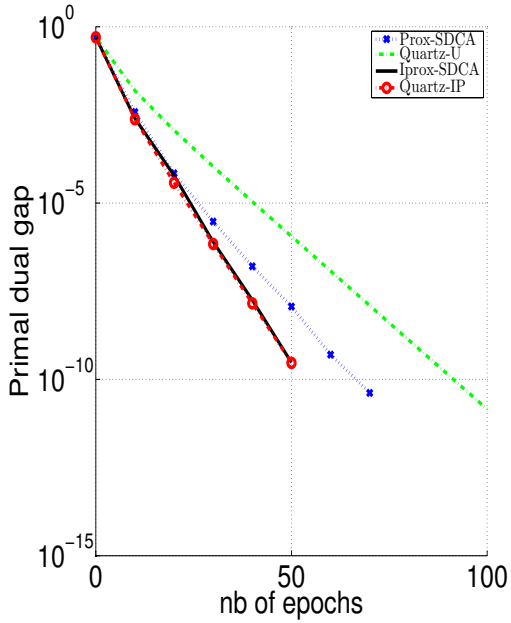
Experimental results



(a) cov1; $n = 522911$;



(b) w8a; $n = 49749$;



(c) ijcnn1; $n = 49990$;

Quartz:

$$w^t = (1-\theta)w^{t-1} + \theta \nabla g^\star\left(\frac{1}{\lambda n} A\alpha^t\right)$$

SDCA:

$$w^t = \nabla g^\star\left(\frac{1}{\lambda n} A\alpha^t\right)$$

Quartz Specialized to Mini-batching

Lemma (Richtarik & Takac '12, Fercoq & Richtarik '13)

If $\hat{S}$ is a τ-nice sampling, then

$$\mathbb{E}\left\|\sum_{i\in\hat{S}} A_i\alpha_i\right\|^2 \leq \sum_{i=1}^{n} p_i v_i |\alpha_i|^2, \quad \forall \alpha = (\alpha_1,\ldots,\alpha_n)\in\mathbb{R}^n,$$

holds for

$$v_i = \sum_{j=1}^{d}\left(1+\frac{(\omega_j-1)(\tau-1)}{n-1}\right)A_{ji}^\top A_{ji}, \qquad i\in[n], \qquad (1)$$

where for each $j\in[d]$, ω_j is the number of nonzero blocks in the j-th row of matrix $A=(A_1,\ldots,A_n)$, i.e.,

$$\omega_j \overset{def}{=} |\{i\in[n] : A_{ji}\neq 0\}|, \qquad j\in[d]. \qquad (2)$$

Iteration Complexity

For standard mini-batching of size τ:

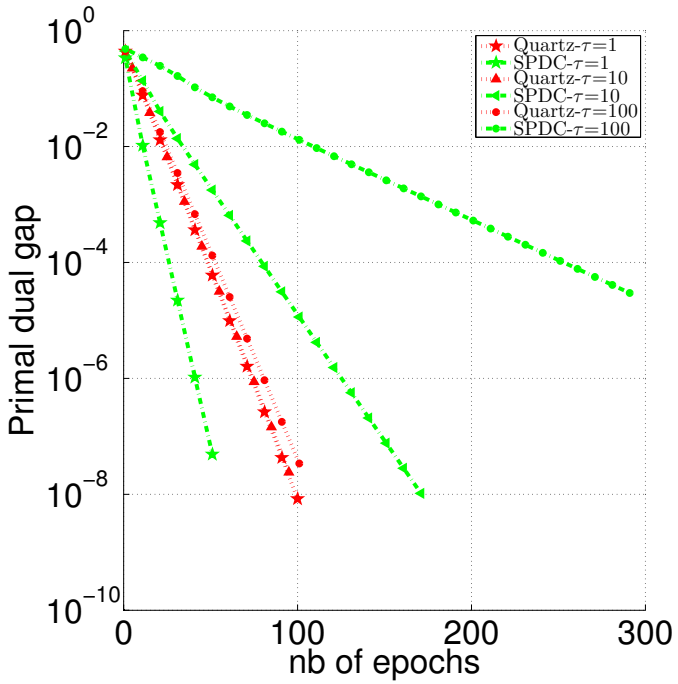
Fully sparse data $(\tilde{\omega} = 1)$	$\left(\frac{n}{\tau} + \frac{\max_i \|A_i\|^2}{\lambda\gamma\tau} \right) \log\left(\frac{P(w^0) - D(\alpha^0)}{\epsilon} \right)$
Fully dense data $(\tilde{\omega} = n)$	$\frac{n}{\tau} + \frac{\max_i \|A_i\|^2}{\lambda\gamma}$
Any data $(1 \leq \tilde{\omega} \leq n)$	$\left(\frac{n}{\tau} + \frac{\left(1 + \frac{(\tilde{\omega}-1)(\tau-1)}{n-1}\right) \max_i \|A_i\|^2}{\lambda\gamma\tau} \right) \log\left(\frac{P(w^0) - D(\alpha^0)}{\epsilon} \right)$

