Stokes-multiplier expansion in an inhomogeneous differential equation with a small parameter

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Abstract

Accurate approximations to the solutions of a second-order inhomogeneous equation with a small parameter ε are derived using exponential asymptotics. The subdominant homogeneous solutions that are switched on by an inhomogeneous solution through a Stokes phenomenon are computed. The computation relies on a resurgence relation, and it provides the ε-dependent Stokes multiplier in the form of a power series. The ε-dependence of the Stokes multiplier is related to constants of integration that can be chosen arbitrarily in the WKB-type construction of the homogeneous solution.

The equation under study governs the evolution of special solutions of the Boussinesq equations for rapidly rotating, strongly stratified fluids. In this context, the switching-on of subdominant homogeneous solutions is interpreted as the generation of exponentially small inertia-gravity waves.

Keywords: exponential asymptotics, Stokes phenomenon, WKB expansion, inertia-gravity waves.

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1 Introduction

This paper is concerned with the small-ε asymptotics of solutions of the linear inhomogeneous differential equation

$$\varepsilon^2\left(\frac{\mathrm{d}^2\zeta}{\mathrm{d}t^2}-\frac{2t}{1+t^2}\frac{\mathrm{d}\zeta}{\mathrm{d}t}\right)+\left((1+\varepsilon)\left(1+\frac{2\varepsilon}{1+t^2}\right)+\frac{1+t^2}{\beta^2}\right)\zeta=\frac{1+t^2}{\beta^2}, \tag{1.1}$$

where $\beta>0$ is a fixed parameter and $\varepsilon>0$. This equation, which is a limiting case of that examined by McWilliams & Yavneh (1998) and Vanneste & Yavneh (2004), arises in geophysical fluid dynamics; it governs the evolution of the amplitude of special solutions of the Boussinesq equations for a rotating stratified fluids. The limit $\varepsilon\downarrow 0$ is the so-called geostrophic limit, which corresponds to fast rotation and strong stratification, and is directly relevant to the earth's atmosphere and oceans. See Appendix A for details.

Like the low-order models proposed by Lorenz (1980), equation (1.1) provides a toy model for studying a general issue of central importance in geophysical fluid dynamics, namely the spontaneous generation of fast oscillations with $O(\varepsilon^{-1})$ frequencies (physically inertia-gravity waves) by slow, or balanced, motion with $O(1)$ frequency. To address this issue, one can consider solutions that evolve from balanced initial conditions, that is, from initial conditions that are free of oscillations. After some time, oscillations appear spontaneously; estimating their amplitude asymptotically then gives some insight into the mechanism of spontaneous generation. We refer the interested reader to Vanneste & Yavneh (2004) and Vanneste (2004) for background and references on the spontaneous generation of fast oscillations in geophysical fluids.

In the context of (1.1), the balanced motion is represented by a particular integral of (1.1) defined by its asymptotic expansion, while the fast oscillations are represented by the homogeneous solutions. The spontaneous generation of fast oscillations can then be identified as an instance of the Stokes phenomenon (e.g. Paris & Wood, 1995): the (subdominant) oscillations are switched on by the (dominant) balanced solution when t crosses a Stokes line, and their amplitude is exponentially small in ε. Estimating this amplitude thus amounts to the calculation of Stokes multipliers.

In their analysis of (1.1), Vanneste & Yavneh (2004) derived a leading-order approximation to the amplitude of the fast oscillations using the Kruskal–Segur technique of matched asymptotics in the complex t-plane (e.g. Hakim, 1998). In the present paper, we revisit the problem and apply the somewhat more sophisticated technique of exponential asymptotics based on resurgence. The upshot is a complete asymptotic expansion for the fast oscillations that are switched on; only a few terms in this expansion turn out to provide a remarkably accurate estimate for moderately small values of ε, as is confirmed by comparison with numerical solutions of (1.1).

In addition to its practical value, the asymptotic analysis of (1.1) has an interest from a more mathematical viewpoint, since it illustrates the structure of Stokes multipliers in problems with small parameters. The Stokes multipliers give the amplitude of the subdominant terms switched on by the Stokes phenomenon. In problems where the independent variable is the asymptotic parameters, these are simply constants; here, however, because the asymptotics is in terms of an independent parameter ε, the Stokes multipliers are functions of ε. What emerges from our analysis is that these functions can be conveniently computed as powers series.

The dependence of the Stokes multiplier on ε can be related to the fact that the (subdominant) homogeneous solutions are defined up to arbitrary ε-dependent factors. As a result, the coefficients in their power-series expansion in ε are defined up to arbitrary constants which appear as constants of integration (cf. Dingle, 1973). The choice of these constants affects the resurgence relation relating the expansions of the dominant and subdominant solutions. The appearance of an expansion for the Stokes multiplier does not seem to have been noted in earlier applications of exponential asymptotics to differential equations with small parameters.

Since this paper discusses exponential asymptotics for an inhomogeneous linear differential equation, the results are related to the ones in Howls & Olde Daalhuis (2003), in which hyperasymptotic expansions are given for solutions of inhomogeneous linear differential equations with a singularity of rank one. As in that paper, the solutions of (1.1) that are switched on when Stokes lines are crossed are homogeneous solutions. We start our analysis of (1.1) by obtaining their small-ε asymptotics.

