Two-dimensional flows in slowly deforming domains

Adiabatic invariance and geometric angle

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Motivation: effect of slow, conservative perturbations on perfect fluids

Simplest model:

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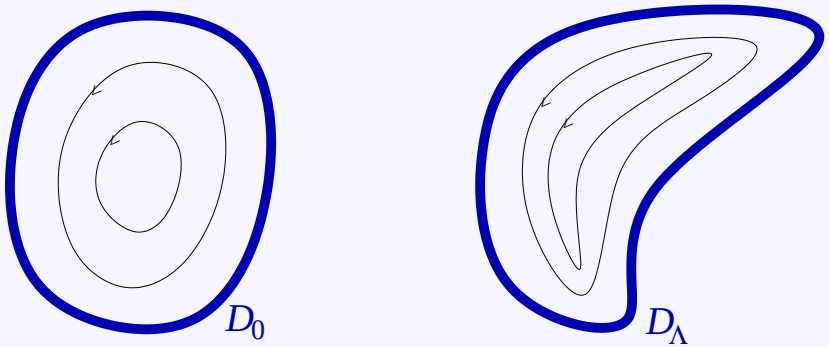

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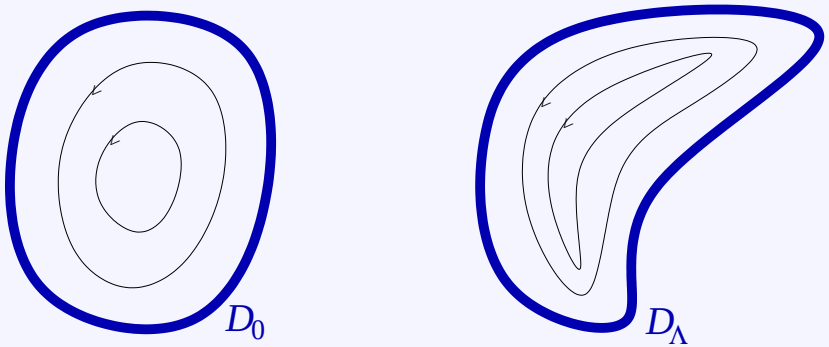


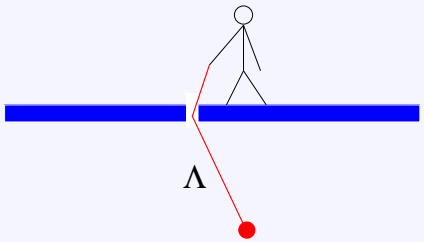
Two issues:

- Eulerian flow $u(x,t)$
- (Lagrangian) fluid-particle positions

for $O(1)$ deformations of the domain.

Flow in slowly deforming domain

Finite-dimensional analogue: Hamiltonian system with slowly variable parameters, e.g. pendulum with length $\Lambda(\epsilon t),\ \epsilon \ll 1$.



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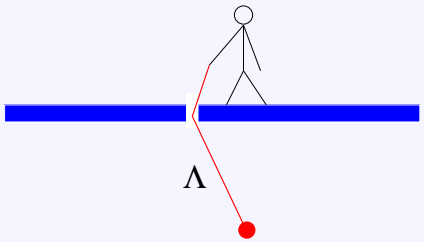

Amplitude (energy):

- determined by the adiabatic invariance of action I, $\Delta I = O(\epsilon)$.
- adiabatic invariance of I stems from invariance of $p\mathrm{d}q$.
- energy depends on Λ instantaneously.

