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Persistence of steady flows of a two-dimensional perfect fluid in deformed domains

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Abstract

The robustness of steady solutions of the Euler equations for two-dimensional, incompressible and inviscid fluids is examined by studying their persistence for small deformations of the fluid-domain boundary. Starting with a given steady flow in a domain D_0, we consider the class of flows in a deformed domain D that can be obtained by rearrangement of the vorticity by an area-preserving diffeomorphism.

We provide conditions for the existence and (local) uniqueness of a steady flow in this class when D is sufficiently close to D_0 in $C^{k,\alpha}$, $k \geqslant 3$ and $0 < \alpha < 1$. We consider first the case where D_0 is a periodic channel and the flow in D_0 is parallel and show that the existence and uniqueness are ensured for flows with non-vanishing velocity. We then consider the case of smooth steady flows in a more general domain D_0. The persistence of the stability of steady flows established using the energy–Casimir or, in the parallel case, the energy–Casimir–momentum method, is also examined. A numerical example of a steady flow obtained by deforming a parallel flow is presented.

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1. Introduction

The dynamics of an incompressible inviscid fluid is governed by the Euler equation, which takes a particularly simple form in two dimensions. In terms of the streamfunction ψ, related

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to the velocity field $\boldsymbol{U}$ by $\boldsymbol{U} = \nabla^\perp \psi := (-\partial_y \psi, \partial_x \psi)$, and the vorticity $\omega = \Delta\psi$, this equation reads

$$\partial_t \omega + \nabla^\perp \psi \cdot \nabla \omega = 0. \tag{1.1}$$

It immediately shows that (for smooth $\boldsymbol{U}$), the vorticity is obtained from the initial vorticity $\omega_{t=0}$ by a smooth rearrangement, i.e.

$$\omega = \omega_{t=0} \circ g_t^{-1}, \tag{1.2}$$

where g_t is an area-preserving diffeomorphism. Considering (1.1) as a dynamical system, its fixed points are steady flows; they are characterized by the existence of a scalar function F, possibly multivalued, which relates the vorticity and streamfunction,

$$\psi = F \circ \omega. \tag{1.3}$$

There are several known ways of obtaining such steady flows. First, any parallel or axisymmetric flow, with its vorticity and streamfunction depending on a single (cross-stream) variable y or r, is evidently steady. These symmetric cases are very special, however, and they do not admit generalizations to more complicated domain shapes. Steady flows can also be found using their characterization as energy extrema under rearrangements of the vorticity: starting with a given vorticity distribution, a relaxation process leads to a configuration that extremizes the energy and is therefore a steady flow (see Shepherd (1990), Moffatt (1992) and references therein). Compared to the method described in this paper, this relaxation process has the advantage that it works for very general domains, but it has the disadvantage that it produces only a small subset of all steady flows, namely (stable) steady flows that are also global extrema for a given vorticity distribution. Alternatively, for a fixed energy, steady flows can be characterized as minima with respect to a partial ordering defined using the notion of polymorphism (Shnirelman 1993).

Finally, one may attempt to solve the equation $\psi = F(\Delta\psi)$ or, equivalently, $\Delta\psi = F^{-1}(\psi)$, directly for a fixed F in a given domain. Interesting solutions are known for certain F^{-1} (Stuart 1967, Mallier and Maslowe 1993, Crowdy 1997). More generally, conditions for the existence of solutions to the semilinear elliptic equation $\Delta\psi = F^{-1}(\psi)$ can be established (Pokhozhaev 1965, Taylor 1996 (chapter 14), Kuzin and Pokhozhaev 1997); again, these conditions are generally quite restrictive. It is worth noting that the situation is much simpler for potential flows: the unique steady flow in any given domain (given the boundary conditions) is obtained by solving Laplace's equation there.

When a steady flow is found, an important question concerns its persistence when small perturbations of the parameters on which it depends are introduced. This paper addresses this question by considering what might be regarded as the most natural form of perturbations, namely changes in the shape of the fluid domain. Thus, given a steady flow ψ_0 in a domain D_0, we look for a steady flow Ψ in a (given) domain D which is obtained from D_0 by a small area-preserving deformation.

Without further constraints, this problem does not have a unique solution: even in a fixed domain, steady flows are not locally unique since the equation $\psi = F(\omega)$ admits continuous families of solutions as F is varied. We therefore impose an additional constraint by requiring the steady flow Ψ to be isovortical to ψ_0, that is we require its vorticity $\Delta\Psi$ to be a smooth area-preserving rearrangement of $\Delta\psi_0$. With this restriction, the problem can be rephrased in terms of the area-preserving diffeomorphism g that effects the rearrangement from D_0 to D. The steadiness condition (1.3) translates into a partial differential equation for g, and the existence and uniqueness (in a certain sense) of solutions to this equation ensure the existence and uniquess of an isovortical steady flow in D.

This constraint is not the only possibility: for example, one may choose to rearrange the streamfunction instead of the vorticity (e.g. Arnold and Khesin (1998), sections II.2.A–C). Our choice is motivated by the fact that the isovortical steady flow in D can in principle be obtained dynamically with arbitrary accuracy by a slow deformation of the fluid domain starting with D_0 and ending with D. Indeed, formal perturbation theory indicates that, starting with an initially steady flow, an adiabatically slow deformation of a fluid domain $D(t)$ leads to a flow that, to leading order, satisfies the steadiness condition (1.3) and is isovortical to the initial flow at each time t. In other words, the leading-order formal approximation to the exact time-dependent flow in a deforming domain $D(t)$ is given by a steady isovortical flow of the type considered in this paper. To see this, write $D = D(\epsilon t)$ for some $\epsilon \ll 1$, and expand the streamfunction and vorticity according to $\psi = \psi^{(0)} + \epsilon\psi^{(1)} + \cdots$ and $\omega = \omega^{(0)} + \epsilon\omega^{(1)} + \cdots$, where $\psi^{(0)}$, $\psi^{(1)}$, etc, depend on time through ϵt. At leading order, the steadiness condition $\psi^{(0)} = F \circ \omega^{(0)}$ for some F is found, while the next-order equation, written as

$$\partial_{\epsilon t}\omega^{(0)} + \nabla^{\perp}(\psi^{(1)} - F'(\omega^{(0)})\psi^{(1)}) \cdot \nabla\omega^{(0)} = 0,$$

indicates that the leading-order vorticity, $\omega^{(0)}$, is rearranged. The area-preserving diffeomorphism g which, in this setting, depends on time only through its dependence on $D(t)$, can therefore be viewed as an asymptotic approximation to the exact, time-dependent area-preserving diffeomorphism g_t in (1.2).

We start the paper in section 2 by deriving a nonlinear partial differential equation for g. This equation also involves the function F, relating the streamfunction and vorticity in D, which is determined by a solvability condition. The persistence of a given steady flow under domain deformations is then considered in the next two sections. It is established by proving the existence of a solution g, unique up to diffeomorphisms along lines of constant vorticity, when the small boundary deformation is sufficiently small and certain hypotheses hold. Section 3 is devoted to a particular case of practical importance, the persistence of parallel channel flows, with vorticity $\omega(y)$, while section 4 is devoted to a general class of flows with no particular symmetries. In each case, our main result states the following. Consider a steady flow defined by a $C^{k,\alpha}$ streamfunction in a smooth bounded domain D_0 (see (3.14) for the definition of $C^{k,\alpha}$). Provided that some hypotheses (H0–H3 below) hold, for any domain D sufficiently close to D_0 in $C^{k,\alpha}$ there is a diffeomorphism g that maps the vorticity of the flow in D_0 to the vorticity of a steady flow in D. We note that the persistence results provide a novel approach for the derivation of steady flows: starting with a known steady flow, one can derive a sequence of steady flows by successive deformations of the domain boundary. Provided that none of the hypotheses for persistence are violated in the process, large deformations of the boundary can be achieved in principle.

The stability of a wide class of two-dimensional steady flows can be established using the energy–Casimir approach (cf Holm *et al* (1985), section II.4 in Arnold and Khesin (1998)). When such flows persist, their stability also persists for sufficiently small boundary deformation; this is because the stability condition depends only on the streamfunction–vorticity relation F which is continuous in the boundary deformation. The persistence of the stability of certain parallel (or axisymmetric) flows is more subtle, however, when their stability is established using the energy–Casimir–momentum method which relies crucially on the translational (or rotational) invariance of the flow and the associated momentum conservation. In section 5 we discuss how the energy–Casimir–momentum method can be adapted to bound the growth rate of the perturbation by a norm of the boundary deformation.

To illustrate the theoretical results, we present in section 6 the numerical computation of a steady flow obtained from a parallel channel flow by a small sinusoidal deformation of the boundary. Interestingly, the iterative algorithm used for this computation is very similar to the iteration used to establish the persistence results. A discussion in section 7 concludes the paper.

