Gravity-wave scattering by turbulence in the atmosphere and ocean

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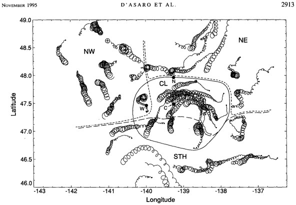
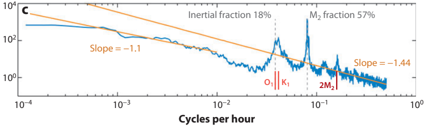
Atmosphere–ocean as two-time-scale systems

Mid-latitude dynamics: a mixture of

1. balanced motion
   - time scales $\gg 1$ day,
   - near geostrophic and hydrostatic balance.
2. inertia-gravity waves (IGWs)
   - fast oscillations, with $T \lesssim 1$ day,
   - generated by topography, convection, winds, tides, spontaneously...



d'Asaro et al 1995



E_K vs ω at 27°N

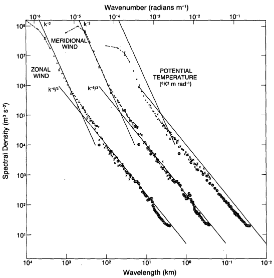
Phillips & Rintoul 2000; Ferrari & Wunsch 2009

Energy spectra

Nastrom–Gage spectrum: still puzzling after all these years.

Atmospheric spectra, from aircraft measurements (8-12 km altitude)

G. D. NASTROM AND K. S. GAGE



- large scales, $E(k) \sim k^{-3}$: balanced dynamics, ✓
- mesoscales, $E(k) \sim k^{-5/3}$:
  - 3D turbulence, ✗
  - small-scale convection, ✗
  - stratified turbulence. ?

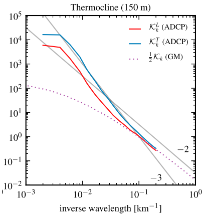
Energy spectra

Ocean spectra: ship tracks

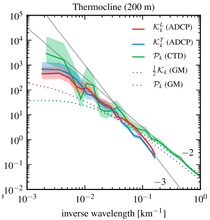
- large-scale $E(k) \propto k^{-3}$, ✓
- sub-mesoscales $E(k) \propto k^{-2}$. ✗

Callies & Ferrari 2013

Gulf Stream



North Pacific



Callies, Bühler & Ferrari:

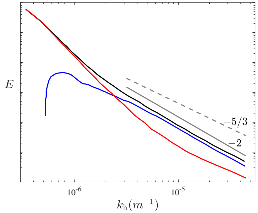
Callies et al 2014–18

- revisit aircraft and ship-track data,
- 2 velocity components enable Helmholtz decomposition,
- assume linear IGWs to decompose spectrum.

$$\boldsymbol{u} = \underbrace{\nabla\phi + \overbrace{\nabla^\perp\psi}^{\text{balanced}}}_{\text{IGWs}}, \qquad E(k) = E_{\text{bal}}(k) + E_{\text{IGW}}(k).$$

Energy spectra

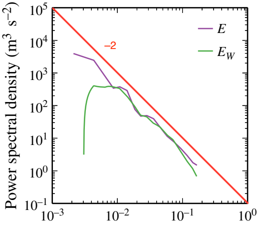
Atmosphere



E_{tot}, E_{IGW}, E_{bal}

adapted from Callies et al 2014

Ocean



E_{tot}, E_{IGW}

Bühler et al 2014

Conclusions:

- observed spectra consistent with IGWs dominating shallow part,
- no explanation for the spectral shape, k^{-2}, $k^{-5/3}$,
- controversial.

Li & Linborg 2018

IGWs in geostrophic turbulence

Hypothesis:

- IGWs forced at large scales,
- cascade to small scales through interactions with the balanced flow,
- catalytic interaction, Lelong & Riley 1991, Bartello 1995
- wave–wave interactions negligible.

Mathematical problem:

- predict IGW scattering resulting from interaction with geostrophic turbulence,
- waves in random media. Rhyzhik, Keller & Papanicolaou 1996

IGWs in turbulence

Balanced flow:

- geostrophic, $\boldsymbol{u} = (-\psi_y, \psi_x, 0)$, $b \propto \psi_z$,
- homogeneous, stationary ψ random field, with given correlation.