2 Homogeneous solutions

Homogeneous solutions of (1.1) satisfy

$$\varepsilon^2\left(\frac{\mathrm{d}^2\zeta}{\mathrm{d}t^2}-\frac{2t}{1+t^2}\frac{\mathrm{d}\zeta}{\mathrm{d}t}\right)+\left((1+\varepsilon)\left(1+\frac{2\varepsilon}{1+t^2}\right)+\frac{1+t^2}{\beta^2}\right)\zeta=0, \tag{2.1}$$

and have formal expansions of the form

$$\zeta_{\mathrm{h}}(t)\sim \mathrm{e}^{f(t)/\varepsilon}\sum_{n=0}^{\infty}b_n(t)\varepsilon^n, \tag{2.2}$$

where $f(t)$ satisfies the equation

$$f'(t)^2+\frac{1+t^2+\beta^2}{\beta^2}=0. \tag{2.3}$$

The coefficients $b_n(t)$ satisfy the recurrence relation

$$2f'(t)b'_{n+1}(t)+\left(f''(t)-\frac{2t}{1+t^2}f'(t)+\frac{3+t^2}{1+t^2}\right)b_{n+1}(t) \\ =-b''_n(t)+\frac{2t}{1+t^2}b'_n(t)-\frac{2}{1+t^2}b_n(t), \tag{2.4}$$

for $n \geq -1$, where we take $b_{-1}(t)=0$. It is clear from (2.2)–(2.3) that the homogeneous solutions describe fast oscillations with $O(\varepsilon^{-1})$ frequency.

From (2.3) it follows that the differential equation has simple turning points at

$$t_p=\mathrm{i}\sqrt{1+\beta^2}, \qquad t_m=-\mathrm{i}\sqrt{1+\beta^2}. \tag{2.5}$$

It is convenient to define for $j=1,2$ and $k=p,m$ the functions

$$f_{jk}(t)=(-1)^j\frac{\mathrm{i}}{\beta}\int_{t_k}^{t}\sqrt{1+\beta^2+\tau^2}\,d\tau \\ =(-1)^j\frac{\mathrm{i}}{2\beta}\left(t\sqrt{1+\beta^2+t^2}+(1+\beta^2)\ln\left(\frac{t+\sqrt{1+\beta^2+t^2}}{t_k}\right)\right), \tag{2.6}$$

which satisfy (2.3). These are multi-valued functions with branch points at the turning points. We take as the branch cuts the half lines $t=\pm r\mathrm{i}\sqrt{1+\beta^2}$, $r>1$. Note that with these definitions we have the relations $f_{1p}=-f_{2p}$, $f_{1m}=-f_{2m}$, and for the principle branch of these functions

$$f_{jp}(t)=f_{jm}(t)+(-1)^j\frac{\pi(1+\beta^2)}{2\beta}, \qquad j=1,2. \tag{2.7}$$

With the particular choices $f(t)=f_{jk}(t)$, the recurrence relation (2.4) can be solved. Suitably normalised, the first term reads

$$b_{jk0}(t)=\left(\frac{t+\sqrt{1+\beta^2+t^2}}{t_k}\right)^{(-1)^j\mathrm{i}\beta/2}\frac{\sqrt{1+\beta^2+t^2}+(-1)^j\mathrm{i}\beta t}{(1+\beta^2+t^2)^{1/4}}. \tag{2.8}$$

The next terms are found by integration (2.4) and have the form

$$b_{jk,n+1}(t)=-b_{jk0}(t)\int^t\frac{b''_{jkn}(\tau)-\frac{2\tau}{1+\tau^2}b'_{jkn}(\tau)+\frac{2}{1+\tau^2}b_{jkn}(\tau)}{2f'_{jk}(\tau)b_{jk0}(\tau)}\,d\tau, \tag{2.9}$$

for $n=0,1,2,\cdots$. Note that in these expressions we do not specify the constants of integration; changing these constants of integration amounts to multiplying the homogeneous solutions by an arbitrary function of ε (cf. Dingle, 1973).

With the notation introduced, we define four special solutions of (2.1) via their asymptotic expansions:

$$\zeta_{jk}(t) \sim \mathrm{e}^{f_{jk}(t)/\varepsilon} \sum_{n=0}^{\infty} b_{jkn}(t)\varepsilon^n, \qquad j = 1, 2 \text{ and } k = p, m, \tag{2.10}$$

as $\varepsilon \downarrow 0$. For real t, the functions $\zeta_{1p}(t)$ and $\zeta_{2m}(t)$ are the recessive solutions and uniquely determined by (2.10); the functions $\zeta_{1m}(t)$ and $\zeta_{2p}(t)$ are dominant but can be uniquely defined using the Borel–Laplace transform (e.g. Balser, 2000) or by insisting that (2.10) holds in a sufficiently large sector of the complex t-plane.

3 Particular integral

A particular integral of (1.1) that is free of fast oscillations can be defined by the formal asymptotic expansion

$$\zeta_{\mathrm{inh}}(t) \sim \sum_{n=0}^{\infty} a_n(t)\varepsilon^n, \tag{3.1}$$

as $\varepsilon \downarrow 0$. We define ζ_{inh} completely by requiring that (3.1) be the complete asymptotic expansion for $t < 0$; that is, ζ_{inh} does not contains exponentially small terms for $t < 0$. Equivalently, ζ_{inh} can be defined as the Borel–Laplace transform (e.g. Balser, 2000) of the right-hand side of (3.1).

Substituting the expansion (3.1) into (1.1), we obtain the recurrence relation

$$\begin{aligned} a_0(t) &= \frac{1+t^2}{1+\beta^2+t^2}, \qquad a_1(t) = \frac{-\beta^2(3+t^2)}{(1+\beta^2+t^2)^2}, \\ a_n(t) &= \frac{-\beta^2\left((1+t^2)a_{n-2}''(t) - 2ta_{n-2}'(t) + 2a_{n-2}(t) + (3+t^2)a_{n-1}(t)\right)}{(1+\beta^2+t^2)(1+t^2)}, \end{aligned} \tag{3.2}$$

$n \geq 2$, from which the $a_n(t)$ can be derived. Note that in contrast to the coefficients $b_{jkn}(t)$, these coefficients do not involve arbitrary constants of integration; they are uniquely determined by (3.2).