2. Formulation of the problem

We start with a steady incompressible flow in a bounded domain $D_0 \subset \mathbb{R}^2$, with streamfunction ψ_0 and vorticity $\omega_0 = \Delta\psi_0$. Steadiness implies that the streamfunction and vorticity are functionally related, that is

$$\psi_0 = F_0 \circ \omega_0, \tag{2.1}$$

for some function F_0. In general, F_0 is multivalued. However, we will assume that $\nabla\psi_0 \neq 0$ in D_0, except at a single (elliptic) point; this ensures that ${F_0}^{-1}$ is single valued.

Taking the Laplacian of (2.1) then gives

$$\omega_0 = \Delta(F_0 \circ \omega_0). \tag{2.2}$$

Let the boundary $\partial D_0 = \bigcup_i \partial D_0^i$, where each ∂D_0^i denotes a connected component of ∂D_0. The boundary condition that $\nabla^\perp\psi_0$ has no normal component is equivalent to $\psi_0 = \text{const} = \psi_\text{b}^i$ on ∂D_0^i; (2.1) then implies that also $\omega_0 = \text{const} =: \omega_\text{b}^i$ on ∂D_0^i.

Now consider a domain D, close (in a sense to be made precise) to D_0, with the same area and topology as D_0. We examine the existence of steady flows in D. As discussed in the introduction, we focus on flows whose vorticity is obtained from ω_0 by an area-preserving rearrangement; thus, we consider steady flows in D with vorticity Ω given by

$$\Omega = \omega_0 \circ g^{-1}, \tag{2.3}$$

where the diffeomorphism

$$g : D_0 \to D : (x, y) \mapsto (X, Y)$$

satisfies the following conditions:

(i) g maps ∂D_0 to ∂D; and
(ii) g is area-preserving, i.e. $\det\nabla g = 1$ (we assume that g also preserves orientation, which is automatic for g near identity).

The steadiness of the new flow implies a relation similar to (2.2),

$$\Omega = \Delta(F \circ \Omega) =: \Delta\Psi, \tag{2.4}$$

for some function F. Since $\Omega = F^{-1}(\Psi)$ with Ψ constant on ∂D^i, Ω is also constant on ∂D^i. Because g maps ∂D_0 to ∂D, this constant is simply ω_b^i, so that the boundary condition reads $\Omega = \omega_\text{b}^i$ on ∂D^i. Together (2.2)–(2.4) give

$$\omega_0 \circ g^{-1} = \Delta(F \circ \omega_0 \circ g^{-1}). \tag{2.5}$$

To solve (2.5), we pull it back to the original domain D_0, and find

$$\omega_0 = \omega_0 \circ g^{-1} \circ g = [\Delta(F \circ \omega_0 \circ g^{-1})] \circ g. \tag{2.6}$$

Defining a g-dependent operator Δ_g by

$$\Delta_g f = \left[\Delta(f \circ g^{-1})\right] \circ g$$

for any function f on D_0, we rewrite (2.6) in the compact form

$$\omega_0 = \Delta_g(F \circ \omega_0). \tag{2.7}$$

This equation is central to this paper. It is a partial differential equation for both the diffeomorphism g and the function F, whose solution depends on the domain D through the boundary conditions for g. Clearly, when $D = D_0$, a solution is $g = \mathrm{id}$ and $F = F_0$. In the next sections we show that a unique solution continues to exist for D close enough to D_0 under some additional hypotheses.

Δ_g can be expressed explicitly in terms of g. We use Cartesian coordinates (x, y) in D_0 and (X, Y) in D, and functions (u, v) on D_0 such that

$$(X, Y) := g(x, y) = (x + u(x, y), y + v(x, y)). \tag{2.8}$$

We emphasize that (u, v) is a finite displacement field, not a velocity field. By definition Δ_g is just the Laplacian in D, $\partial_{XX} + \partial_{YY}$, pulled back to D_0 (i.e. expressed in terms of (x, y)).

Now, using $\det \nabla g = 1$, g is seen to satisfy

$$\begin{aligned} \partial_Y y &= \partial_x X = 1 + u_x, & \partial_X y &= -\partial_x Y = -v_x, \\ \partial_X x &= \partial_y Y = 1 + v_y, & \partial_Y x &= -\partial_y X = -u_y. \end{aligned} \tag{2.9}$$

With a slight abuse of notation, we then have

$$\begin{aligned} \Delta_g = \partial_{XX} + \partial_{YY} &= [\partial_X x \partial_x + \partial_X y \partial_y]^2 + [\partial_Y x \partial_x + \partial_Y y \partial_y]^2 \\ &= [(1 + v_y)\partial_x - v_x \partial_y]^2 + [-u_y \partial_x + (1 + u_x)\partial_y]^2. \end{aligned} \tag{2.10}$$

With this result, (2.7) takes a completely explicit form as an equation for (u, v) (in place of g) and F. It must be supplemented by the area-preservation condition $\det \nabla g = 1$, or, for (u, v),

$$u_x + v_y + u_x v_y - u_y v_x = 0. \tag{2.11}$$

We now turn to the boundary conditions. First, we must have

$$g : \partial D_0 \to \partial D, \tag{2.12}$$

which will be made explicit in sections 3 and 4. Next, F must be fixed for boundary values of Ω. In place of F, it is convenient to work with the function χ defined by

$$\chi := \psi_* - \psi_0, \tag{2.13}$$

where

$$\psi_* := \Psi \circ g = F \circ \omega_0 \tag{2.14}$$

is the pull-back of the streamfunction $\Psi = F \circ \Omega$ from D to D_0; cf (2.7), which now reads

$$\omega_0 = \Delta \psi_0 = \Delta_g(\psi_0 + \chi). \tag{2.15}$$

We note that this definition implies that χ is a function of ψ_0 only, in the sense that their gradients are parallel. The constancy of Ψ on ∂D, required for the velocity to be tangent to ∂D, implies that

$$\chi = \mathrm{const}^i \qquad \text{on } \partial D_0^i. \tag{2.16}$$

For a simply-connected domain, the constant is arbitrary and will be taken to be zero. When the domain is multiply connected, extra constraints are needed in addition to the conservation of vorticity to determine the deformed flow uniquely. We impose the conservation of circulation: the circulation along each connected piece of the boundary ∂D_0 is unchanged under the domain deformation. Therefore the constants must be chosen such that

$$\oint_{\partial D^i} \partial_N \Psi \, \mathrm{d}L = \oint_{\partial D_0^i} \partial_n \psi_0 \, \mathrm{d}l, \tag{2.17}$$

where $\mathrm{d}l$ denotes the (differential) arclength along ∂D_0^i and ∂_n denotes the derivative normal to ∂D_0^i; similarly for $\mathrm{d}L$ and ∂_N with respect to ∂D^i. When the left-hand side integral is pulled back to ∂D_0^i, this provides constraints on χ on the boundary that are sufficient to fix all but one of the constants in (2.16). The last constant can be set to zero.

Clearly, the diffeomorphism g cannot be uniquely determined by (2.7). This is because displacements along lines of constant vorticity ω_0 or, equivalently, along streamlines $\psi_0 = \text{const}$ have no effect on the rearranged flow and so can be arbitrary. To remove this arbitrariness, we shall impose the additional constraints (3.3) or (4.17), which are chosen to simplify the computation.

This completes the specification of our problem: a steady flow in D isovortical to the original flow is found if the unknowns (u, v, χ) satisfy the nonlinear equations (2.7) and (2.11), with Δ_g given in (2.10), the boundary conditions (2.12), (2.16) and (2.17) and the constraint (3.3) or (4.17).

Before addressing this nonlinear problem, it is instructive to consider its linearized version, that is, we assume that the displacements (u, v) are infinitesimal. Neglecting nonlinear terms, (2.11) is automatically satisfied by taking

$$(u, v) = \nabla^{\perp}\phi,$$

for some ϕ. Up to quadratic and higher-order terms, the vorticity and streamfunction in D (which we identify with D_0 in the present case of infinitesimal displacements) are given by

$$\Omega = \omega_0 - \nabla^{\perp}\phi \cdot \nabla\omega_0,$$
$$\Psi = \psi_0 - \nabla^{\perp}\phi \cdot \nabla\psi_0 + \chi.$$

Introducing this into (2.7) gives

$$(\Delta - \omega_0')\nabla^{\perp}\psi_0 \cdot \nabla\phi + \Delta\chi = 0, \tag{2.18}$$

where $\omega_0' := \nabla\omega_0/\nabla\psi_0 = (F_0^{-1})' \circ \psi_0$. With $\nabla^{\perp}\phi$ specified on the boundary, this equation can be solved for ϕ and χ using a solvability condition described in section 4.

Returning to the nonlinear problem, we employ an iterative procedure both to compute the numerical solution in section 6 and, augmented with a contraction mapping argument (cf, e.g. section 8.1 in Kolmogorov and Fomin (1970)), to prove the existence and uniqueness of the solution in sections 3 and 4. Denoting the unknowns by w and the nonlinear problem by $N(w) = 0$, let $w^0 = 0$ and the successive iterations w^n satisfy

$$w^{n+1} = Tw^n := w^n - (\mathrm{D}N)^{-1}N(w^n), \tag{2.19}$$

where $(\mathrm{D}N)$ is the linearization of N at $w = 0$. The iteration (2.19) converges if the mapping T is contracting, i.e. if there exists $0 \leqslant \theta < 1$ such that

$$|T\hat{w} - T\tilde{w}| \leqslant \theta|\hat{w} - \tilde{w}| \tag{2.20}$$

for any $\hat{w}$, $\tilde{w}$ in some norm $|\cdot|$.

