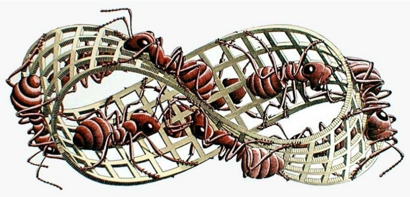
Vortex dynamics with a twist

fluid dynamics on a Möbius strip

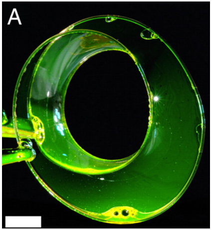
Jacques Vanneste
School of Mathematics and Maxwell Institute
University of Edinburgh, UK
www.maths.ed.ac.uk/˜vanneste



Motivation

Study two-dimensional fluid dynamics on a Möbius strip:

- application of ‘modern’ differential geometry to fluid dynamics,
- effect of the non-orientability,
- dynamics of soap films,
- pretty pictures.



Goldstein et al

Geophysical and astrophysical applications:

- hmm. . . exoplanets (maybe).

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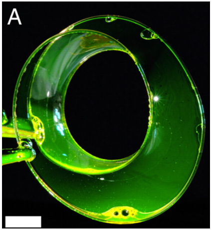

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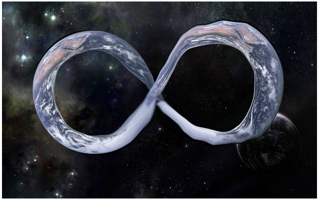


Möbius Strip Earth Society

Which Möbius strip?

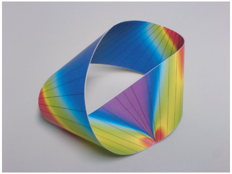
- Flat: periodic channel $(x, y) \in [0, 2\pi] \times [-1, 1]$ with

$$u(x + \pi, y) = u(x, -y), \quad v(x + \pi, y) = -v(x, -y).$$

  Too simple! Not embedded in $\mathbb{R}^3$, no pretty pictures.

- Genuine strip, made of inextensible material,
  - zero Gaussian curvature,
  - without external stresses,
    minimising elastic energy

    Starosin & Van Der Heijden 2007



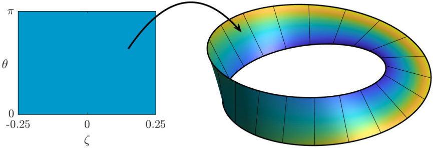
  Too complicated! No closed form expression.

- Ruled surface: simple and pretty.

Which Möbius strip?

Ruled surface parameterised by $-d \leq \zeta \leq d$ and $0 \leq \theta \leq \pi$:

$$x = (1 + \zeta \cos\theta)\cos(2\theta),$$
$$y = (1 + \zeta \cos\theta)\sin(2\theta),$$
$$z = \zeta \sin\theta,$$



Twist, $\boldsymbol{x}(\zeta, \pi) = \boldsymbol{x}(-\zeta, 0)$: not a global coordinate chart.

Metric: pulling back $g = dx \otimes dx + dy \otimes dy + dz \otimes dz$,

$$g = d\zeta \otimes d\zeta + \left(4 + 8\cos\theta\,\zeta + (3 + 2\cos(2\theta))\zeta^2\right) d\theta \otimes d\theta .$$

Area element:

$$\mu = |g|^{1/2} d\zeta \wedge d\theta ,$$

A pseudo 2-form! Can be interpreted as the normal to M.

Vortex dynamics on a manifold

Write the Euler equations as

$$(\partial_t + \mathcal{L}_{\boldsymbol{u}})\nu = d\pi, \qquad \operatorname{div} \boldsymbol{u} = 0,$$

with the momentum 1-form

$$\nu = g(\boldsymbol{u}, \cdot).$$

Kelvin-friendly formulation, in Cartesian coordinates,

$$D_t(u\,dx + v\,dy + w\,dz) = d(-p + |\boldsymbol{u}|^2/2).$$

Applying d takes the curl:

$$(\partial_t + \mathcal{L}_{\boldsymbol{u}})d\nu = 0,$$

where $d\nu$ is the vorticity 2-form.

Vortex dynamics on a manifold

In 2 dimensions:

$$\underbrace{d\nu}_{\text{2-form}} = \underbrace{\omega}_{\text{pseudoscalar}} \times \underbrace{\mu}_{\text{pseudo-2-form}} .$$

Incompressibility: introduce a (pseudoscalar) streamfunction such that

$$\boldsymbol{u} \lrcorner \mu = -d\psi .$$

This leads to the vorticity formulation

$$(\partial_t + \mathcal{L}_{\boldsymbol{u}})\omega = 0 , \qquad \omega = \Delta\psi .$$

Vortex dynamics on a Möbius strip

In (ζ, θ) coordinates:

$$\partial_t \omega + |g|^{-1/2} \partial(\psi, \omega) = 0,$$
$$|g|^{-1/2} \left(\partial_\zeta \left(|g|^{1/2} \partial_\zeta \psi \right) + \partial_\theta \left(|g|^{-1/2} \partial_\theta \psi \right) \right) = \omega.$$

Boundary conditions:

$$\psi(\zeta, \theta + \pi, t) = -\psi(-\zeta, \theta, t) \quad \text{and} \quad \psi(\zeta = \pm d, \theta, t) = \pm C(t).$$

To determine $C(t)$, use circulation conservation:

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = 0 \quad \text{with} \quad \Gamma = \sum_{\pm} \int_0^\pi |g|^{1/2}(\pm d, \theta) \partial_\zeta \psi(\pm d, \theta, t)\, d\theta.$$

Vortex dynamics on a Möbius strip

Effect of non-orientability

Since ω satisfies

$$\omega(\zeta, \theta + \pi, t) = -\omega(-\zeta, \theta, t),$$

the vorticity $d\nu$ always vanishes somewhere.

Only pseudo 2-forms can be integrated:

$$\int_M \mu, \; \int_M \omega^2 \mu, \; \int_M \omega^4 \mu, \cdots$$

The Casimirs

$$\int_M f(\omega)\mu$$

are well defined and invariant only for *even* $f(\cdot)$.

No Stokes theorem to relate the circulation Γ to an integral of ω.

Numerical simulations

Discretise:

- in space, using a finite-difference scheme,
- in time, using 3rd-order Adams–Bashforth.

To control small scales:

- add (unphysical) dissipative term $\epsilon\Delta\omega$ to the vorticity equation,
- use free slip boundary conditions,
- account for the change in Γ.

Visualise:

- the vorticity 2-form $d\nu$ is coordinate-independent,
- can be represented as a field of vectors normal to M,
- easier to use a colour plot of the pseudoscalar ω,
- change sign at the 'seam'.

Movie time

Vortex along the (single) boundary: Escher's ant.

Movie time

Vortex collision

Movie time

Shear instability

Movie time

Turbulence: with circulation around the strip.

Conclusions

- Fluid dynamics on manifolds:
  - is fun,
  - is made easy by exterior calculus.
- Non-orientability:
  - has little impact on familiar vortex-dynamics phenomena,
  - only affects global properties,
  - probably impact the condensate that results from the inverse energy cascade.