“π is irrational”

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First Attempt

Theorem (J. H. Lambert, 1761)

$$\tan r \notin \mathbb{Q}, \;\; r \in \mathbb{Q} \setminus \{0\}.$$

Corollary: $\tan \frac{\pi}{4} = 1 \Rightarrow \pi \notin \mathbb{Q}$.
Proof:

$$\begin{aligned}\tan r =& \frac{r - \frac{r^3}{3!} + \frac{r^5}{5!} - \frac{r^7}{7!} + \cdots}{1 - \frac{r^2}{2!} + \frac{r^4}{4!} - \frac{r^6}{6!} + \cdots} \\ =& \frac{r}{1 + \frac{-r^2}{3 + \frac{-r^2}{5 + \frac{-r^2}{7 + \frac{-r^2}{9 + \cdots}}}}}.\end{aligned}$$

For a continued fraction $y = [b_0, a_1, b_1, a_2, b_2, \cdots],\ a_i, b_i \in \mathbb{Z}$,

1. $$y = b_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \cdots}}} = b_0 + \frac{\lambda_1 a_1}{\lambda_1 b_1 + \frac{\lambda_1 \lambda_2 a_2}{\lambda_2 b_2 + \frac{\lambda_2 \lambda_3 a_3}{\lambda_3 b_3 + \cdots}}};$$

2. if $b_0 = 0,\ |a_i| < |b_i|,\ \forall i \geqslant 1$, then
   1. $|y| \leqslant 1$;
   2. writing
   $$y_n := \frac{a_n}{b_n + \frac{a_{n+1}}{b_{n+1} + \frac{a_{n+2}}{b_{n+2} + \cdots}}},$$
   if $\exists N > 0$ s.t. $|y_n| \neq 1,\ \forall n \geqslant N$, then y is irrational.

$$\tan\frac{p}{q} = \cfrac{p/q}{1+\cfrac{-p^2/q^2}{3+\cfrac{-p^2/q^2}{5+\cfrac{-p^2/q^2}{7+\cdots}}}} = \cfrac{p}{q+\cfrac{-p^2}{3q+\cfrac{-p^2}{5q+\cfrac{-p^2}{7q+\cdots}}}}.$$

Define

$$y_n := \cfrac{-p^2}{(2n+1)q+\cfrac{-p^2}{(2n+3)q+\cfrac{-p^2}{(2n+5)q+\cdots}}},$$

and choose n large s.t. $p^2 < 2nq \Rightarrow |-p^2| < (2(n+k)+1)q$, then $|y_{n+k}| \leqslant 1, \ k \geqslant 0$. Moreover,

$$|y_n| = \frac{p^2}{(2n+1)q + y_{n+1}} \leqslant \frac{p^2}{2nq} < 1, \ \forall n \geqslant N.$$

□

A Mordern (Nicer) Approach

Theorem (I. Niven, 1947)

$$\cos r \notin \mathbb{Q}, \ \ r \in \mathbb{Q} \setminus \{0\}.$$

Lemma

If $h(x) = x^n g(x)/n!$ *where* $g \in \mathcal{P}[\mathbb{Z}]$, *then* $h^{(j)}(0) \in \mathbb{Z}$, $\forall j \geqslant 0$, *and is divisible by* $n+1$ *except* $j = n$. *But* $(n+1) \mid h^{(n)}(0)$ *if* $g(0) = 0$.

Lemma

If $f(x)$ *is a polynomial in* $(r-x)^2$, *then* $f^{(j)}(r) = 0$, $\forall j$ *odd.*

Proof: since cos is even, let $r = a/b,\ a, b > 0$. Define

$$f(x) := \frac{x^{p-1}(a-bx)^{2p}(2a-bx)^{p-1}}{(p-1)!},$$

where p is an odd prime to be chosen. Further define

$$F(x) := f(x) - f^{(2)}(x) + f^{(4)}(x) - \cdots - f^{(4p-2)}(x),$$

and we have $\frac{\mathrm{d}}{\mathrm{d}x}(F'(x)\sin x - F(x)\cos x) = f(x)\sin x$, and

$$\int_0^r f(x)\sin x \mathrm{d}x = F'(r)\sin r - F(r)\cos r + F(0). \qquad (*)$$

For the RHS of $(*)$,

- rearrange $f(x) = (r-x)^{2p}(r^2 - (r-x)^2)^{p-1} b^{3p-1}/(p-1)!$, Lemma 2 $\Rightarrow$ $F'(r) = 0$;
- Lemma 1 $\Rightarrow$ $p \mid f^{(j)}(0) \in \mathbb{Z}$, $\forall j$, except $j = p-1$.

In fact, $p \nmid f^{(p-1)}(0) = a^{2p}(2a)^{p-1}$, if we choose $p > a$. Thus $p \nmid F(0) =: q$, $(p, q) = 1$.

Note that $f(r-x) = x^{2p}(a^2 - b^2x^2)^{p-1} b^{p+1}/(p-1)!$. By Lemma 1, $p \mid f^{(j)}(r) \in \mathbb{Z}$, since $x \mid g(x) = x^{p+1}(a^2 - b^2x^2)^{p-1} b^{p+1}$. Thus $F(r) = pm$ for some integer m.