In the next two sections we establish this contraction property when ∂D and ∂D_0 are sufficiently close in $C^{k,\alpha}$, with $k \geqslant 2$ and $0 < \alpha < 1$. We first consider the particular case of parallel flows when D_0 is a channel and then the case of flows in more general simply-connected domains.

3. Steady flows in a deformed channel

In this section, we take the domain $D_0 = [0, \ell) \times [0, 1]$, with periodicity in x assumed, and consider a parallel shear flow given by the streamfunction $\psi_0(y)$ and vorticity $\omega_0(y) = \partial_{yy}\psi_0(y)$. Obviously such a flow is steady. Because we consider ψ_0 and ω_0 as functions of y, we do not make direct use of their functional relationship F_0 which can therefore be multivalued.

Let us put the nonlinear problem for (u, v, χ) in a form that is suitable for an iteration procedure. Evaluating (2.15) explicitly in the case of a parallel flow gives

$$\begin{aligned}0 = \chi_{yy} - v_{xx}\partial_y\psi_0 + u_{xy}\partial_y\psi_0 + 2u_x\partial_{yy}\psi_0 - v_{xx}\partial_y\chi + u_{xy}\partial_y\chi + 2u_x\partial_{yy}\chi \\ - v_y v_{xx}\partial_y\psi_* + v_x v_{xy}\partial_y\psi_* + v_x^2\partial_{yy}\psi_* + u_x u_{xy}\partial_y\psi_* + u_x^2\partial_{yy}\psi_* - u_y u_{xx}\partial_y\psi_*.\end{aligned} \tag{3.1}$$

Note that both the linear and nonlinear parts of this expression are anisotropic, in the sense that they contain $\partial_{xx}u$ and $\partial_{xy}u$ terms but not $\partial_{yy}u$ and similarly for v. Note also that, by definition, χ is a function of y only.

The problem is most easily formulated by introducing the (non-unique) decomposition of the displacement (u, v) into two functions η and ϕ, with

$$\begin{aligned}u &= \eta_x - \phi_y, \\ v &= \eta_y + \phi_x.\end{aligned} \tag{3.2}$$

To remove the arbitrariness of g up to displacements along streamlines, i.e. in the x-direction, we impose the constraint

$$\int_0^\ell \phi(x, y)\,\mathrm{d}x = 0. \tag{3.3}$$

After some computation, we can write (3.1) as

$$\chi_{yy} - \partial_y\psi_0\Delta\phi_x - 2\partial_{yy}\psi_0\phi_{xy} = -2\partial_{yy}\psi_0\eta_{xx} + \Phi_{\mathrm{nl}}(\eta, \phi, \chi), \tag{3.4}$$

where $\Phi_{\mathrm{nl}}(\cdots)$ contains only terms nonlinear in (η, ϕ, χ),

$$\begin{aligned}\Phi_{\mathrm{nl}}(\eta, \phi, \chi) = v_{xx}\chi_y - u_{xy}\chi_y - 2u_x\chi_{yy} + v_y v_{xx}\partial_y\psi_* - v_x v_{xy}\partial_y\psi_* - v_x^2\partial_{yy}\psi_* \\ -u_x u_{xy}\partial_y\psi_* - u_x^2\partial_{yy}\psi_* + u_y u_{xx}\partial_y\psi_*.\end{aligned} \tag{3.5}$$

In terms of η and ϕ, the area-preservation condition (2.11) becomes

$$\Delta\eta = u_y v_x - u_x v_y. \tag{3.6}$$

For the channel domain, it is convenient to let the boundaries of the deformed domain D be defined by the graphs

$$\begin{aligned}Y &= b^0(X) \qquad \text{for } y = 0, \\ Y &= 1 + b^1(X) \qquad \text{for } y = 1.\end{aligned} \tag{3.7}$$

Area preservation dictates that

$$\int_0^\ell (b^1(x) - b^0(x))\,\mathrm{d}x = 0. \tag{3.8}$$

Without loss of generality, we can take $\int_0^\ell b^0(x)\,\mathrm{d}x = 0$. The boundary conditions (2.12) then take the form

$$v(x, i) = b^i(x + u(x, i)) \tag{3.9}$$

for $i = 0, 1$. In terms of η and ϕ, this reads

$$\phi_x(x, i) + \eta_y(x, i) = b^i(x + u(x, i)). \tag{3.10}$$

Integrating this along the boundary, $y = i$, we find

$$0 = \int_0^\ell [-\eta_y + b^i(x + u(x, i))]\,\mathrm{d}x.$$

Therefore, using the arbitrariness in the decomposition (3.2), we can take, $\eta_y = \mathrm{const}^i$,

$$\eta_y(x, i) = \frac{1}{\ell}\int_0^\ell b^i(x + u(x, i))\,\mathrm{d}x, \tag{3.11}$$

as a boundary condition for (3.6). With this choice, the value of ϕ_x on the boundary is determined from (3.10) which becomes

$$\phi_x(x,i) = b^i(x+u(x,i)) - \frac{1}{\ell}\int_0^\ell b^i(x+u(x,i))\,\mathrm{d}x. \tag{3.12}$$

The boundary conditions for χ are obtained by pulling the left-hand side of (2.17) back into D_0; using the fact that $|\partial_N\Psi| = |\nabla_X\Psi|$ and following a computation similar to (2.7)–(2.10), we find

$$\ell\partial_y\psi_0 = \int_0^\ell \partial_y\psi_0\,\mathrm{d}x = \int_0^\ell [(1+u_x)^2 + v_x^2]\partial_y\psi_*\,\mathrm{d}x. \tag{3.13}$$

We have two differential equations (3.4) and (3.6) for the three unknowns (η,ϕ,χ), with the boundary conditions (3.11), (3.12) and (3.13). The solution is determined uniquely using the conditions that χ is a function of y only and that ϕ has zero x-average (3.3). Using the usual norm in $C^{k,\alpha}$, namely, for f sufficiently smooth in D_0,

$$|f|_{k,\alpha} := |f|_{C^{k,\alpha}(D_0)} := \sum_{m=0}^{k}\sup_{\boldsymbol{x}\in D_0}|\nabla^m f(\boldsymbol{x})| + \sup_{\boldsymbol{x}\neq\boldsymbol{x}'}\frac{|\nabla^k f(\boldsymbol{x}) - \nabla^k f(\boldsymbol{x}')|}{|\boldsymbol{x}-\boldsymbol{x}'|^\alpha}, \tag{3.14}$$

we now state the main result of this section.

Theorem 1. *Let $k \geqslant 2$ and $0<\alpha<1$ be fixed, and let $\psi_0 \in C^{k,\alpha}(D_0)$ define a steady flow independent of x with $|\partial_y\psi_0| \geqslant c_\psi > 0$. Let $b^i \in C^{k,\alpha}(\partial D_0)$ define the boundary of the deformed domain D. Then there exists an $\varepsilon_0(\psi_0) > 0$ such that for*

$$|b|_{C^{k,\alpha}(\partial D_0)} \leqslant \varepsilon_0,$$

there is a $g : D_0 \to D$, unique up to displacements along streamlines, and a unique function $\chi(y)$ that give a steady flow in D, with vorticity $\Omega = \omega_0\circ g^{-1}$ and streamfunction $\Psi = (\psi_0+\chi)\circ g^{-1}$. Moreover, $\chi \in C^{k,\alpha}(D_0)$ and $u_x,\ v_x \in C^{k-1,\alpha}(D_0)$ for $g = (x+u,\ y+v)$.

We first fix some notations. Let $|\cdot|_{k,\alpha;\partial D_0} := |\cdot|_{C^{k,\alpha}(\partial D_0)}$ and $\|b\| := |b|_{k,\alpha;\partial D_0}$; by $|f|_{k,1}$ we mean the usual Lipschitz norm in an appropriate space, and $|f|_k$ denotes the usual C^k norm. We will often regard $\chi(y)$ as a function in D_0 which does not depend on x. It is understood that all constants denoted by c, c' and c_j may depend on the initial domain D_0 and k (and α) in addition to the parameters explicitly shown; with an abuse of notation, c will be used to denote various constants which may not be the same each time the letter is used.

In the proof of this theorem and in section 4 we will need to use two basic results on the solution of elliptic partial differential equations which we cite here.