Remark: it might seem that the $a_n(t)$ have poles at $t = \pm\mathrm{i}$. However, a local analysis at these points shows that these are regular points of the differential equation (1.1), and also that all the $a_n(t)$ are regular there. Hence, the only singularities of $a_n(t)$ are poles at the turning points t_k, $k = p, m$.

4 Stokes lines and Stokes multipliers

Through the Stokes phenomenon, the inhomogeneous solution (3.1) switches on homogeneous solutions of the form (2.2) when t crosses a Stokes line. To determine the Stokes lines, and to evaluate the corresponding Stokes multipliers, we now use the large-n asymptotics of the coefficients $a_n(t)$. Appendix B reviews the connection between the Stokes phenomenon and the growth of the coefficients in the divergent asymptotic expansions. For our purpose, it is sufficient to observe that when we substitute the ansatz

$$a_n(t) \sim K \sum_{s=0}^{\infty} \frac{\tilde{b}_s(t)\Gamma(n-s+\alpha)}{\left(-\tilde{f}(t)\right)^{n-s+\alpha}}, \tag{4.1}$$

as $n \to \infty$, into recurrence relation (3.2), then we find that $\tilde{f}(t)$ satisfies (2.3) and that the coefficients $\tilde{b}_s(t)$ satisfy recurrence relation (2.4). Equation (4.1) is a typical resurgence relation, which connects the late coefficients in the expansion of a particular solution to the coefficients of another solution.

From the remark following (3.2) we know that, in the complex t-plane, $a_n(t)$ has only singularities at the turning points t_p and t_m. On the other hand, according to (4.1), the singularities of $a_n(t)$ are located at the zeros of $\tilde{f}(t)$. It follows that the only candidates for $\tilde{f}(t)$ in (4.1) are the $f_{jp}(t)$ and $f_{jm}(t)$ defined above. The connection between resurgence and the Stokes phenomenon indicates that the only homogeneous functions of the form (2.2) that can be switched on by $\zeta_{\mathrm{inh}}(t)$ when a Stokes line is crossed behave like $\exp(\tilde{f}(t)/\varepsilon)$, where $\tilde{f}(t)$ is one of these four candidates. On a Stokes line these functions must be maximally subdominant; thus, the Stokes lines are

$$\{t \in \mathbb{C} \,|\, \Im f_{jk}(t) = 0\}, \qquad j = 1, 2 \text{ and } k = p, m, \tag{4.2}$$

and are illustrated in Figure 1.

The $\tilde{b}_s(t)$ that correspond to $f_{jk}(t)$ are the $b_{jks}(t)$ defined in (2.9). These, however, are defined up to arbitrary constants of integrations. But, the asymptotic validity of (4.1) requires a suitable choice for these integration constants. This important point is discussed in details in section 5.

Assuming that (4.1) holds, we determine the constants α and K by observing that as $t \to t_k$, $k = p, m$,

$$a_{2n}(t) \sim \frac{9\Gamma(3n)}{\Gamma(n)} \left(\frac{-\beta^2}{6t_k} \right)^{n+1} (t - t_k)^{-(3n+1)}, \tag{4.3}$$

$$a_{2n+1}(t) \sim (2 - \beta^2) \frac{3\Gamma(3n+2)}{2t_k\Gamma(n+1)} \left(\frac{-\beta^2}{6t_k} \right)^{n+1} (t - t_k)^{-(3n+2)}. \tag{4.4}$$

The reader can check this by substituting these relations in (3.2). Using (2.6), (2.8) and (4.3), we find that, for even n, both sides of (4.1) grow like $(t - t_k)^{-(3n/2+1)}$ as $t \to t_k$ provided that $\alpha = 1/2$. For odd n, the growth of the right-hand side with $\alpha = 1/2$ is apparently faster than the growth in $(t - t_k)^{-(3n+1)/2}$ expected from (4.4). This apparent mismatch is resolved by noting that the a_n are in fact obtained by summing four series of the form (4.1) (see (4.5) below), and that the dominant contributions cancel out for odd n, leading to a growth consistent with (4.4).

We summarise the results so far in

$$a_n(t) \sim \sum_{j=1,2} \sum_{k=p,m} \frac{K_{jk}}{2\pi \mathrm{i}} \sum_{s=0}^{\infty} \frac{b_{jks}(t)\Gamma(n - s + \frac{1}{2})}{(-f_{jk}(t))^{n-s+1/2}}, \tag{4.5}$$

as $n \to \infty$. Now we let $t \to t_k$ in (4.5) and compare the results with (4.3) and (4.4). The result is

$$K_{1p} = K_{2m} = \frac{-\mathrm{i}\sqrt{\beta\pi/2}}{(1+\beta^2)^{3/4}}, \qquad K_{2p} = \mathrm{i}K_{1p}, \qquad K_{1m} = -\mathrm{i}K_{1p}. \tag{4.6}$$

For the moment we assume that we have taken the correct constants of integration in the $b_{jks}(t)$ for (4.5) to hold. We now analyse what happens when t crosses the imaginary axis along the real axis. The part of the imaginary axis between the two turning points is a double Stokes line. When it is crossed, the two terms $K_{1p}\varepsilon^{-1/2}\zeta_{1p}(t)$ and $-K_{2m}\varepsilon^{-1/2}\zeta_{2m}(t)$ are switched on, with $\zeta_{1p}(t)$ and $\zeta_{2m}(t)$ exponentially small in ε, since $\Re f_{1p}(t) = \Re f_{2m}(t) = -\pi(1+\beta^2)/(4\beta) < 0$ for real t. (The opposite signs of the two terms switched on arise from the fact that t rotates around t_p and t_m in opposite senses as the Stokes lines are crossed.) In addition, since a double Stokes line is being crossed, there is also a switch on of an extra term that is exponentially smaller than (the already exponentially small) $K_{1p}\varepsilon^{-1/2}\zeta_{1p}(t) - K_{2m}\varepsilon^{-1/2}\zeta_{2m}(t)$; we will ignore this extra term. (For more details on double Stokes lines see for example Voros (1983)).