IGWs:

- intrinsic frequency

$$f \leq \omega = \sqrt{f^2 \cos^2\theta + N^2 \sin^2\theta} \leq N,$$

with $\theta = \tan^{-1}(k_\mathrm{h}/k_\mathrm{v})$, angle between $\boldsymbol{k}$ and $\boldsymbol{z}$,

- fast compared with balanced flow,

$$\mathrm{Ro} = U/(fL) \ll 1.$$

IGWs in turbulence

Boussinesq simulation

ζ_{geo} w_{IGW}

IGWs in turbulence

Asymptotic reduction using $\mathrm{Ro} = \epsilon \ll 1$:

Write governing equations for IGWs as

$$\partial_t \phi + \mathcal{L}(\nabla)\phi + \epsilon^{1/2}\mathcal{N}(\boldsymbol{x}/\epsilon, \nabla)\phi = 0,$$

with spec $\mathcal{L}(\nabla) = \{\mathrm{i}\omega\}$.

Define the Wigner matrix

$$W(\boldsymbol{x}, \boldsymbol{k}, t) = \int \phi(\boldsymbol{x} + \epsilon \boldsymbol{y}/2, t)\phi^*(\boldsymbol{x} - \epsilon \boldsymbol{y}/2, t)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{y}}\,\mathrm{d}\boldsymbol{y}$$

and write its evolution. Solve pertubatively using multiple scale method: $\boldsymbol{\xi} = \boldsymbol{x}/\epsilon$:

$$W(\boldsymbol{x}, \boldsymbol{k}, t) = W_0(\boldsymbol{x}, \boldsymbol{k}, t) + \epsilon^{1/2} W_1(\boldsymbol{x}, \boldsymbol{\xi}, \boldsymbol{k}, t) + \cdots$$

IGWs in turbulence

To leading order,

$$W_0(\boldsymbol{x},\boldsymbol{k},t) = a(\boldsymbol{x},\boldsymbol{k},t)\boldsymbol{b}(\boldsymbol{x},\boldsymbol{k})\boldsymbol{b}^*(\boldsymbol{x},\boldsymbol{k}),$$

where $a(\boldsymbol{x},\boldsymbol{k},t)$ is a wavenumber-resolving energy density.

Next order, using ergodicity: kinetic equation

$$\frac{\partial a}{\partial t} + \nabla_{\boldsymbol{k}}\omega \cdot \nabla_{\boldsymbol{x}} a - \nabla_{\boldsymbol{x}}\omega \cdot \nabla_{\boldsymbol{k}} a = \int_{\mathbb{R}^3} \sigma(\boldsymbol{k},\boldsymbol{k}')a(t,\boldsymbol{x},\boldsymbol{k}')\mathrm{d}\boldsymbol{k}' - \Sigma(\boldsymbol{k})a(t,\boldsymbol{x},\boldsymbol{k}),$$

- $\sigma(\boldsymbol{k},\boldsymbol{k}')$ is the differential scattering cross-section,
- $\Sigma(\boldsymbol{k}) = \int \sigma(\boldsymbol{k},\boldsymbol{k}')\mathrm{d}\boldsymbol{k}'$, total cross-section.

IGWs in turbulence

The differential scattering cross section $\sigma(\boldsymbol{k}, \boldsymbol{k}')$ encodes the impact of turbulence on IWGs:

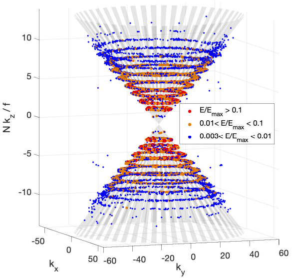
$$\sigma(\boldsymbol{k}, \boldsymbol{k}') \propto \delta(\omega(\boldsymbol{k}) - \omega(\boldsymbol{k}'))\, E(\boldsymbol{k} - \boldsymbol{k}'),$$

with $E(\boldsymbol{k})$ the turbulent energy spectrum.

- scattering results from resonant triads:

$$\text{IGWs} + \text{flow} = \text{IGWs},$$
$$\omega + 0 = \omega.$$

- energy transfers restricted to constant-frequency cone.



Danioux & V 2016, Savva & V 2018, Savva, Kafiabad & V in prep

Wavenumber diffusion

Assume further

- statistical homogeneity, $\nabla_{\boldsymbol{x}}\omega = \nabla_{\boldsymbol{x}} a = 0$,
- IGW scale $\ll$ flow scale.