Lemma 1 (cf e.g. prob. 6.2 in Gilbarg and Trudinger (1977)). *Suppose that in a bounded $C^{k,\alpha}$ domain D_0 the homogeneous equation*

$$Lu = \Delta u + p\cdot\nabla u + qu = 0, \qquad u = 0 \text{ on } \partial D_0,$$

where p and q are in $C^{k-2,\alpha}(\bar{D}_0)$ with $k \geqslant 2$, has only the trivial solution $u=0$ (this holds in particular if $q \leqslant 0$). Then the solution u of the inhomogeneous equation

$$Lu = \Delta u + p\cdot\nabla u + qu = f, \qquad u = \varphi \text{ on } \partial D_0, \tag{3.15}$$

with $\varphi \in C^{k,\alpha}(\partial D_0)$ and $f \in C^{k-2,\alpha}(D_0)$, is unique and satisfies

$$\begin{aligned}|u|_{k,\alpha} &\leqslant c'(D_0,q)\left(\max_{\bar{D}_0}|u| + |\varphi|_{k,\alpha} + |f|_{k-2,\alpha}\right)\\ &\leqslant c(D_0,q)(|\varphi|_{k,\alpha} + |f|_{k-2,\alpha}).\end{aligned} \tag{3.16}$$

We note that 'domain' in Gilbarg and Trudinger (1977) is an open connected subset of $\mathbb{R}^2$, but the result is readily applicable to the channel in this section: the problem (and solution) in an annulus in $\mathbb{R}^2$ can be smoothly deformed into that in a channel by modifying the coefficients p and q.

Lemma 2. *In a bounded $C^{k,\alpha}$ domain D_0, $k \geqslant 2$, the solution u of the Neumann problem*

$$\Delta u = f \tag{3.17}$$

in D_0 and $\partial u/\partial n = \varphi$ on ∂D_0, with $f \in C^{k-2,\alpha}(\bar{D}_0)$ and $\varphi \in C^{k-1,\alpha}(\partial D_0)$, and

$$\int_{D_0} f\,\mathrm{d}x\,\mathrm{d}y = \int_{\partial D_0} \varphi\,\mathrm{d}l$$

satisfies

$$|u|_{k,\alpha} \leqslant c'(|\varphi|_{k-1,\alpha} + |f|_{k-2,\alpha}) \tag{3.18}$$

when one requires that its integral over D_0 vanish. Moreover, u is unique.

The bound (3.18) follows from the estimate (cf theorem 3.3.1 in Ladyzhenskaya and Ural'tseva (1968)),

$$|u|_{k,\alpha} \leqslant c'\left(\max_{\bar{D}_0}|u| + |\varphi|_{k-1,\alpha} + |f|_{k-2,\alpha}\right), \tag{3.19}$$

and the fact that in a bounded domain $\max_{\bar{D}_0}|u| \leqslant c''(|\varphi|_0 + |f|_0)$ when the integral of u over D_0 is required to vanish (this follows from the existence of Green's function for the Neumann problem). Uniqueness follows from the fact that the only solutions to the problem $\Delta u = 0$ in D_0 with $\partial u/\partial n = 0$ on ∂D_0 are constants.

We shall also need Hölder estimates for compositions of functions. Assuming that the domains of definition of the functions are sufficiently regular, we obtain by elementary means

$$|f \circ h|_{k,\alpha} \leqslant C_k'|f|_{k,1}|h|_{k,\alpha}(1 + |h|_k)^k + |f|_0, \tag{3.20}$$

where the norms are taken in the relevant domains. When $f(w_0) = 0$ for some w_0 in the range of h, in place of the last term we can write $2[f]_{0,1}|h|_0$, where $[f]_{0,1}$ is the Lipschitz constant of f, in which case (3.20) becomes

$$|f \circ h|_{k,\alpha} \leqslant C_k|f|_{k,1}|h|_{k,\alpha}(1 + |h|_k)^k. \tag{3.21}$$

For $k \geqslant 1$, de la Llave and Obaya (1999, case ii.3 of theorem 4.3) give the estimate

$$|f \circ h|_{k,\alpha} \leqslant C_k|f|_{k,\alpha}(1 + |h|_{k,\alpha}^{k+\alpha}), \tag{3.22}$$

provided that the domain of definition D_0 is 'compensated', meaning that there exists a constant κ_0 such that for any $x, y \in D_0$, their arclength distance $d_{D_0}(x, y) \leqslant \kappa_0\|x - y\|_{\mathbb{R}^2}$. This is a mild restriction (non-compensated domains such as $\{(x, y) : a < x^2 + y^2 < b,\ y \neq 0 \text{ when } x > 0\}$ are non-generic) and is assumed in all cases where this result is used below. In what follows, we shall need both (3.21), which tends to 0 as $|h| \to 0$, and (3.22), which assumes less regularity of f.

Proof of theorem 1. We take $\|b\| \leqslant 1$ and use the iterations (2.19). In the first part of the proof, we set up the iteration and show that $w^n = (\chi^n, \eta^n, \phi^n)$ is bounded by $\|b\|$ throughout; this result is then used in the second part to show that the contraction condition (2.20) is satisfied, thus proving convergence.

Suppose that at the beginning of iteration n we have

$$|\chi^n|_{k,\alpha} \leqslant 1, \qquad |\eta^n|_{k,\alpha} \leqslant 1, \qquad |\eta_x^n|_{k,\alpha} \leqslant 1 \qquad \text{and} \qquad |\phi_x^n|_{k,\alpha} \leqslant 1. \tag{3.23}$$

With $w^0 = (\chi^0, \eta^0, \phi^0) = 0$, this is trivially satisfied for $n = 0$. It then follows that u_x^n and v_x^n are in $C^{k-1,\alpha}(D_0)$. The extra x-derivatives in (3.23) have been introduced to account for the anisotropy of (3.1). We note that (3.3) implies that $|\phi^n|_{k,\alpha} \leqslant c|\phi_x^n|_{k,\alpha}$.

The iteration corresponding to (3.6) and (3.11) is

$$\Delta\eta^{n+1} = u_y^n v_x^n - u_x^n v_y^n \qquad \text{in } D_0, \tag{3.24a}$$

$$\eta_y^{n+1} = \gamma_0^n := \frac{1}{\ell}\int_0^\ell b^0(x + u^n(x,0))\,\mathrm{d}x \qquad \text{at } y = 0, \tag{3.24b}$$

$$\eta_y^{n+1} = \gamma_0^n + \frac{1}{\ell}\int_{D_0} [u_y^n v_x^n - u_x^n v_y^n]\,\mathrm{d}x\,\mathrm{d}y \qquad \text{at } y = 1. \tag{3.24c}$$

For the iteration, the boundary conditions on $y = 1$ have been chosen to ensure the solvability of the interior equation (3.24*a*). Upon convergence, the equivalence of (3.11) and (3.24*c*) on $y = 1$ follows from area-preservation. We first note that the right-hand side of the interior equation (3.24*a*) is bounded as

$$|u_x^n v_y^n - u_y^n v_x^n|_{k-2,\alpha} \leqslant c(|\eta^n|_{k,\alpha}^2 + |\phi^n|_{k,\alpha}^2). \tag{3.25}$$

Turning to the boundary conditions (3.24*b*), we compute at $y = 0$

$$\int_0^\ell b^0(x + u^n(x,0))\,\mathrm{d}x = \int_0^\ell [b^0(x) + (b^0)'(x')u^n(x,0)]\,\mathrm{d}x$$
$$= \int_0^\ell (b^0)'(x')u^n(x,0)\,\mathrm{d}x \tag{3.26}$$

for some $x'(x) \in [0,\ell)$. By Cauchy–Schwarz, the last expression is bounded in absolute value by $\ell(|\eta^n|_{1;\partial D_0}^2 + |\phi^n|_{1;\partial D_0}^2 + |b|_1^2)$. Note that (3.25) has also provided a bound for the boundary conditions (3.24*c*), which is just a constant so that only the $|\cdot|_0$ norm is needed. Lemma 2 then gives

$$|\eta^{n+1}|_{k,\alpha} \leqslant c_1(|\eta^n|_{k,\alpha}^2 + |\phi^n|_{k,\alpha}^2 + \|b\|^2). \tag{3.27}$$

Now we take ∂_x (3.24*a*),

$$\Delta\eta_x^{n+1} = \partial_x(u_x^n v_y^n - u_y^n v_x^n), \tag{3.28}$$

whose right-hand side is bounded as

$$|\partial_x(u_x^n v_y^n - u_y^n v_x^n)|_{k-2,\alpha} \leqslant c(|\eta_x^n|_{k,\alpha}^2 + |\eta^n|_{k,\alpha}^2 + |\phi_x^n|_{k,\alpha}^2).$$

Using lemma 2 on (3.28) with $\partial_y\eta_x^n = 0$ as boundary conditions, and adding the resulting inequality to (3.27), we arrive at

$$|\eta^{n+1}|_{k,\alpha} + |\eta_x^{n+1}|_{k,\alpha} \leqslant c_1(|\eta_x^n|_{k,\alpha}^2 + |\eta^n|_{k,\alpha}^2 + |\phi_x^n|_{k,\alpha}^2 + \|b\|^2). \tag{3.29}$$

This is the first important estimate needed to establish the boundedness of $|w^n|$.