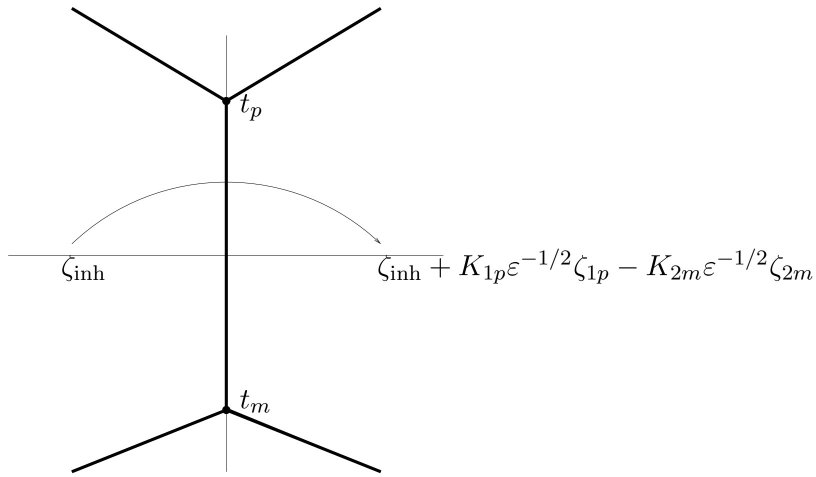

Figure 1: The Stokes lines and the Stokes phenomenon

Let $\zeta(t)$ be a solution of the inhomogeneous differential equation (1.1) that has $\zeta_{\rm inh}(t)$ as its complete asymptotic expansion for $t < 0$, then $\zeta(t)$ has $\zeta_{\rm inh}(t) + K_{1p}\varepsilon^{-1/2}\zeta_{1p}(t) - K_{2m}\varepsilon^{-1/2}\zeta_{2m}(t)$ as its asymptotic expansion for $t > 0$ (cf. Appendix B). The dominant part of $K_{1p}\varepsilon^{-1/2}\zeta_{1p}(t) - K_{2m}\varepsilon^{-1/2}\zeta_{2m}(t)$ is found from (2.2), (2.6), (2.8) and (4.6) to be given by

$$-\sqrt{\frac{2\beta\pi}{\varepsilon}}\,\mathrm{e}^{-\pi(1+\beta^2+\beta\varepsilon)/(4\beta\varepsilon)}\,\frac{\sqrt{1+\beta^2+t^2}\,\sin R(t,\varepsilon)+\beta t\cos R(t,\varepsilon)}{(1+\beta^2+t^2)^{1/4}\,(1+\beta^2)^{3/4}}, \tag{4.7}$$

where

$$\begin{aligned} R(t,\varepsilon) = \frac{1}{2\beta\varepsilon}&\left[t\sqrt{1+\beta^2+t^2} + (1+\beta^2)\ln\left(\frac{t+\sqrt{1+\beta^2+t^2}}{\sqrt{1+\beta^2}}\right)\right] \\ &+ \frac{\beta}{2}\ln\left(\frac{t+\sqrt{1+\beta^2+t^2}}{\sqrt{1+\beta^2}}\right). \end{aligned} \tag{4.8}$$

This dominant part depends only on the first coefficients b_{jk0}, which are entirely defined by (2.8) and is unaffected by the choice of constants of integrations for the b_{jks}, $s > 0$. To obtain higher-order corrections, however, it is necessary to determine the correct constants of integration in (2.9) which ensure that (4.5) is valid. This is considered in the next section.

5 The missing constants of integration

We first concentrate our analysis on the coefficient $b_{jk1}(t)$. Note that changing the constant of integration in (2.9) has the same effect as replacing $b_{jk1}(t)$ by $b_{jk1}(t)+\alpha_{jk1}b_{jk0}(t)$, where α_{jk1} is just a constant. Taking $b_{jk1}(t)+\alpha_{jk1}b_{jk0}(t)$ as the new $b_{jk1}(t)$ and determining all the other b-coefficients via recurrence relation (2.9) has the same effect as replacing each $b_{jk,s+1}(t)$ by $b_{jk,s+1}(t)+\alpha_{jk1}b_{jks}(t)$. The effect on (4.5) is

$$a_n(t) \sim \sum_{j=1,2}\sum_{k=p,m}\frac{K_{jk}}{2\pi\mathrm{i}}\left(\sum_{s=0}^{\infty}\frac{b_{jks}(t)\Gamma(n-s+\frac{1}{2})}{(-f_{jk}(t))^{n-s+(1/2)}}+\alpha_{jk1}\sum_{s=0}^{\infty}\frac{b_{jks}(t)\Gamma(n-s-\frac{1}{2})}{(-f_{jk}(t))^{n-s-(1/2)}}\right), \tag{5.1}$$

as $n\to\infty$.