Suppose $\cos r = d/k,\ d, k \in \mathbb{Z},\ k > 0$. Then

$$k \int_0^r f(x) \sin x \mathrm{d}x = -pmd + kq.$$

Choose $p > k$ s.t. $p \nmid kq \Rightarrow -pmd + kq \neq 0$. Also one estimates

$$\left| k \int_0^r f(x) \sin x \mathrm{d}x \right| \leqslant kr \sup_{0<x<r} f(x) < k \frac{r^{4p-1} b^{3p-1}}{(p-1)!} \to 0,$$

as $p \to \infty$. Choose p large s.t. it lies in $(-1, 1)$. $\square$

Corollary: π is irrational. If we are only interested in π, consider (Y. Iwamoto, 1949)

$$f(x) = \frac{x^n(1-x)^n}{n!},$$

for some integer n to be chosen, and

$$F(x) = b^n \left(\pi^{2n} f(x) - \pi^{2n-2} f^{(2)}(x) + \cdots + (-1)^n f^{(2n)}(x)\right),$$

if assuming $\pi^2 = a/b$. Same type of argument gives irrationality of π^2.

Corollary: other tirgonometric functions are irrational on $\mathbb{Q} \setminus \{0\}$, and their inverses are either 0 or irrational on $\mathbb{Q}$.

What about e?

Fourier, 1815: Write $e = 1 + \frac{1}{1!} + \cdots + \frac{1}{n!} + R$. Assume $e = a/b$ and fix $n > b$, then $bn!\left(1 + \frac{1}{1!} + \cdots + \frac{1}{n!}\right) \in \mathbb{N}^+$, but

$$bn!R < b\left(\frac{1}{n+1} + \frac{1}{(n+1)^2} + \cdots\right) = \frac{b}{n} < 1.$$

Theorem (J. F. Koksma, 1949)

$$e^r \notin \mathbb{Q},\ \ r \in \mathbb{Q} \setminus \{0\}, and\ \log r \notin \mathbb{Q},\ \ r \in \mathbb{Q} \setminus \{1\}.$$

Proof: it suffices to prove for $r \in \mathbb{N}^+$. Assume $e^r = a/b$. Use Iwamoto's $f(x)$ and define

$$F(x) = r^{2n} f(x) - r^{2n-1} f'(x) + \cdots - r f^{(2n-1)}(x) + f^{(2n)}(x),$$

$F(0), F(1) \in \mathbb{Z}$ (Lemma 1). Note $\frac{\mathrm{d}}{\mathrm{d}x}(e^{rx}F(x)) = r^{2n+1} e^{rx} f(x)$, and

$$b r^{2n+1} \int_0^1 e^{rx} f(x) \mathrm{d}x = aF(1) - bF(0) \in \mathbb{Z},$$

but we have $0 < f(x) < 1/n!$ for $0 < x < 1$ and hence

$$0 < b r^{2n+1} \int_0^1 e^{rx} f(x) \mathrm{d}x < b e^r \frac{r^{2n+1}}{n!} < 1,$$

if we choose n sufficiently large. □

Transcendental?

Charles Hermite, 1873: "e is transcendental".

Proof: supppose $a_m e^m + a_{m-1} e^{m-1} + \cdots + a_1 e + a_0 = 0, \; a_j \in \mathbb{Z}$.
Same type of argument works if applied to

$$f(x) = \frac{x^{p-1}(x-1)^p(x-2)^p \cdots (x-m)^p}{(p-1)!},$$
$$F(x) = f(x) + f'(x) + \cdots + f^{(mp+p-1)}(x).$$

Compute $\frac{\mathrm{d}}{\mathrm{d}x}(e^{-x}F(x))$ to get $\int_0^j e^{-x} f(x) \mathrm{d}x = F(0) - e^{-j}F(j)$.

$$\sum_{j=0}^{m} a_j e^j \int_0^j e^{-x} f(x) \mathrm{d}x = -\sum_{j=0}^{m} \sum_{i=0}^{mp+p-1} a_j f^{(i)}(j). \qquad (**)$$

Apply Lemma 1 to $f(x), f(x+1), \cdots, f(x+m)$ with $n = p-1$,

- $f^{(i)}(j) \in \mathbb{Z},\ \forall i = 0, \cdots, mp+p-1,\ j = 0, \cdots, m;$
- $p \mid f^{(i)}(j)$ except for $(i,j) = (p-1, 0)$.

Indeed, $p \nmid f^{(p-1)}(0) = (-1)^p(-2)^p \cdots (-m)^p$, if choose $p > m$. Further choose $p > |a_0| \ \Rightarrow\ p \nmid a_0 f^{(p-1)}(0)$, thus the RHS of $(**)$ is a non-zero integer. But

$$|\mathrm{LHS}| \leqslant \sum_{j=0}^{m} |a_j| e^m m \cdot 1 \cdot \frac{m^{mp+p-1}}{(p-1!)} < 1,$$

if p is chosen sufficiently large. □

Ultimate Use of Hermite's Technique

Theorem (F. von Lindemann, 1882; K. Weierstrass, 1885)

Given any distinct (complex) algebraic no's $\alpha_1, \cdots, \alpha_m$, *the values* $e^{\alpha_1}, \cdots, e^{\alpha_m}$ *are linearly independent over the field of algebraic no's.*

Corollaries:

- e^α is transcendental for algebraic $\alpha \neq 0$.
- If π were algebraic, then $e^{i\pi} + 1 = 0$.
- If $\sin \alpha$ were algebraic for algebraic $\alpha \neq 0$, then $e^{i\alpha} - e^{-i\alpha} - 2i \sin \alpha e^0 = 0$. Same for other trigonometrics.
- All the values of $\log \alpha$ and other inverse trigonometric functions are transcendental for algebraic $\alpha \neq 1$.

Reference Books

1. Ivan Niven, Irrational Numbers, The Mathematical Association of America, 1956.
2. Martin Aigner and G unter M. Ziegler, Proofs from The Book, Springer, 1988.
3. Julian Havil, The Irrationals, Princeton University Press, 2012.