From (3.4) we obtain the iteration

$$\chi_{yy}^{n+1} - \partial_y\psi_0\Delta\phi_x^{n+1} - 2\partial_{yy}\psi_0\phi_{xy}^{n+1} = -2\partial_{yy}\psi_0\eta_{xx}^{n+1} + \Phi_{\mathrm{nl}}(u^n, v^n, \chi^n). \tag{3.30}$$

Integrating this in x gives

$$\chi_{yy}^{n+1} = \frac{1}{\ell}\int_0^\ell \Phi_{\mathrm{nl}}(u^n, v^n, \chi^n)\,\mathrm{d}x =: \bar{\Phi}_{\mathrm{nl}}(u^n, v^n, \chi^n), \tag{3.31}$$

which can now be solved for χ^{n+1} subject to the boundary conditions $\chi^{n+1}(y = 0) = 0$ and $\partial_y\chi^{n+1}(y = 1) = \xi^{n+1}$ where the constant ξ^{n+1} is given by (cf (3.13))

$$\xi^{n+1} = -\frac{1}{\ell}\partial_y(\psi_0 + \chi^n)\int_0^\ell [(u_x^n)^2 + (v_x^n)^2]\,\mathrm{d}x. \tag{3.32}$$

Using (3.23*a*), this implies

$$|\xi^{n+1}| \leqslant C(|\phi_x^n|_1^2 + |\eta_x^n|_1^2). \tag{3.33}$$

We note that $\Phi_{\rm nl}(\cdots)$ in (3.30) contains only quadratic and cubic terms in (u^n, v^n, χ^n), with the cubic terms boundable by the quadratic ones thanks to hypothesis (3.23). A straightforward computation then shows that

$$|\Phi_{\rm nl}(u^n, v^n, \chi^n)|_{k-2,\alpha} \leqslant c(\psi_0)(|\chi^n|_{k,\alpha}^2 + |\eta^n|_{k,\alpha}^2 + |\eta_x^n|_{k,\alpha}^2 + |\phi_x^n|_{k,\alpha}^2). \tag{3.34}$$

This, combined with (3.31), (3.33) and lemma 1 with $q = 0$ gives

$$|\chi^{n+1}|_{k,\alpha} \leqslant c_2(\psi_0)(|\chi^n|_{k,\alpha}^2 + |\eta_x^n|_{k,\alpha}^2 + |\eta^n|_{k,\alpha}^2 + |\phi_x^n|_{k,\alpha}^2). \tag{3.35}$$

This is the second important estimate.

The iteration for ϕ^{n+1} follows from the zero-mean part of (3.30) and from (3.12),

$$\begin{aligned} &\partial_y\psi_0\Delta\phi_x^{n+1} + 2\partial_{yy}\psi_0\partial_y\phi_x^{n+1} = 2\partial_{yy}\psi_0\eta_{xx}^{n+1} - (\Phi_{\rm nl} - \bar{\Phi}_{\rm nl})(u^n, v^n, \chi^n) \qquad \text{in } D_0 \\ &\phi_x^{n+1}(x, i) = b^i(x + u^n(x, i)) - \frac{1}{\ell}\int_0^\ell b^i(x + u^n(x, i))\,\mathrm{d}x \qquad \text{for } i = 0, 1, \end{aligned} \tag{3.36}$$

which is to be regarded as an equation for ϕ_x^{n+1}. We first estimate the nonlinear boundary term,

$$|u^n|_{k,\alpha;\partial D_0} \leqslant c|u_x^n|_{k-1,\alpha;\partial D_0} \leqslant c|u_x^n|_{k-1,\alpha} \leqslant c(|\eta_x^n|_{k,\alpha} + |\phi_x^n|_{k,\alpha}).$$

Using (3.22) and (3.27), the boundary conditions (3.36*b*) are bounded as

$$\begin{aligned} |\phi_x^{n+1}|_{k,\alpha;\partial D_0} &\leqslant c\|b\|(1 + |u^n|_{k,\alpha;\partial D_0}^{k+\alpha}) \\ &\leqslant c\|b\| + c(|\eta^n|_{k,\alpha}^2 + |\eta_x^n|_{k,\alpha}^2 + |\phi_x^n|_{k,\alpha}^2), \end{aligned} \tag{3.37}$$

where we have used (3.23) to arrive at the second inequality. Another application of lemma 1 to (3.36) combined with (3.34) and (3.37) leads to the third important estimate,

$$|\phi_x^{n+1}|_{k,\alpha} \leqslant c_3(\psi_0)(\|b\| + |\chi^n|_{k,\alpha}^2 + |\eta_x^n|_{k,\alpha}^2 + |\eta^n|_{k,\alpha}^2 + |\phi_x^n|_{k,\alpha}^2). \tag{3.38}$$

We remark that the hypothesis $\partial_y\psi_0(y) \neq 0$ is essential here to ensure ellipticity (i.e. that c_3 is finite).

Now let

$$\|w^n\| := |\chi^n|_{k,\alpha}^2 + |\eta^n|_{k,\alpha}^2 + |\eta_x^n|_{k,\alpha}^2 + |\phi_x^n|_{k,\alpha}^2;$$

adding the squares of the three estimates (3.29), (3.35) and (3.38), we find

$$\|w^{n+1}\| \leqslant c_4(\|b\|^2 + \|w^n\|^2), \tag{3.39}$$

where we take $c_4 \geqslant 1$. Remembering that $\|w^0\| = 0$, the hypothesis (3.23) is satisfied for all n if we take

$$\|b\| \leqslant 1/(2c_4),$$

which also implies the bound

$$\|w^n\| \leqslant 2c_4\|b\|^2 \tag{3.40}$$

for all n.

To prove contraction, we turn to (2.20) with $w = (\chi, \phi, \eta)$ and consider two realizations $\tilde{w}$ and $\hat{w}$, which satisfy the boundedness conditions above. The computation proceeds much in the same fashion as the boundedness estimates above, with the bound (3.40) playing an important role.

In analogy with (3.27), we compute

$$\Delta(\tilde{\eta}^{n+1} - \hat{\eta}^{n+1}) = \tilde{u}_y^n\tilde{v}_x^n - \tilde{u}_x^n\tilde{v}_y^n - \hat{u}_y^n\hat{v}_x^n + \hat{u}_x^n\hat{v}_y^n,$$

which together with lemma 2 gives

$$|\tilde\eta^{n+1} - \hat\eta^{n+1}|_{k-1,\alpha} \leqslant c(|\tilde\eta^{n} - \hat\eta^{n}|_{k-1,\alpha} + |\tilde\phi^{n} - \hat\phi^{n}|_{k-1,\alpha}) \times (|\tilde\eta^{n}|_{k-1,\alpha} + |\hat\eta^{n}|_{k-1,\alpha} + |\tilde\phi^{n}|_{k-1,\alpha} + |\hat\phi^{n}|_{k-1,\alpha}) \leqslant c\|b\|(|\tilde\eta^{n} - \hat\eta^{n}|_{k-1,\alpha} + |\tilde\phi^{n} - \hat\phi^{n}|_{k-1,\alpha}).$$

The boundedness results (3.29), (3.35) and (3.38) have been used to arrive at the second inequality. Adding an analogous estimate for $|\tilde\eta^{n+1}_x - \hat\eta^{n+1}_x|_{k-1,\alpha}$, we have

$$|\tilde\eta^{n+1} - \hat\eta^{n+1}|_{k-1,\alpha} + |\tilde\eta^{n+1}_x - \hat\eta^{n+1}_x|_{k-1,\alpha} \leqslant c_5\|b\|(|\tilde\eta^{n} - \hat\eta^{n}|_{k-1,\alpha} + |\tilde\eta^{n}_x - \hat\eta^{n}_x|_{k-1,\alpha} + |\tilde\phi^{n}_x - \hat\phi^{n}_x|_{k-1,\alpha}). \tag{3.41}$$

Similarly, from

$$\tilde\chi^{n+1}_{yy} - \hat\chi^{n+1}_{yy} = \bar\Phi_{\rm nl}(\tilde w^n) - \bar\Phi_{\rm nl}(\hat w^n)$$

we obtain

$$|\tilde\chi^{n+1} - \hat\chi^{n+1}|_{k-1,\alpha} \leqslant c_6\|b\|(|\tilde\chi^{n} - \hat\chi^{n}|_{k-1,\alpha} + |\tilde\phi^{n}_x - \hat\phi^{n}_x|_{k-1,\alpha} + |\tilde\eta^{n}_x - \hat\eta^{n}_x|_{k-1,\alpha} + |\tilde\eta^{n} - \hat\eta^{n}|_{k-1,\alpha}). \tag{3.42}$$

On the boundary ∂D_0, we have

$$\tilde\phi^{n+1}_x(x,i) - \hat\phi^{n+1}_x(x,i) = b^i(x + \tilde u^n(x,i)) - b^i(x + \hat u^n(x,i)) - \int_0^\ell [b^i(x + \tilde u^n(x,i)) - b^i(x + \hat u^n(x,i))]\,{\rm d}x,$$

which, following the computation leading to (3.37), gives

$$|\tilde\phi^{n+1}_x - \hat\phi^{n+1}_x|_{k-1,\alpha;\partial D_0} \leqslant c_7\|b\|(|\tilde\phi^{n} - \hat\phi^{n}|_{k-1,\alpha} + |\tilde\eta^{n} - \hat\eta^{n}|_{k-1,\alpha}) \tag{3.43}$$

on the boundary ∂D_0 and

$$|\tilde\phi^{n+1}_x - \hat\phi^{n+1}_x|_{k-1,\alpha} \leqslant c_8\|b\|(|\tilde\chi^{n} - \hat\chi^{n}|_{k-1,\alpha} + |\tilde\phi^{n}_x - \hat\phi^{n}_x|_{k-1,\alpha} + |\tilde\eta^{n}_x - \hat\eta^{n}_x|_{k-1,\alpha} + |\tilde\eta^{n} - \hat\eta^{n}|_{k-1,\alpha}) \tag{3.44}$$

in the interior D_0.