The second step is the correction of $b_{jk2}(t)$ by $b_{jk2}(t)+\alpha_{jk2}b_{jk0}(t)$, where α_{jk2} is a constant. The effect is again that we replace all the higher $b_{jk,s+2}(t)$ by $b_{jk,s+2}(t)+\alpha_{jk2}b_{jks}(t)$. Continuing this process, we observe that adding the constants α_{jks} to $b_{jks}(t)$, $s=1,2,\cdots$ leads to the replacement of each $b_{jks}(t)$ by

$$\sum_{\ell=0}^{s}\alpha_{jk\ell}b_{jk,s-\ell},$$

where we have taken $\alpha_{jk0}=1$. The effect on the resurgence relation (4.5) is its replacement by

$$a_n(t) \sim \sum_{j=1,2}\sum_{k=p,m}\frac{K_{jk}}{2\pi\mathrm{i}}\sum_{\ell=0}^{\infty}\alpha_{jk\ell}\sum_{s=0}^{\infty}\frac{b_{jks}(t)\Gamma(n-s-\ell+\frac{1}{2})}{(-f_{jk}(t))^{n-s-\ell+(1/2)}}, \tag{5.2}$$

as $n\to\infty$. Note that we can write (5.2) as

$$a_n(t) \sim \sum_{j=1,2}\sum_{k=p,m}\sum_{\ell=0}^{\infty}\frac{K_{jk\ell}}{2\pi\mathrm{i}}\sum_{s=0}^{\infty}\frac{b_{jks}(t)\Gamma(n-s-\ell+\frac{1}{2})}{(-f_{jk}(t))^{n-s-\ell+(1/2)}}, \tag{5.3}$$

as $n\to\infty$, where we take $K_{jk\ell}=K_{jk}\alpha_{jk\ell}$. Hence, $K_{jk0}=K_{jk}$.

The analysis above shows that taking the correct constants of integration in (2.9) for (4.5) to be valid, is equivalent to taking the correct constants $K_{jk\ell}$, $\ell=1,2,3,\cdots$ in (5.3). Thus, instead of trying to determine the constants of integration in (2.9), we will restrict ourselves to determining the correct constants $K_{jk\ell}$ in (5.3); that is, we will fix the constants of integrations in the b_{jks} arbitrarily (so that (4.5) does not hold in general), and compute the $K_{jk\ell}$ for (5.3) to hold. Of course, once the $K_{jk\ell}$ are known, one can redefine a set of b_{jks} by

$$\frac{1}{K_{jk}}\sum_{\ell=0}^{s} K_{jk\ell} b_{jk,s-\ell}$$

so that (4.5) holds. In fact, a deeper analysis suggests that there is no alternative to the computation of the $K_{jk\ell}$; in particular, there is no obvious starting point for the integrals in (2.9) which would guarantee that (4.5) holds.

In this paper we are in the fortunate situation that we were able, in the previous section, to determine the exact value of $K_{jk0} = K_{jk}$. We now show how one can determine all the missing constants. The method is very similar to the one discussed in Olde Daalhuis (1999). We use a truncated version of (5.3); to be more precise, we take L terms in the ℓ sum in (5.3). Suppose that we want to determine the constants $K_{jk\ell}$ for $j = 1, 2$, $k = p, m$ and $\ell = 0, 1, \cdots, L-1$ numerically. These are $4L$ unknowns. Note that for large n and bounded t the truncated version of (5.3) is an 'equation' in which the only unknowns are exactly these $K_{jk\ell}$. Thus by taking either $4L$ different values for n, or for t, we obtain $4L$ equations with $4L$ unknowns. Since we can take n as large as we want, we will be able to determine these $4L$ constants to any precision.

In the previous section we used (4.5) to determine the switching-on when the double Stokes line is crossed. Since the constants of integration had not yet been determined, we were only able to derive the switched-on terms to leading-order. We can now use (5.3) to calculate higher-order approximations to these terms. Using (B.4)–(B.5), we can write the terms switched on when the double Stokes line is crossed as $\tilde{K}_{1p}(\varepsilon)\zeta_{1p}(t) - \tilde{K}_{2m}(\varepsilon)\zeta_{2m}(t)$, where now the Stokes multipliers $\tilde{K}_{1p}(\varepsilon)$ and $\tilde{K}_{2m}(\varepsilon)$ are functions of ε:

$$\tilde{K}_{jk}(\varepsilon) = \sum_{\ell=0}^{\infty} K_{jk\ell}\varepsilon^{\ell-(1/2)}. \tag{5.4}$$

The final result in the previous section is still the correct dominant term that is switched on when the double Stokes line is crossed. Why do we need all these extra terms in the expansion of the Stokes multipliers $\tilde{K}_{1p}(\varepsilon)$ and $\tilde{K}_{2m}(\varepsilon)$? We illustrate in the next section that with these extra terms we can obtain results that are also valid when ε is not very small. In fact, one example shows good results for the case $\varepsilon = 1$.

In the previous section a careful analysis near the turning points was necessary to determine the exact values for the K_{jk0}. This required considering complex values of t. Here we note that when we are only interested in determining the $K_{jk\ell}$ numerically (to any precision) we can confine our calculations to the real t-axis. In fact, the numerical method given above works better when we take real t, because, for a given ℓ, the four terms in the $4L$ equations for the $K_{jk\ell}$ have similar order of magnitude.

6 A numerical illustration

In the numerical illustration we take $\beta = 1$. For the constant of integration in (2.9) we take $+\infty$ as one of the limits of integration. Thus the b-coefficients are now well defined. We define $\zeta_{\rm inh}(t)$ as the solution of (1.1) that has (3.1) as its complete asymptotic expansion for $t < 0$. Hence, for $t < 0$ we can approximate this function via asymptotic expansion (3.1) in which we take the optimal number of terms. (See Olde Daalhuis (1998) for more detail on the optimal number of terms in asymptotic expansions.) From (5.3) it follows that the optimal number of terms in the approximation

$$\zeta_{\rm inh}(t) \approx \sum_{n=0}^{N-1} a_n(t)\varepsilon^n, \tag{6.1}$$

is $N = [|f_{jk}(t)/\varepsilon|]$, where $[\cdot]$ denotes the integer part. Note that this N depends on t, but that for real t it does not depend on the choice of j and k. In practice, we take at least three terms, that is, we take

$$N = \max\left([|f_{jk}(t)/\varepsilon|]\,, 3\right). \tag{6.2}$$

In this way the discontinuities in t of the approximant are less conspicuous.