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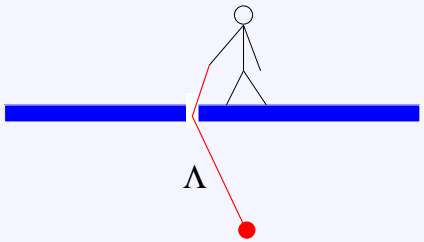

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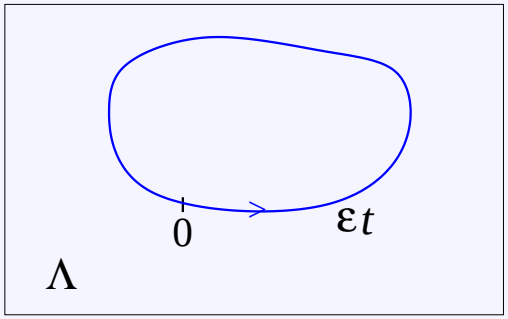
Angle $\langle\theta\rangle$:

- dynamical angle $\int \omega\,\mathrm{d}t$
- + Hannay–Berry (geometric) angle
- depends on path in parameter space

Flow in slowly deforming domains



Geometric angle depends only on the curve in parameter space, not on the speed at which this curve is traced.

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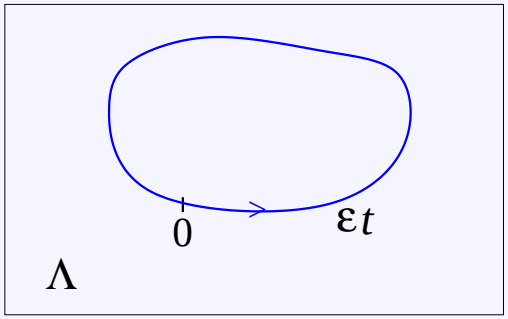

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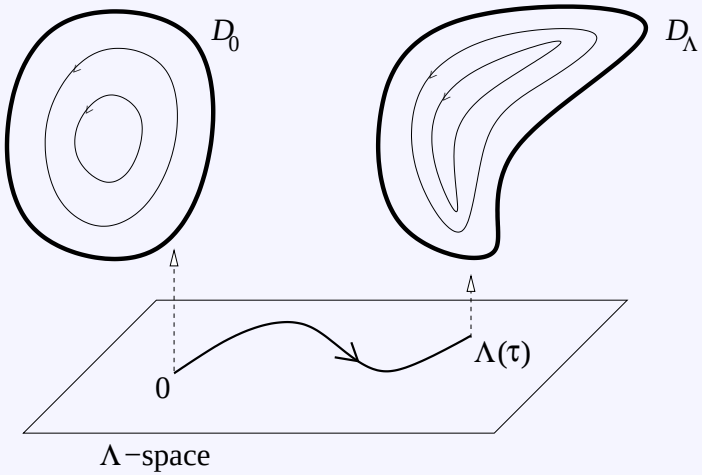
Analogy f-d system/fluid:

f-d system	fluid
amplitude	Eulerian flow
angle	particle position

Formulation

Domain defined by parameters: $\Lambda = \Lambda(\epsilon t),\ \epsilon \ll 1$.



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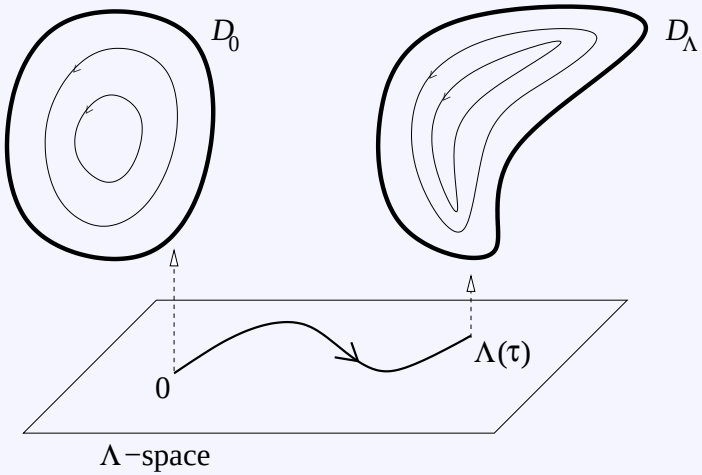

Perfect, incompressible flow in 2D: $\partial_t u + u \cdot \nabla u = -\nabla p, \quad \operatorname{div} u = 0$.