Comparison with Accelerated Mini-Batch P-D methods

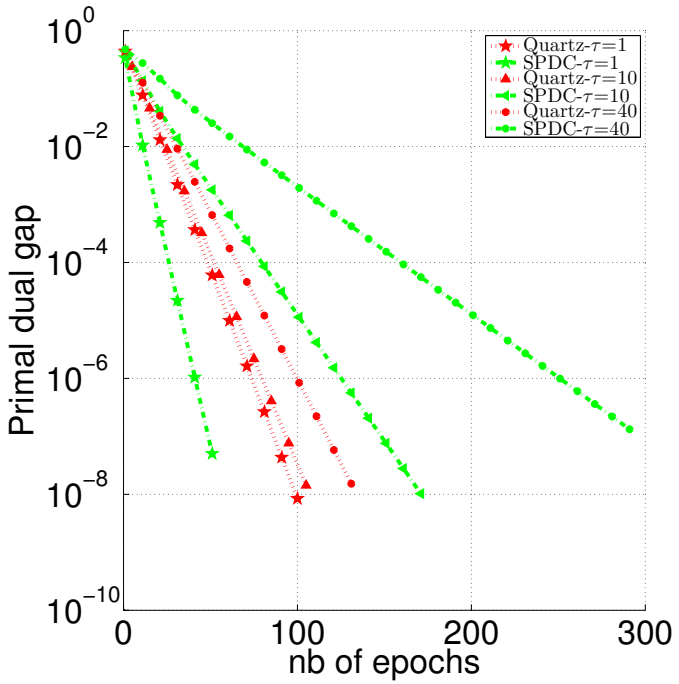
Algorithm	Iteration complexity	g
ASDCA [S-Shwartz & Zhang '13]	$\max\left\{\frac{n}{\tau}, \sqrt{\frac{n}{\lambda\gamma\tau}}, \frac{1}{\lambda\gamma\tau}, \frac{n^{\frac{1}{3}}}{(\lambda\gamma\tau)^{\frac{2}{3}}}\right\}$	$\frac{1}{2}\|\cdot\|^2$
SPDC [Zhang & Xiao '14]	$\frac{n}{\tau} + \sqrt{\frac{n}{\lambda\gamma\tau}}$	general
QUARTZ with τ-nice sampling	$\frac{n}{\tau} + \left(1 + \frac{(\tilde{\omega}-1)(\tau-1)}{n-1}\right)\frac{1}{\lambda\gamma\tau}$	general

Acc-Prox-SDCA [Shalev-Shwartz & Zhang '13]; APCG [Lin, Lu & Xiao '14]

Experimental results



(d) $n = 10^5$; sparsity 0.1%



(e) $n = 10^5$; sparsity 1%

Figure 1: Comparison of Quartz with SPDC for different mini-batch size τ in the regime $\kappa = 10n$.

Conclusion

- Summary
  - Quartz: randomized dual coordinate ascent method
  - arbitrary sampling
  - direct primal-dual analysis
  - additional data-driven speedup
- Future work
  - Quartz as a randomized SOR:

$$w^t = (1-\theta)w^{t-1} + \theta \nabla g^*(\bar{\alpha}^{t-1})$$

$$\alpha_i^t = \begin{cases} \left(1-\theta p_i{}^{-1}\right)\alpha_i^{t-1} + \theta p_i{}^{-1}\left(-\nabla\phi_i(A_i^\top w^t)\right) & , \quad i \in S_t \\ \alpha_i^{t-1} & , \quad i \notin S_t \end{cases}$$

  - Accelerated mini-batch dual coordinate ascent+data-driven speedup