For $t > 0$ we have to incorporate the Stokes phenomenon and approximate $\zeta_{\rm inh}(t)$ by

$$\begin{aligned}\zeta_{\rm inh}(t) \approx \sum_{n=0}^{N-1} a_n(t)\varepsilon^n &+ {\rm e}^{f_{1p}(t)/\varepsilon} \sum_{\ell=0}^{L-1} K_{1p\ell} \sum_{r=0}^{R-1} b_{1pr}(t)\varepsilon^{\ell+r-(1/2)} \\ &+ {\rm e}^{f_{2m}(t)/\varepsilon} \sum_{\ell=0}^{L-1} K_{2m\ell} \sum_{r=0}^{R-1} b_{2mr}(t)\varepsilon^{\ell+r-(1/2)},\end{aligned} \tag{6.3}$$

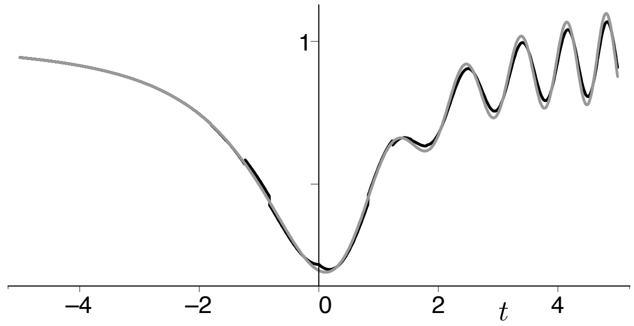

Figure 2: The exact (grey) and approximation (black) with $\varepsilon = 0.5$ and $L = R = 1$.

for some fixed values of L and R which we do not attempt to optimise.

In our first illustrations we take only the leading-order Stokes multipliers and only the first b-coefficients, that is we set $L = R = 1$. We take $t = -5$ and $\varepsilon = 1/2$ and compute $\zeta_{\text{inh}}(t)$ and its derivative via (6.1). To compute the 'exact' $\zeta_{\text{inh}}(t)$ numerically we take these two values and integrate the differential equation (1.1) in the positive t direction. This gives the grey curve in Figure 2. The black curve in this Figure is (6.1) for $t < 0$ and (6.3) with $L = R = 1$ for $t > 0$. Note that the black curve has discontinuous jumps because that the optimal number of terms N changes with t. If we did not incorporate the Stokes phenomenon then the black curve would be symmetric in t, and there would be no oscillations on the right-hand side of Figure 2. Note that, in this first illustration, the small parameter $\varepsilon = 0.5$ is not very small; taking only the dominant terms of the parts that are switched on gives nonetheless a very good approximation for $t > 0$.

In the second illustration we take the even larger value $\varepsilon = 1$, and the result is given in Figure 3. This time, the leading-order approximation is not as good for $t > 0$. More terms in the expansion of the Stokes multipliers, and more b-coefficients are needed to obtain a better approximation. We now fix $L = R = 5$. To compute the Stokes multipliers we use the approximation

$$a_n(t) \approx \sum_{j=1,2} \sum_{k=p,m} \sum_{\ell=0}^{4} \frac{K_{jk\ell}}{2\pi \mathrm{i}} \sum_{r=0}^{4} \frac{b_{jkr}(t)\Gamma(n-r-\ell+\frac{1}{2})}{(-f_{jk}(t))^{n-r-\ell+(1/2)}}. \tag{6.4}$$

In this equation, the $a_n(t)$ and $b_n(t)$ are calculated numerically from their recurrence relation,

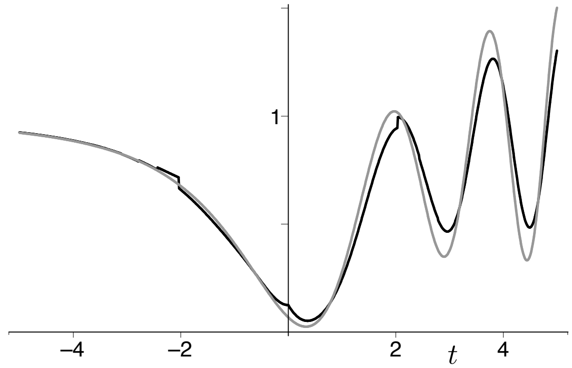

Figure 3: The exact (grey) and approximation (black) with $\varepsilon = 1$ and $L = R = 1$.

and the $K_{jk\ell}$ (including K_{jk0}) are regarded as unknowns. Hence, (6.4) is one equation with 20 unknowns. By taking $t = 1/2$ and for n the values $181, 182, \cdots, 200$, we obtain 20 equations and solve them. The result is

$$\begin{aligned} K_{1p0} &= -0.7452250447\mathrm{i}, \\ K_{1p1} &= -0.3105104338 - 0.4657656523\mathrm{i}, \\ K_{1p2} &= -0.0232886541 + 0.3383269912\mathrm{i}, \\ K_{1p3} &= 0.1744022316 - 0.1406351894\mathrm{i}, \\ K_{1p4} &= -0.1578173268 - 0.0112755970\mathrm{i}, \end{aligned} \tag{6.5}$$

and $\mathrm{i}K_{1p\ell}$ and $\mathrm{i}K_{2m\ell}$ are complex conjugates, as can be expected from symmetry.

We take these Stokes multipliers and use them in (6.3) with $L = R = 5$. The result is the black curve in Figure 4 which provides an excellent approximation to the exact solution.