Kafiabad, Savva & V 2019

Scattering equation reduces to diffusion equation

$$\partial_t a = \nabla_{\boldsymbol{k}} \cdot (\boldsymbol{D} \cdot \nabla_{\boldsymbol{k}} a) \,,$$

where $\boldsymbol{D}(\boldsymbol{k})$ the diffusivity tensor

$$D_{ij}(\boldsymbol{k}) = -\frac{1}{2} k_m k_n \int_{-\infty}^{\infty} \frac{\partial^2 \Pi_{mn}}{\partial x_i \partial x_j}(\boldsymbol{c}(\boldsymbol{k})s)\, \mathrm{d}s, \quad \Pi_{mn}(\cdot) = \langle U_m(\boldsymbol{x}+\cdot) U_n(\boldsymbol{x}) \rangle$$

McComas & Bretherton 1977, Müller 1976, 1977, Bal et al 2010

Wavenumber diffusion

Key properties:

- $\boldsymbol{D} \cdot \boldsymbol{c} = 0$: diffusion along the constant-frequency cone,
- no energy transfer between the two nappes of the cone.

Use spherical coordinates (k, ϕ, θ) and assume $\partial_\phi a = 0$.

Energy density $e(k,t) \propto k^2 a(k,t)$ satisfies

$$\partial_t e = \partial_k \left(Q\, k^5 \partial_k \left(k^{-2} e \right) \right),$$

where $Q = Q[N, f, \theta, E_{\text{geo}}(K_\text{h}, K_\text{v})]$.

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Wavenumber diffusion

Predictions

- energy confined to one nappe of cone (e.g., upward propagating IGWs),
- energy spreading on the cone with time scale $(Qk)^{-1} \propto k^{-1}\mathrm{Ro}^{-2}$.

Initial-value problem:

- for $e(k,0) = \delta(k - k_*)$,

$$e(k,t) = \tfrac{1}{2}k_*^{-2}\int_0^\infty J_4(k^{-1/2}\lambda)J_4(k_*^{-1/2}\lambda)\mathrm{e}^{-Q\lambda^2 t/4}\lambda\,\mathrm{d}\lambda$$
$$\sim k^{-2}t^{-5}.$$

Wavenumber diffusion

Numerical simulations

Check predictions against 3D Boussinesq simulations:

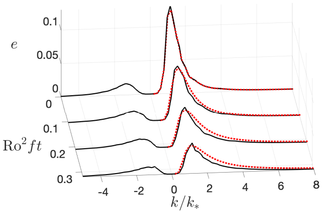
- pseudospectral, 768^3 resolution, Bartello 1995, Waite & Bartello 2004
- $\mathrm{Ro} = \zeta_{\mathrm{rms}}/f = 0.1$, $N/f = 32$,
- IGWs added to well-developed QG turbulence.

Numerical simulations

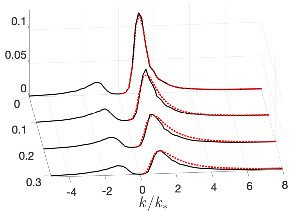
Initial-value problem

With peak $k_{\text{geo,h}} \approx 4$, take $k_{*\text{h}} = 16$.

Match spectra after a short transient during which diffusion approximation breaks down.



$\omega = 2f$



$\omega = 3f$

Wavenumber diffusion

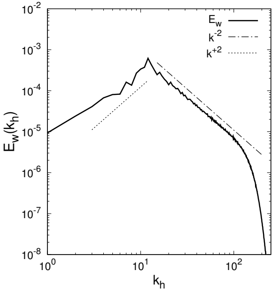
Forced problem:

- for $F(k) = \delta(k - k_*)$,

$$e(k) \propto \begin{cases} k^2 & \text{for } 0 < k < k_* \\ k^{-2} & \text{for } k > k_* \end{cases},$$

- constant flux $F = 4Qk^2 e(k)$,
- consistent with observed spectra.

Simulation with $k_{*h} = 12$.



Conclusions

- Statistical theory of linear IGWs in weak turbulent flow.
- Energy exchanged on constant-frequency cone through wave + wave + flow resonances (catalytic interactions).
- Diffusion approximation for $k_{\mathrm{geo}} \ll k_{\mathrm{IGW}} \ll \omega/U$:
  - k^{-2} equilibrium spectrum, consistent with observations,
  - forward scale cascade, isotropisation.
- Slow diffusion of frequency.
- For $k_{\mathrm{IGW}} \sim k_{\mathrm{geo}}$, scattering equation

$$\partial_t a(\boldsymbol{k},t) = \int \sigma(\boldsymbol{k},\boldsymbol{k}')a(\boldsymbol{k}',t)\,\mathrm{d}\boldsymbol{k}' - \int \sigma(\boldsymbol{k},\boldsymbol{k}')\,\mathrm{d}\boldsymbol{k}'\,a(\boldsymbol{k},t)\,,$$

  - energy exchanges between two nappes of cone.