Vorticity formulation: $\omega = \operatorname{curl} u$, $u = \nabla^\perp \psi = (-\partial_y \psi, \partial_x \psi)$,

$$\partial_t \omega + [\psi, \omega] = 0, \quad \Delta \psi = \omega,$$

where $[\psi, \omega] = \partial_{x_1} \psi \partial_{x_2} \omega - \partial_{x_2} \psi \partial_{x_1} \omega$.

Formulation

Boundary condition: ∂D_Λ is a material curve

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The vorticity is rearranged: $\omega(x,t) = \omega_0(g_t^{-1}x)$, i.e.

$$\omega = \omega_0 \circ g_t^{-1},$$

where g_t is an area-preserving diffeomorphism, with $\dot{g}_t = v$.

Steady Flows: $[\psi, \omega] = 0$

$$\Rightarrow \quad \psi = F_\Lambda(\omega) \qquad \text{in } D_\Lambda\,.$$

Vorticity and streamfunction are functionaly related.

Perturbation expansion

Expand in power series:

$$\omega = \omega^{(0)} + \epsilon\omega^{(1)} + \cdots, \quad \psi = \psi^{(0)} + \epsilon\psi^{(1)} + \cdots$$

and substitute into 2D Euler.

Assuming all coefficients depend on ϵt, we find:

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At leading order, $[\omega^{(0)}, \psi^{(0)}] = 0$ (instantaneously) steady flow,

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At the next order, $\partial_{\epsilon t}\omega^{(0)} + [\psi^{(1)}, \omega^{(0)}] + [\psi^{(0)}, \omega^{(1)}] = 0.$

Rewriting as

$$\partial_{\epsilon t}\omega^{(0)} + [\phi, \omega^{(0)}] = 0, \quad \text{with} \quad \phi = \psi^{(1)} - F'(\omega^{(0)})\omega^{(1)}$$

shows that $\omega^{(0)}$ is rearranged by velocity $\nabla^{\perp}\phi$.

Eulerian flow

The leading-order flow $\omega^{(0)}$ can be found by imposing that it is:

- steady, $\psi^{(0)} = \Delta^{-1}\omega^{(0)} = F(\omega^{(0)})$ for some $F(\cdot)$.
- a rearrangement, $\omega^{(0)}(x,t) = \omega_0(g_\Lambda^{-1}x)$ for an area-preserving diffeomorphism g_Λ.

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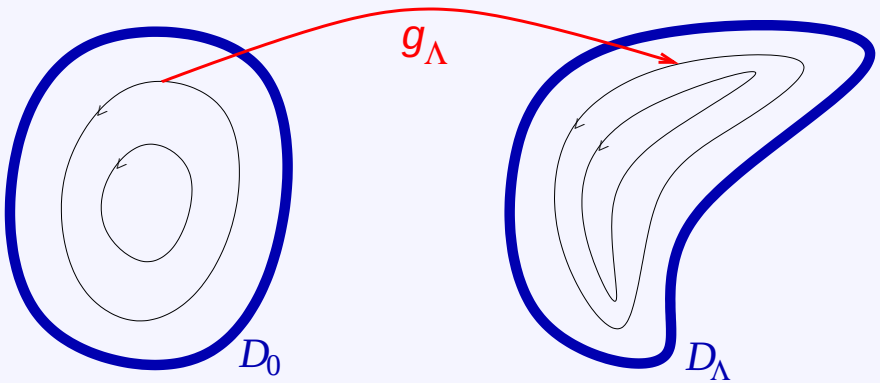