7 Discussion

We have studied the exponential asymptotics of (1.1) using the resurgence relationswhich relate the late terms in the asymptotic expansion of its particular integral to the early terms in the expansion of its homogeneous solutions. Our approach, applicable to a wide class of problems with

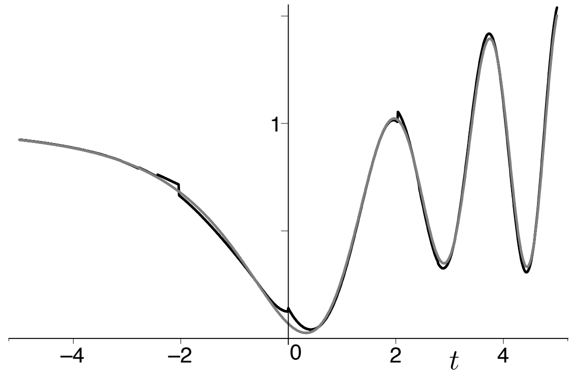

Figure 4: The exact (grey) and approximation (black) with $\varepsilon = 1$ and $L = R = 5$.

small parameters, highlights the relationship between (i) the constants of integrations that appear when constructing the homogeneous solutions, and (ii) the ε-dependence of the Stokes multipliers. The constants of integration can be chosen arbitrarily (reflecting the fact that the homogeneous solutions are defined up to an arbitrary function of ε). Different choices lead to different forms of the Stokes multipliers, so that the subdominant terms that are switched on are left invariant.

The particular equation studied in this paper describes the spontaneous generation of inertia-gravity waves in a model of geophysical fluid. The results of Vanneste & Yavneh (2004) about the exponential smallness of this generation are recovered here using a different exponential-asymptotic technique. A useful extension is the computation of the switched-on terms describing the inertia-gravity waves to higher-order in ε: this makes it possible to estimate the wave amplitude with good accuracy for the values of the 'small' parameters as large as 1, that is, for values of the Rossby number large enough for the waves to have amplitudes similar to that of the balanced motion which generates them.

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A Derivation of (1.1)

Equation (1.1) follows from an exact reduction of the partial differential equations governing the dynamics of a rotating stratified fluid. Specifically, introducing solutions of the form

$$(u, v, w, b) = (\Sigma y, 0, 0, 0) + \Re\left\{[\hat{u}(t), \hat{v}(t), \hat{w}(t), \hat{b}(t)]\mathrm{e}^{\mathrm{i}[k(x-\Sigma yt)+mz]}\right\}$$

into the hydrostatic Boussinesq equations (e.g. Gill, 1982) leads to a set of ordinary differential equations for $\hat{u}(t)$, $\hat{v}(t)$, $\hat{w}(t)$ and $\hat{b}(t)$. In the above, (u, v, w) are the three components of the fluid velocity, and b is the buoyancy; their form is that of a wave, with time-dependent wavevector $(k, -k\Sigma t, m)$, superimposed on a horizontal Couette flow, with shear Σ. A single ordinary differential equation can be derived for $\zeta(t) = \mathrm{i}k\hat{v}(t) + \mathrm{i}\Sigma kt\hat{u}(t)$; when suitably non-dimensionalised, it takes the form (1.1), where $\varepsilon = |\Sigma|/f$ and $\beta = fm/(Nk)$ and we have assumed that $\Sigma < 0$. Here the Coriolis parameter f and Brunt–Väisälä frequency N characterise the fluid's rotation and stratification.

McWilliams & Yavneh (1998) and Vanneste & Yavneh (2004) provide a detailed derivation of a slightly more general version of (1.1) obtained when the hydrostatic approximation is not made. Their equation reduces to (1.1) in the limit $m/k \to \infty$, $N/f \to \infty$ with β fixed.

B Asymptotic expansions of Gevrey order one

In the discussion below the set of variables $\mathbf{x}$ is kept constant.

An asymptotic expansion of the form

$$y(\mathbf{x}, \varepsilon) \sim \mathrm{e}^{f(\mathbf{x})/\varepsilon} \sum_{n=0}^{\infty} a_n(\mathbf{x})\varepsilon^{n-\alpha}, \qquad \text{as } \varepsilon \to 0 \text{ in sector } \beta_1 < \mathrm{ph}\,\varepsilon < \beta_2 \tag{B.1}$$

is of Gevrey order one (see, e.g., Balser, 2000) if for all positive integers N and all ε in subsectors of the form $|\varepsilon| \le r$, $\beta_1 < \gamma_1 \le \mathrm{ph}\,\varepsilon \le \gamma_2 < \beta_2$ we have the estimates

$$\left| \mathrm{e}^{-f(\mathbf{x})/\varepsilon}\varepsilon^{\alpha} y(\mathbf{x}, \varepsilon) - \sum_{n=0}^{N-1} a_n(\mathbf{x})\varepsilon^n \right| \le \frac{CN!}{\alpha^N}\varepsilon^N, \tag{B.2}$$

where C and α depend only on r, γ_1 and γ_2.