Adding (3.41), (3.42) and (3.44), we find, with $|w|_{k,\alpha} := |\chi|_{k,\alpha} + |\eta|_{k,\alpha} + |\eta_x|_{k,\alpha} + |\phi_x|_{k,\alpha}$,

$$|\tilde w^{n+1} - \hat w^{n+1}|_{k-1,\alpha} \leqslant c_9\|b\||\tilde w^{n} - \hat w^{n}|_{k-1,\alpha},$$

whence contraction follows, provided we take $\|b\|$ sufficiently small. This proves the convergence of the iteration (2.19) to the unique solution of the nonlinear equations for $w = (\chi, \eta, \phi)$ and with it, the theorem. □

Remarks.

1. The solution of the linearized problem (2.18) is simply the first iterate (χ^1, η^1, ϕ^1); it can be verified from the foregoing that $\eta^1 = 0$ and $\chi^1 = 0$, with the latter resulting from the symmetry of the initial domain D_0 (i.e. $\chi^1 \neq 0$ for a general domain D_0). The equation for ϕ^1 is then

$$\partial_y\psi_0\Delta\phi^1_x + 2\partial_{yy}\psi_0\phi^1_x = 0, \tag{3.45}$$

with inhomogeneous boundary conditions $\phi^1_x(x,i) = b^i(x)$, $i = 0, 1$. Rewritten as

$$\partial_y\psi_0\Delta(\partial_y\psi_0\phi^1_x) - \partial_{yyy}\psi_0(\partial_y\psi_0\phi^1_x) = 0, \tag{3.46}$$

this can be recognized as Rayleigh's equation (Drazin and Reid 1981) for zero phase speed perturbations to the shear flow $\partial_y\psi_0$.

2. It is also interesting to obtain bounds on the individual components of w. This can be done as follows. Referring back to (3.38), (3.40) gives

$$|\phi_x^n|_{k,\alpha} \leqslant c\|b\|. \tag{3.47}$$

Using this bound in (3.29), we find

$$|\eta^{n+1}|_{k,\alpha} + |\eta_x^{n+1}|_{k,\alpha} \leqslant c(|\eta^n|_{k,\alpha}^2 + |\eta_x^n|_{k,\alpha}^2 + \|b\|^2), \tag{3.48}$$

which with $\eta^0 = 0$ implies that

$$|\eta^n|_{k,\alpha} + |\eta_x^n|_{k,\alpha} \leqslant c\|b\|^2, \tag{3.49}$$

valid for all n, provided that $\|b\|$ is sufficiently small. Finally, a similar application of (3.47) and (3.49) in (3.35) gives

$$|\chi^n|_{k,\alpha} \leqslant c\|b\|^2 \tag{3.50}$$

for all n, again for $\|b\|$ sufficiently small. Thus, we observe that, in the decomposition of the diffeomorphism $g = \mathrm{id} + \nabla^\perp\phi + \nabla\eta$, the divergence-free component ϕ dominates the curl-free component η, with the former scaling as $\|b\|$ and the latter as $\|b\|^2$. The change in the vorticity–streamfunction relationship F–F_0, which is proportional to χ, is also of second order in $\|b\|$. This, however, arises from the translational symmetry of the initial domain D_0: as will be apparent in the next section (cf (4.29)), for a generic domain $|\chi|$ scales as $\|b\|$.

4. More general unperturbed domain

The general case where the domain D_0 is curved proceeds in essentially the same way as the channel case of the previous section, with a few extra complications which we treat in this section. We limit ourselves to domains which are topologically equivalent to a disc and to flows whose streamlines have the simplest topology in these domains; that is, the streamlines consist of nested simple loops, with a single stagnation point at the centre. With this topology, and with the hypothesis $\nabla\psi_0 \neq 0$ (or, more precisely, H2 below) which we will make, each value of ψ_0 identifies a single streamline so that F_0^{-1} is single valued.

To establish the existence and uniqueness of a solution to equation (2.7) for the diffeomorphism g such that $\omega_0 \circ g^{-1}$ is a steady flow in D, we shall need additional assumptions on the flow in D_0, given by H1–H3 below. As in the channel case, the solution is anisotropic in the sense that g admits one more derivative in the direction of the basic velocity $\boldsymbol{U}_0$. This requires an extra differentiability of the unperturbed streamfunction ψ_0, which is why theorem 2 of this section requires one more derivative than theorem 1 in the channel case.

In analogy with (3.2), we write

$$g = \mathrm{id} + \nabla\eta + \nabla^\perp\phi \tag{4.1}$$

for two scalar functions η and ϕ (as before, this decomposition is not unique). Let $\psi_* := F \circ \omega_0$ and $\chi := \psi_* - \psi_0$ as in the previous section. We introduce the notation

$$\begin{aligned} \partial_s &= \nabla^\perp\psi_0 \cdot \nabla, \\ \mathrm{d}s &= |\nabla\psi_0|^{-1}\,\mathrm{d}l, \end{aligned} \tag{4.2}$$

where l is the arclength along $\psi_0 = \mathrm{const}$.

We first consider the boundary conditions (2.12). Let ∂D_0 be defined by $B_0(x, y) = 0$ and ∂D by $B(x, y) = B_0(x, y) + b(x, y) = 0$, where both B_0 and B are defined in a sufficiently large neighbourhood of $\partial D_0 \cup \partial D$, denoted by ND, in which they have non-vanishing gradients.

We shall choose b to be small and smooth, and such that the area-preservation condition is satisfied. With this, (2.12) can be written in the form

$$0 = B(x+u, y+v) = (B_0 + b)(x+u, y+v) \qquad (x, y) \in \partial D_0. \tag{4.3}$$

Noting that $B_0(x, y) = 0$ for $(x, y) \in \partial D_0$, we rewrite (4.3) as

$$u\partial_x B_0 + v\partial_y B_0 = \beta(u, v; x, y), \tag{4.4}$$

where $(x, y) \in \partial D_0$ and (u, v) are evaluated on ∂D_0 and where

$$\beta(u, v; x, y) := -B_0(x+u, y+v) + u\partial_x B_0 + v\partial_y B_0 - b(x+u, y+v) \tag{4.5}$$

groups all the terms in $B_0(x+u, y+v)$ that are nonlinear in (u, v) and $b(x+u, y+v)$. Choosing B_0 such that $|\nabla B_0| = 1$ on ∂D_0 (and such that $\nabla B_0 \cdot \nabla\psi_0 > 0$), (4.4) is equivalent to

$$\frac{\partial\phi}{\partial l} + \frac{\partial\eta}{\partial n} = \beta(u, v; x, y), \tag{4.6}$$

where l and n are orthonormal coordinates, respectively, tangent and normal to the boundary. In (4.6), which is the analogue of (3.9), all quantities are evaluated on the (unperturbed) boundary, $(x, y) \in \partial D_0$.

Separating the average-along-streamlines part of $\beta(u, v; x, y)$, we take $\partial\eta/\partial n$ constant on the boundary and write the boundary conditions for η and ϕ in the form

$$\left.\begin{aligned} \frac{\partial\eta}{\partial n} &= \oint \beta(u, v; x, y)\,\mathrm{d}s \Big/ \oint \mathrm{d}s =: \bar\beta(u, v) \\ \partial_s\phi &= |\nabla\psi_0|^{-1}(\beta - \bar\beta)(u, v; x, y) =: \tilde\beta(u, v; x, y) \end{aligned}\right\} \quad \text{on } \partial D_0. \tag{4.7}$$

For χ, we take $\chi = 0$ on ∂D_0, thus choosing the (arbitrary) constant value of Ψ on ∂D to be the constant value of ψ_0 on ∂D_0.

The interior equation for η is, as before,

$$\Delta\eta = u_y v_x - u_x v_y. \tag{4.8}$$

Moving on to (2.7), after some manipulation it can be expressed in the form

$$\Delta\chi + [\Delta - \omega_0']\partial_s\phi = \Phi^\sharp. \tag{4.9}$$

Here $\omega_0' := \nabla\omega_0/\nabla\psi_0 = (F_0^{-1})' \circ \psi_0$ and

$$\Phi^\sharp = -2[\eta_{xx}\partial_{yy}\psi_0 + \eta_{yy}\partial_{xx}\psi_0 - 2\eta_{xy}\partial_{xy}\psi_0] + \Phi_{\mathrm{nl}}(\chi, u, v),$$

where

$$\begin{aligned} \Phi_{\mathrm{nl}}(\chi, u, v) = &-(2\eta_{yy} - 2\phi_{xy})\chi_{xx} - (2\eta_{xx} + 2\phi_{xy})\chi_{yy} - (2\phi_{yy} + 2\phi_{xx} + 4\eta_{xy})\chi_{xy} \\ &- (u_y^2 + v_y^2)\partial_{xx}\psi_* - (u_x^2 + v_x^2)\partial_{yy}\psi_* + (u_x u_y + v_x v_y)\partial_{xy}\psi_* \\ &- [u_y\partial_s u_x - u_x\partial_s u_y - v_x\partial_s v_y + v_y\partial_s v_x]\psi_*' + \chi'\partial_s\Delta\phi \end{aligned} \tag{4.10}$$

only contains terms quadratic or higher in (u, v, χ). We have denoted $\psi_*' := \nabla\psi_*/\nabla\psi_0$ and $\chi' := \nabla\chi/\nabla\psi_0$, both of which are well defined since the gradients are collinear.