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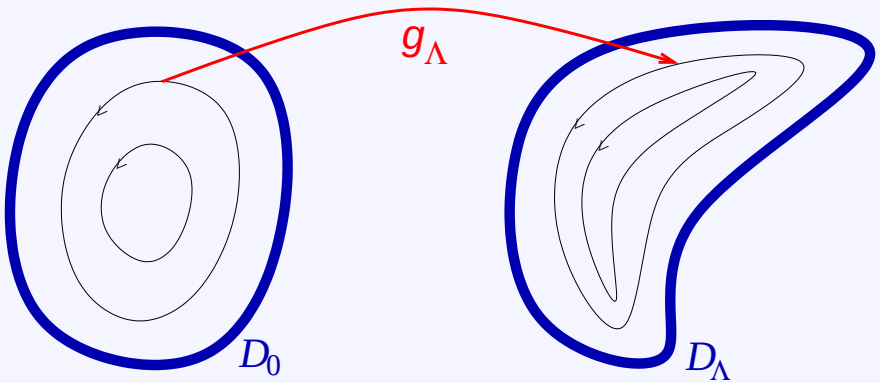

The map g_Λ

- satisfies $\det g_\Lambda = 1$,
- maps ∂D_0 to ∂D_Λ.
- depends on t only through Λ

Eulerian flow (continued)

The existence of g_Λ also answers the question of robustness of steady flows to domain deformation:

given a steady flow in the domain D_0, does it persist when the domain is deformed to D_Λ? (Wirosoetisno & V, 2005)

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The problem is written entirely in terms of g_Λ and F:

- g_Λ satisfies the nonlinear PDE,

$$\omega_0 = \Delta(F \circ \omega_0 \circ g_\Lambda^{-1}) \circ g_\Lambda,$$

- F is determined by a solvability condition.

Eulerian flow (continued)

Use an implicit-function-theorem argument to show:

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For adiabatic deformations, the constraint $T < \infty$ is a requirement of slowness.

Eulerian flow (continued)

How can we find g_Λ?

- The PDE may be solved numerically, e.g. using an iterative scheme.
- This is simpler if D_0 is a channel or a disc, but convergence appears limited to very small boundary deformations.
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Lie series: write g_Λ as the flow at δ of vector field $\nabla^\perp \varphi(\delta)$, and expand φ to find

$$f(g_\Lambda x) = f(x) + \delta[\varphi_1, f](x) + \frac{\delta^2}{2}([\varphi_1, [\varphi_1, f]](x) + [\varphi_2, f](x)) + \cdot$$

This leads to a sequence of linear problems for the φ_i and $F = F_0 + \delta F_1 + \cdots$.

Eulerian flow (continued)

Example: deformation of an axisymmetric flow in a disc.

Take D_0 to be the unit disc, and $\psi(0) = \psi(r, 0) = r^{1/2}$.

The deformed domain D_Λ is defined by

$$r = 1 + \delta \sum_m \Lambda_m \exp(\mathrm{i}m\sigma) + O(\delta^2)$$

We find that:

$$\varphi_1 = \sum_m \frac{\mathrm{i}}{m} r_m^\beta \Lambda_m \exp(\mathrm{i}m\sigma), \quad \text{with} \quad \beta_m = \sqrt{m^2 - 3/4} + 3/2 \quad \text{and} \quad F_1 = 0.$$

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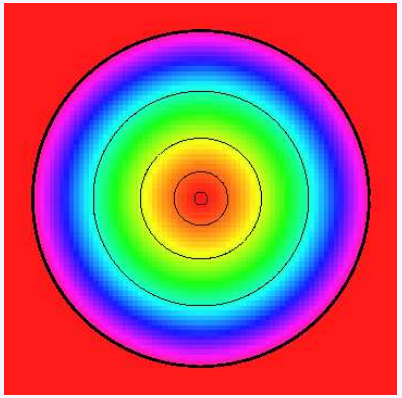

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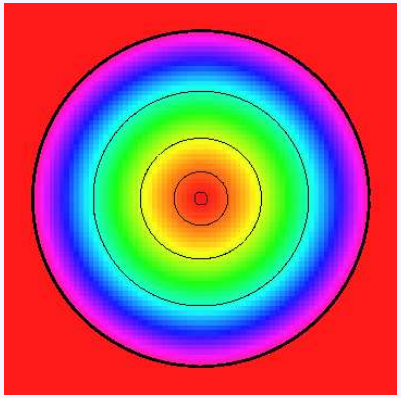