We assume that we are dealing with a linear finite dimensional problem in which all small ε asymptotic expansions are of Gevrey order one. Let

$$\tilde{y}(\mathbf{x},\varepsilon) = \mathrm{e}^{f_0(\mathbf{x})/\varepsilon} \sum_{n=0}^{\infty} a_{n0}(\mathbf{x})\varepsilon^{n-\alpha_0}, \tag{B.3}$$

be a formal solution of our problem. Suppose $y_1(\mathbf{x},\varepsilon)$ and $y_2(\mathbf{x},\varepsilon)$ are two distinct solutions of our problem, each having $\tilde{y}(\mathbf{x},\varepsilon)$ as their complete asymptotic expansion in open ε-sectors S_1 and S_2, respectively. Since we assume that our problem is linear $y_1(\mathbf{x},\varepsilon) - y_2(\mathbf{x},\varepsilon)$ is also a solution. In the case that the sectors S_1 and S_2 overlap, $y_1(\mathbf{x},\varepsilon) - y_2(\mathbf{x},\varepsilon)$ has as asymptotic expansion

$$y_1(\mathbf{x},\varepsilon) - y_2(\mathbf{x},\varepsilon) \sim \sum_{j\neq 0} \nu_j K_j(\varepsilon) \mathrm{e}^{f_j(\mathbf{x})/\varepsilon} \sum_{n=0}^{\infty} a_{nj}(\mathbf{x})\varepsilon^{n-\alpha_j}, \tag{B.4}$$

where $\nu_j = \pm 1$, as $\varepsilon \to 0$ in the sector $S_1 \cap S_2$. Compared with $\tilde{y}(\mathbf{x},\varepsilon)$ the asymptotic expansion with index j in (B.4) is of course exponentially small in $S_1 \cap S_2$. It is switched on by $\tilde{y}(\mathbf{x},\varepsilon)$ when Stokes line $\{\varepsilon \in \mathbb{C} \,|\, \mathrm{ph}\,\varepsilon = \mathrm{ph}\,(f_0(\mathbf{x}) - f_j(\mathbf{x}))\}$ is crossed. In (B.4) the $K_j(\varepsilon)$ are nonzero functions of ε. The dependence on ε is related to the fact that the functions $y_1(\mathbf{x},\varepsilon)$ and $y_2(\mathbf{x},\varepsilon)$ in (B.1) are typically defined up to multiplication by arbitrary functions of ε; only for special choices of these functions can the Stokes multipliers be made ε-independent.

By varying sectors S_1 and S_2 we determine all the asymptotic expansions that can be switched on by $\tilde{y}(\mathbf{x},\varepsilon)$. All these Stokes phenomena are reflected in the growth of the coefficients. One can prove, for example using the Cauchy-Heine transform (see Balser (2000), and for an application Olde Daalhuis & Olver (1994)), that

$$a_{n0}(\mathbf{x}) \sim \sum_j \frac{K_j}{2\pi\mathrm{i}} \sum_{s=0}^{\infty} a_{sj}(\mathbf{x}) \frac{\Gamma\,(n-s+\alpha_j-\alpha_0)}{(f_0(\mathbf{x}) - f_j(\mathbf{x}))^{n-s+\alpha_j-\alpha_0}}, \tag{B.5}$$

as $n \to \infty$. In (B.5) the j-sum is over all the asymptotic expansions that can be switched on by $\tilde{y}(\mathbf{x},\varepsilon)$.

In many problems (for example in this paper) it is relatively easy to determine an asymptotic expansion of the form (B.5) directly from the recurrence relations of the coefficients $a_{n0}(\mathbf{x})$. In this way we are able to determine the asymptotic expansions (and especially the $f_j(\mathbf{x})$) that are switched on when the Stokes lines are crossed.

The main differential equation (1.1) is inhomogeneous. However, we can differentiate this equation and construct a third order linear homogeneous ODE, which has as solutions all the linear combinations of solutions of (1.1) and (2.1). Hence, the results of this Appendix apply to the main problem discussed in this paper.

References

Balser, W. 2000, *Formal power series and linear systems of meromorphic ordinary differential equations*, Universitext, Springer-Verlag, New York.

Dingle, R. B. 1973, *Asymptotic expansions: their derivation and interpretation*, Academic Press.

Gill, A. E. 1982, *Atmosphere-ocean dynamics*, Academic Press.

Hakim, V. 1998, Asymptotic techniques in nonlinear problems: some illustrative examples, *in* C. Godrèche & P. Manneville, eds, Hydrodynamics and nonlinear instabilities, Cambridge University Press, chapter 3, pp. 295–386.

Howls, C. J. & Olde Daalhuis, A. B. 2003, Hyperasymptotic solutions of inhomogeneous to linear differential equations with a singularity of rank one, *Proc. R. Soc. Lond. A* **459**, 2599–2612.

Lorenz, E. N. 1980, Attractor sets and quasi-geostrophic equilibrium, *J. Atmos. Sci.* **37**, 1685–1699.

McWilliams, J. C. & Yavneh, I. 1998, Fluctuation growth and instability associated with a singularity of the balance equations, *Phys. Fluids* **10**, 2587–2596.

Olde Daalhuis, A. B. 1998, Hyperasymptotic solutions of higher order linear differential equations with a singularity of rank one, *Proc. R. Soc. Lond. A* **454**, 1–29.

Olde Daalhuis, A. B. 1999, On the computation of Stokes multipliers via hyperasymptotics, *Sūrikaisekikenkyūsho Kōkyūroku* **1088**, 68–78.

Olde Daalhuis, A. B. & Olver, F. W. J. 1994, Exponentially improved asymptotic solutions of ordinary differential equations. II. Irregular singularities of rank one, *Proc. Roy. Soc. London Ser. A* **445**, 39–56.

Paris, R. B. & Wood, A. D. 1995, Stokes phenomenon demystified, *Bull. Inst. Math. Appl.* **31**, 21–28.

Vanneste, J. 2004, Inertia-gravity-wave generation by balanced motion: revisiting the Lorenz-Krishnamurthy model, *J. Atmos. Sci.* **61**, 224–234.

Vanneste, J. & Yavneh, I. 2004, Exponentially small inertia-gravity waves and the breakdown of quasi-geostrophic balance, *J. Atmos. Sci.* **61**, 211–223.

Voros, A. 1983, The return of the quartic oscillator: the complex WKB method, *Ann. Inst. H. Poincaré Sect. A (N.S.)* **39**, 211–338.