Note the appearance in (4.9) of the linear term $\Delta\chi + [\Delta - \omega_0']\partial_s\phi$ which we have encountered in (2.18) when considering the linearization of the equation for g in section 2. We have written the linear terms involving η on the right-hand side of (4.9) since, as in the channel case, η is quadratic in $|b|$ while ϕ is linear.

The fundamental problem here is to derive both χ and ϕ from (4.9), given right-hand side and boundary conditions. When the initial flow, ψ_0, is a parallel channel flow, as in the previous section, this is straightforward: averaging over x eliminates the terms containing ϕ from the left-hand side and so provides an equation for $\chi(y)$ alone; $\phi(x, y)$ can then be determined by considering the zero-average part of (4.9). An analogous procedure also obtains when the initial flow is an axisymmetric flow in a disc, as interested readers can verify. In the general non-symmetric case here, the solvability of (4.9) dictates that $\partial_s\phi$ have zero average along each streamline and the definition of χ demands that it be a function of ψ_0 only. These two conditions allow the determination of χ and ϕ from (4.9), but this requires three additional hypotheses which we now detail.

The first hypothesis is

H1. *The problem* $(\Delta - \omega_0')u = 0$ *in* D_0 *with* $u = 0$ *on* ∂D_0 *only admits the trivial solution.* A sufficient condition for this is that $\omega_0' > -\lambda_1$, where $\lambda_1 > 0$ is the smallest eigenvalue of $-\Delta$ in D_0 with homogeneous boundary conditions. This latter condition is in turn implied by Arnold stability (cf section 5), which we however do not assume here. Why this hypothesis is redundant in the parallel case is discussed in section 7.

We denote Green's function of $\Delta - \omega_0'$ by $G(x, y; x^*, y^*)$, viz

$$u(x, y) = \int_{D_0} G(x, y; x^*, y^*)[\Delta - \omega_0']u(x^*, y^*)\mathrm{d}x^*\,\mathrm{d}y^* + \oint_{\partial D_0} \frac{\partial G}{\partial n^*}(x, y; x^*, y^*)u(x^*, y^*)\mathrm{d}l^*,$$

for any sufficiently smooth $u(x, y)$ and let

$$[\Delta - \omega_0']^{-1}_{\mathrm{hbc}}\varphi(x, y) := \int_{D_0} G(x, y; x^*, y^*)\varphi(x^*, y^*)\mathrm{d}x^*\,\mathrm{d}y^*.$$

Writing (4.9) as

$$[\Delta - \omega_0'](\partial_s\phi + \chi) + \omega_0'\chi = \Phi^\sharp \tag{4.11}$$

and applying $[\Delta - \omega_0']^{-1}_{\mathrm{hbc}}$, we find

$$\begin{aligned}\partial_s\phi + \chi + &\int_{D_0} G(x, y; x^*, y^*)\omega_0'(x^*, y^*)\chi(x^*, y^*)\mathrm{d}x^*\,\mathrm{d}y^* \\ &= \oint_{\partial D_0} \frac{\partial G}{\partial n^*}(x, y; x^*, y^*)\tilde{\beta}(u, v; x^*, y^*)\mathrm{d}l^* + [\Delta - \omega_0']^{-1}_{\mathrm{hbc}}\Phi^\sharp \\ &=: [\Delta - \omega_0']^{-1}\Phi^\sharp(\chi, u, v),\end{aligned} \tag{4.12}$$

where we have used (4.7*b*) and the fact that $\chi = 0$ on ∂D_0. We stress that the nonlinear dependence on (u, v) on the last line enters through the boundary conditions of $[\Delta - \omega_0']^{-1}$ as well as through $\Phi^\sharp$.

Now divide (4.12) by $|\nabla\psi_0|$ and integrate along contours of constant ψ_0 to eliminate the first term on the left-hand side,

$$\oint_{\psi_0} \partial_s\phi \frac{\mathrm{d}l}{|\nabla\psi_0|} = \oint_{\psi_0} \partial_s\phi\,\mathrm{d}s = 0.$$

Assuming that the right-hand side is given, as is the case in the linearized problem or in the iterative procedure, this turns (4.12) into a one-dimensional Fredholm integral equation of the second kind for $\chi_\psi(\psi_0)$ (by which we mean χ considered as a function of ψ_0), viz

$$\begin{aligned}\mu(\psi_0)\chi_\psi(\psi_0) + &\int_{D_0}\oint_{\psi_0} G(\psi_0, s; {\psi_0}^*, s^*)\,\mathrm{d}s\,\mathrm{d}s^*\omega_0'(\psi_0^*)\chi_\psi(\psi_0^*)\mathrm{d}\psi_0^* \\ &=: (\mu + K)\chi_\psi(\psi_0) = \oint_{\psi_0} [\Delta - \omega_0']^{-1}\Phi^\sharp(\chi, u, v)\mathrm{d}s,\end{aligned} \tag{4.13}$$

where with an abuse of notation we have written G as a function of (ψ_0, s) and where

$$\mu(\psi_0) := \oint_{\psi_0} \mathrm{d}s = \oint_{\psi_0} \frac{\mathrm{d}l}{|\nabla\psi_0|}.$$

We note for future reference that when $\psi_0 \in C^{k+1,\alpha}(D_0)$, $(\mu + K) : C^{k,\alpha}(I_0) \to C^{k,\alpha}(I_0)$ where $I_0 \subset \mathbb{R}$ is the image of D_0 under ψ_0. Equation (4.13) is the analogue of (3.31) obtained in the parallel case. For its solvability, we need two further hypotheses:

H2. *There exists a* $c_\psi > 0$ *such that, for all* ψ_0,

$$\mu(\psi_0) \leqslant \frac{1}{c_\psi}. \tag{4.14}$$

This holds if the vorticity does not vanish at the fixed point of the flow (note that such a fixed point is necessarily elliptic).

H3. *The initial flow,* ψ_0*, is such that* $\nu = 1$ *is not in the spectrum in* $C^{k,\alpha}(I_0)$ *of the homogeneous problem*

$$(\mu + \nu K)u = 0. \tag{4.15}$$

This guarantees the existence of a unique solution to (4.13).

Once the Fredholm equation (4.13) has been solved for χ_ψ (and thus χ), a separate equation for ϕ is obtained by subtracting (4.13)/μ from (4.12),

$$\partial_s\phi + [\Delta - \omega_0']^{-1}_{\text{hbc}}(\omega_0'\chi) - \frac{1}{\mu}\oint_{\psi_0}[\Delta - \omega_0']^{-1}_{\text{hbc}}(\omega_0'\chi)\,\mathrm{d}s$$
$$= [\Delta - \omega_0']^{-1}\Phi^\sharp - \frac{1}{\mu}\oint_{\psi_0}[\Delta - \omega_0']^{-1}\Phi^\sharp\,\mathrm{d}s. \tag{4.16}$$

This determines ϕ up to the addition of an arbitrary function of ψ_0 (corresponding to arbitrary displacements along streamlines); as in the channel case, to fix it we impose the constraint

$$\oint_{\psi_0}\phi\,\mathrm{d}s = 0. \tag{4.17}$$

This completes the formulation of the problem of finding steady flows in general deformed domains. The existence, uniqueness and smoothness of solutions to this problem are given by the following theorem.

Theorem 2. *Let* D_0 *and* D *be* $C^{k+1,\alpha}$ *domains, with the boundary deformation specified by (4.3), where* B_0 *and* b *belong to* $C^{k+1,\alpha}(ND)$*. Let* $k \geqslant 2$, $\psi_0 \in C^{k+1,\alpha}(D_0)$ *define a steady flow, and suppose that H1–H3 are satisfied. Then there exists an* $\varepsilon_0 > 0$ *such that for*

$$|b|_{C^{k+1,\alpha}(ND)} \leqslant \varepsilon_0,$$

there is a $g : D_0 \to D$*, unique up to displacements along streamlines, and a unique* $\chi \in C^{k,\alpha}(D_0)$ *that give a steady flow in* D *with vorticity* $\Omega = \omega_0 \circ g^{-1} \in C^{k-1,\alpha}(D)$ *and streamfunction* $\Psi = (\psi_0 + \chi) \circ g^{-1} \in C^{k+1,\alpha}(D)$.