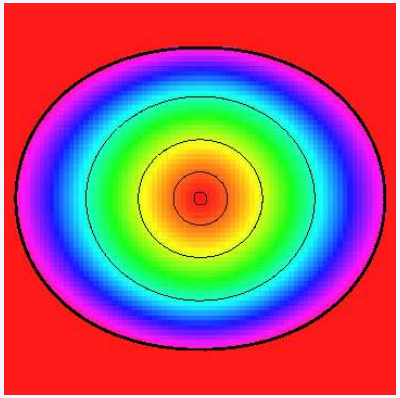

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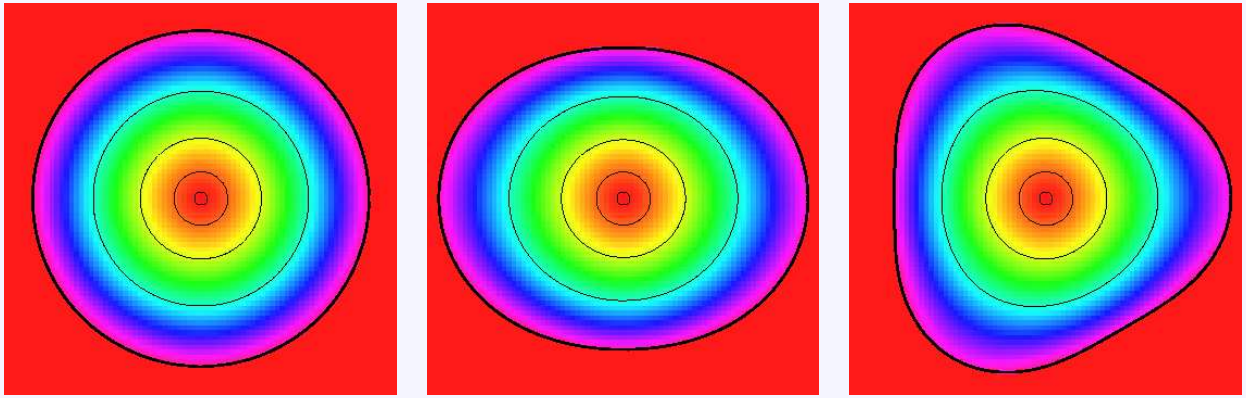

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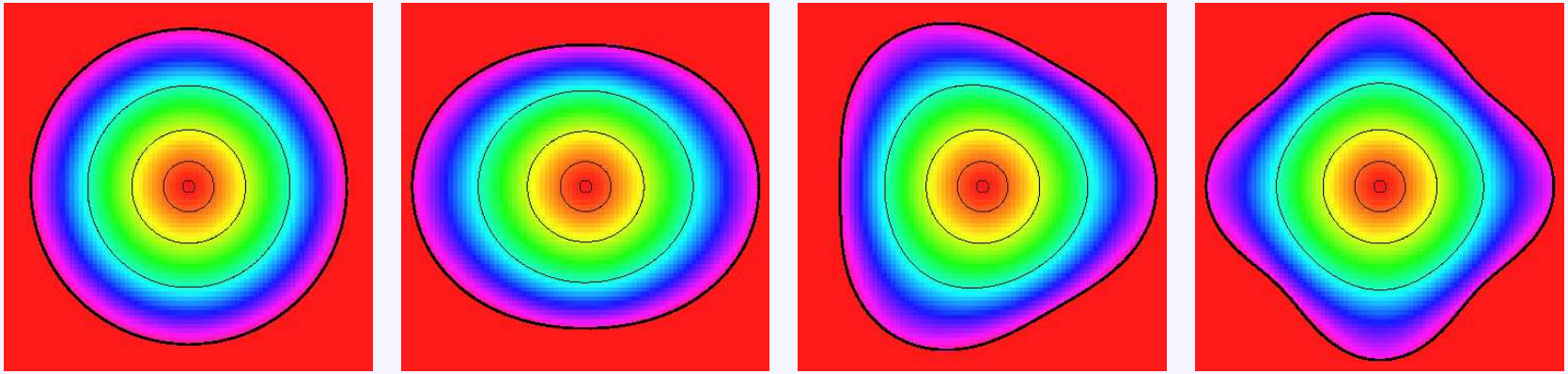

Lagrangian trajectories

For $t = O(\epsilon^{-1})$, position $(x, y)(t)$ of particles is governed by:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{\partial}{\partial y}\left(\psi^{(0)} + \epsilon\psi^{(1)}\right) + O(\epsilon^2), \quad \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\partial}{\partial x}\left(\psi^{(0)} + \epsilon\psi^{(1)}\right) + O(\epsilon^2).$$

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This is a ‘doubly perturbed’ Hamiltonian system with Hamiltonian

$$H(x, y; \Lambda) = \psi^{(0)}(x, y, \Lambda(\epsilon t)) + \epsilon\psi^{(1)}(x, y, \Lambda(\epsilon t)) \quad .$$

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Determination of $\psi^{(1)}$: recall

$$\partial_{\epsilon t}\omega^{(0)} + [\phi, \omega^{(0)}] = 0, \quad \text{with} \quad \phi = \psi^{(1)} - F'(\omega^{(0)})\omega^{(1)}.$$

Since $\omega^{(0)}$ is known, ϕ can be determined up to an arbitrary function of $\omega^{(0)}$:

$$\nabla^{\perp}\phi = \frac{\mathrm{d}}{\mathrm{d}\epsilon t} g_\Lambda x = d_\Lambda g_\Lambda \cdot \dot{\Lambda}.$$

Lagrangian trajectories (continued)

Hence, $\psi^{(1)} = \Delta^{-1}\omega^{(1)}$ is found by solving

$$\psi^{(1)} - F'(\omega^{(0)})\omega^{(1)} = \phi,$$

The gauge freedom in ϕ is fixed by the condition that the total vorticity is rearranged:

$$\iint_{\omega^{(0)}+\epsilon\omega^{(1)}=\Omega} \mathrm{d}x - \iint_{\omega^{(0)}=\Omega} \mathrm{d}x = O(\epsilon^2) \implies \oint_{\omega^{(0)}=\Omega} \omega^{(1)}\mathrm{d}s = 0,$$

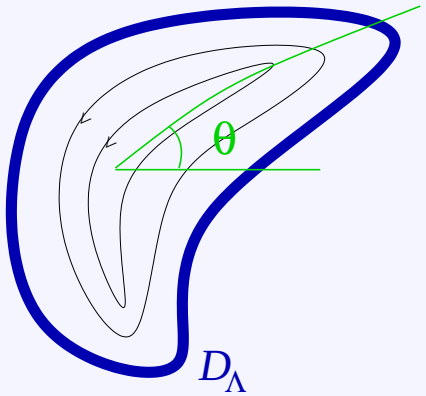
where $\mathrm{d}s = \mathrm{d}l/|\nabla\omega^{(0)}|$.
A consequence is that

$$\oint_{\omega^{(0)}=\Omega} \psi^{(1)}\mathrm{d}s = \oint_{\omega^{(0)}=\Omega} \phi\mathrm{d}s.$$

Lagrangian trajectories (continued)

To find particle trajectories, we use action–angle coordinates:

- $I = A(\omega^{(0)})$, area inside contour, is an adiabatic invariant,
- θ, conjugate to I, gives position along ω-contours.



Note: $2\pi \mathrm{d}s = A'(\omega^{(0)})\mathrm{d}\theta$.

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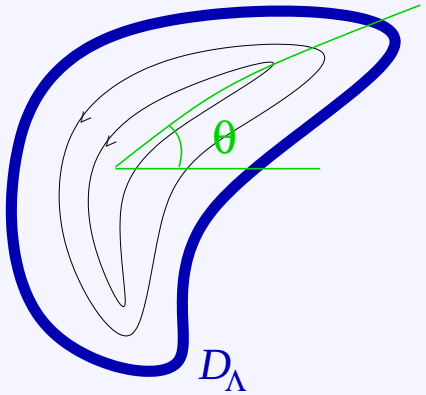

Note: $2\pi \mathrm{d}s = A'(\omega^{(0)})\mathrm{d}\theta$.

Use a generating function: $x_2 = \partial_{x_1} S(x_1, I), \quad \theta = \partial_I S(x_1, I)$, with the new Hamiltonian

$$\bar{H}(I, \theta, \Lambda) = \bar{\psi}^{(0)}(I, \theta, \Lambda) + \epsilon \bar{\psi}^{(1)}(I, \theta, \Lambda) + \partial_t S$$

Lagrangian trajectories (continued)

With $\partial_t S = \partial_t \bar{S} - \bar{x}_1 \partial_t \bar{x}_2$, we find the evolution equation for the angle:

$$\dot{\theta} = \nu + \epsilon \partial_I \left[\partial_t \bar{S} - \bar{x}_1 \partial_t \bar{x}_2 + \bar{\psi}^{(1)} \right],$$

where $\nu = \partial_I \bar{\psi}^{(0)}$ is the frequency.

Lagrangian trajectories (continued)

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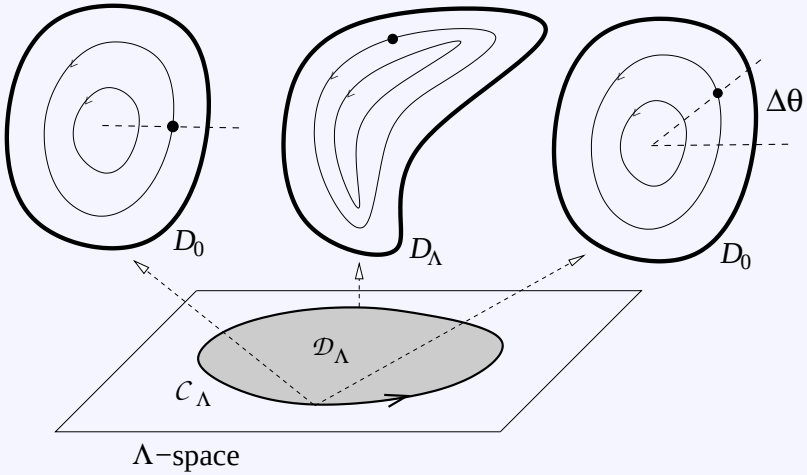

Consider cyclic deformation of the domain, and use averaging:

$$\Delta\theta \sim \frac{\mathrm{d}}{\mathrm{d}I} \int_0^t \left[\psi^{(0)} - \langle \bar{x}_1 \partial_t \bar{x}_2 + \bar{\psi}^{(1)} \rangle \right] \mathrm{d}t', \quad \text{where} \quad \langle \cdot \rangle = \frac{1}{2\pi} \int \cdot \, \mathrm{d}\theta.$$

Lagrangian trajectories (continued)

Now, use:

- $\int \psi^{(1)}\,\mathrm{d}\theta = \int \phi\,\mathrm{d}\theta = \int \Psi \cdot \dot{\Lambda}\,\mathrm{d}\theta$, where $\Psi = \sum_m \Psi_m \mathrm{d}\Lambda_m$ is a function-value one-form (connection), with $\nabla^{\perp}\Psi = d_\Lambda g_\Lambda$,
- $\bar{x}(I,\theta,\Lambda) = g_\Lambda x(I,\theta,0)$,
- Stokes' theorem.

to find

$$\Delta\theta(t) \sim \frac{\mathrm{d}}{\mathrm{d}I}\int_0^t \psi^{(0)}(\Lambda(s))\,\mathrm{d}s + \frac{\mathrm{d}}{\mathrm{d}I}\int_{\mathcal{D}_\Lambda} d_\Lambda\Psi - \frac{1}{2}[\Psi,\Psi]$$

with $[\Psi,\Psi] = \sum_{m,n}[\Psi_m,\Psi_n]d\Lambda_m \wedge d\Lambda_n$

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- $\int \psi^{(1)}\,\mathrm{d}\theta = \int \phi\,\mathrm{d}\theta = \int \Psi \cdot \dot{\Lambda}\,\mathrm{d}\theta$, where $\Psi = \sum_m \Psi_m \mathrm{d}\Lambda_m$ is a function-value one-form (connection), with $\nabla^{\perp}\Psi = d_\Lambda g_\Lambda$,
- $\bar{x}(I,\theta,\Lambda) = g_\Lambda x(I,\theta,0)$,
- Stokes' theorem.

to find

$$\Delta\theta(t) \sim \underbrace{\frac{\mathrm{d}}{\mathrm{d}I}\int_0^t \psi^{(0)}(\Lambda(s))\,\mathrm{d}s}_{\text{dynamical angle}} + \underbrace{\frac{\mathrm{d}}{\mathrm{d}I}\int_{\mathcal{D}_\Lambda} d_\Lambda\Psi - \frac{1}{2}[\Psi,\Psi]}_{\text{geometric angle}}$$

Lagrangian trajectories (continued)

Geometric angle:

- determined completely from the connection Ψ defined by g_Λ with suitable gauge,

< > - +

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Small deformations: $f(g_\Lambda x) = f(x) + \delta[\varphi_1, f](x) + \cdots$, leads to

$$\Psi = \delta d_\Lambda \varphi_1 + \frac{\delta^2}{2}(d_\Lambda \varphi_2 + [d_\Lambda \varphi_1, \varphi_1]) + \cdots$$

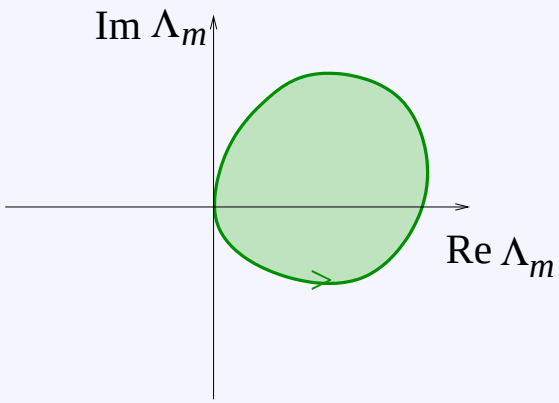
Lagrangian trajectories (continued)

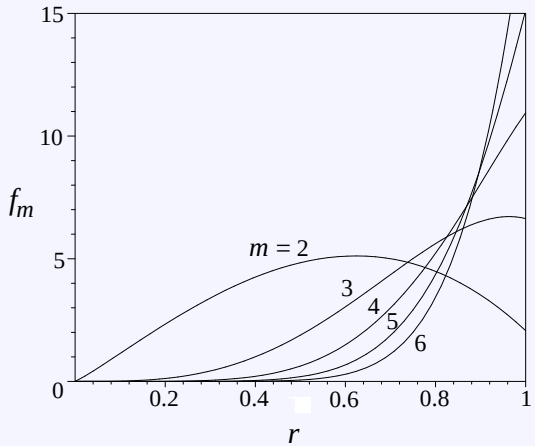
Example: deformed axisymmetric flow:

$r = 1 + \delta \sum_m \Lambda_m \exp(\mathrm{i}m\sigma) + O(\delta^2)$ gives

$$\langle\theta\rangle_{\text{geom}} = \delta^2 \sum_{m>0} f_m(r)\mathcal{A}_m + O(\delta^3),$$

where $\mathcal{A}_m$ is the area enclosed by path of Λ_m in the complex plane.





For $\psi(0) = r^{1/2}$, $f_m(r)$ is a sum of 2 powers of r.

Conclusions

- Procedure for computing flows in slowly deforming domains
- Eulerian flow is quasi-steady
- Eulerian flow depends only on the initial and final domain shapes
- Particle positions defined by their angle along vorticity contours
- Angle depends on the history of the domain, in a geometric manner

Extensions:

- rotation period T unbounded (non-parallel critical levels)
- 3D flows