We note that since the new flow Ψ is (qualitatively) as smooth as the initial flow ψ_0, we can repeat the process as long as H1–H3 continue to hold to obtain larger deformations.

Proof. The proof of this theorem is similar to that of theorem 1, whose notation we keep (here constants may depend on B_0 and ψ_0 as well as on k). Therefore, here we will only treat the extra complications posed by the non-symmetric boundary.

As in the channel case, suppose that at the beginning of iteration n we have

$$|\eta^n|_{k,\alpha} \leqslant C_{ND}, \qquad |\partial_s\eta^n|_{k,\alpha} \leqslant C_{ND} \qquad \text{and} \qquad |\partial_s\phi^n|_{k,\alpha} \leqslant C_{ND}, \tag{4.18}$$

where $C_{ND} \leqslant 1$ is chosen such that $(x + u(x, y), y + v(x, y)) \in ND$ for $(x, y) \in \partial D_0$; since $\psi_0 \in C^{k+1,\alpha}(D_0)$, the last two inequalities imply that $\partial_s u^n$, $\partial_s v^n \in C^{k-1,\alpha}(D_0)$. For χ^n, we suppose that

$$|\chi^n_\psi|_{k,\alpha;I_0} \leqslant C'_{ND} \tag{4.19}$$

and note that $|\chi|_{k,\alpha} \leqslant c|\chi_\psi|_{k,\alpha;I_0}$; we choose C'_{ND} such that $|\chi^n|_{k,\alpha} \leqslant C_{ND}$. With a slight abuse of notation we will henceforth write $|\chi_\psi|_{k,\alpha}$ for $|\chi_\psi|_{k,\alpha;I_0}$.

To estimate the boundary terms, we note using (3.21) that, since $B_0|_{\partial D_0} = 0$,

$$|B_0 \circ (\mathrm{id} + u)|_{k,\alpha;\partial D_0} \leqslant c\,|B_0|_{k,1;ND}|u|_{k,\alpha;\partial D_0}.$$

Using this to bound the first term in (4.5), bounding the second term in the obvious fashion and using (3.22) to bound the last term, we arrive at

$$\begin{aligned} c|\beta(u, v; \cdot)|_{k,\alpha;\partial D_0} &\leqslant |B_0|_{k,1;ND}|(u, v)|_{k,\alpha;\partial D_0} + |B_0|_{k+1,\alpha;\partial D_0}|(u, v)|_{k,\alpha;\partial D_0} \\ &\quad + |b|_{k,\alpha;ND}(1 + |(u, v)|^{k+\alpha}_{k,\alpha;\partial D_0}). \end{aligned} \tag{4.20}$$

We estimate $(u, v)|_{\partial D_0}$ by 'projection' as in the channel case,

$$|u|_{k,\alpha;\partial D_0} \leqslant c|\partial_s u|_{k-1,\alpha;\partial D_0} \leqslant c|\partial_s u|_{k-1,\alpha} \leqslant c(|\partial_s \eta|_{k,\alpha} + |\partial_s \phi|_{k,\alpha})$$

and a similar estimate for $|v|_{k,\alpha;\partial D_0}$. Introducing this into (4.20) and absorbing $|B_0|_{k+1,\alpha;ND}$ into the constant, we arrive at

$$|\beta(u, v; \cdot)|_{k,\alpha;\partial D_0} \leqslant c\|b\|(1 + |\partial_s \eta|^2_{k,\alpha;\partial D_0} + |\partial_s \phi|^2_{k,\alpha;\partial D_0}), \tag{4.21}$$

where $\|b\| := |b|_{k,\alpha;ND}$.

As before, we compute η^{n+1} by solving

$$\begin{aligned} &\Delta \eta^{n+1} = u^n_y v^n_x - u^n_x v^n_y \qquad \text{in } D_0, \\ &\frac{\partial \eta^{n+1}}{\partial n} = \int_{D_0} [u^n_y v^n_x - u^n_x v^n_y]\mathrm{d}x\,\mathrm{d}y \Big/ \oint_{\partial D_0} \mathrm{d}l \qquad \text{on } \partial D_0. \end{aligned} \tag{4.22}$$

As in the channel case, here the boundary conditions are different from (4.7); however, upon convergence these two boundary conditions are equivalent because both consist of a constant which expresses the fact that g is area preserving. Lemma 2 then implies that

$$|\eta^{n+1}|_{k,\alpha} \leqslant c(|\phi^n|^2_{k,\alpha} + |\eta^n|^2_{k,\alpha}).$$

To bound $\partial_s \eta^{n+1}$, we take ∂_s(4.22a) and use the identity

$$(\Delta - \omega_0')\partial_s \varphi = \partial_s \Delta\varphi + 2\nabla^\perp \partial_x \psi_0 \cdot \nabla \partial_x \varphi + 2\nabla^\perp \partial_y \psi_0 \cdot \nabla \partial_y \varphi$$

with $\varphi = \eta^{n+1}$. Noting that the entire right-hand side is bounded in $C^{k-2,\alpha}(D_0)$ by $|\eta^n|^2_{k,\alpha} + |\partial_s \eta^n|^2_{k,\alpha} + |\partial_s \phi^n|^2_{k,\alpha}$, and using the argument leading to (3.29), lemma 2 again gives

$$|\partial_s \eta^{n+1}|_{k,\alpha} \leqslant c_1(|\partial_s \phi^n|^2_{k,\alpha} + |\partial_s \eta^n|^2_{k,\alpha} + |\eta^n|^2_{k,\alpha}). \tag{4.23}$$

Now, the iteration corresponding to (4.9) is

$$\begin{aligned} \Delta \chi^{n+1} + [\Delta - \omega_0']\partial_s \phi^{n+1} &= -2[\eta^{n+1}_{xx}\partial_{yy}\psi_0 + \eta^{n+1}_{yy}\partial_{xx}\psi_0 - 2\eta^{n+1}_{xy}\partial_{xy}\psi_0] + \Phi_{\mathrm{nl}}(\chi^n, u^n, v^n) \\ &=: \Phi^\sharp(\chi^n, u^n, v^n; \eta^{n+1}). \end{aligned} \tag{4.24}$$

Thanks to the anisotropy of the nonlinear terms, Φ_{nl}, in (4.10), we have

$$|\Phi_{\mathrm{nl}}(\phi^n, \eta^n, \chi^n)|_{k-2,\alpha} \leqslant c(|\partial_s \phi^n|^2_{k,\alpha} + |\partial_s \eta^n|^2_{k,\alpha} + |\eta^n|^2_{k,\alpha} + |\chi^n_\psi|^2_{k,\alpha}).$$

Combined with (4.23), the last line of (4.24) is bounded as

$$|\Phi^\sharp(\chi^n, u^n, v^n; \eta^{n+1})|_{k-2,\alpha} \leqslant c_2(|\partial_s \phi^n|^2_{k,\alpha} + |\partial_s \eta^n|^2_{k,\alpha} + |\eta^n|^2_{k,\alpha} + |\chi^n_\psi|^2_{k,\alpha}). \tag{4.25}$$

Unlike in the proof of theorem 1, at this point we need to estimate the boundary contribution, which is the solution, $\partial_s \phi^{n+1}_{\mathrm{b}}$, of the problem

$$\begin{aligned} &[\Delta - \omega_0']\partial_s \phi^{n+1}_{\mathrm{b}} = 0, \\ &\partial_s \phi^{n+1}_{\mathrm{b}}(x, y) = \tilde{\beta}(u^n, v^n; x, y) \qquad (x, y) \in \partial D_0. \end{aligned}$$

Using (4.21),

$$|\partial_s\phi_{\mathrm{b}}^{n+1}|_{k,\alpha;\partial D_0} \leqslant c(\psi_0)\,\|b\|(1+|\partial_s\eta^n|_{k,\alpha}^2+|\partial_s\phi^n|_{k,\alpha}^2). \tag{4.26}$$

Lemma 1 and (4.25) then give

$$|[\Delta-\omega_0']_n^{-1}\Phi^\sharp|_{k,\alpha} \leqslant c_3(|\partial_s\phi^n|_{k,\alpha}^2+|\partial_s\eta^n|_{k,\alpha}^2+|\eta^n|_{k,\alpha}^2+|\chi_\psi^n|_{k,\alpha}^2+\|b\|), \tag{4.27}$$

where the subscript n on $[\Delta-\omega_0']^{-1}$ defined in (4.12), is a reminder that its boundary conditions depend on (u^n, v^n).

We turn to χ, obtained from (4.13),

$$(\mu+K)\,\chi_\psi^{n+1}(\psi_0) = \oint_{\psi_0}[\Delta-\omega_0']_n^{-1}\Phi^\sharp(u^n, v^n, \chi^n; \eta^{n+1})\mathrm{d}s =: \Xi(\psi_0).$$

Using H2 and (4.27),

$$|\Xi|_{C^{k,\alpha}(I_0)} \leqslant c(\psi_0)(|\partial_s\phi^n|_{k,\alpha}^2+|\partial_s\eta^n|_{k,\alpha}^2+|\eta^n|_{k,\alpha}^2+|\chi_\psi^n|_{k,\alpha}^2+\|b\|). \tag{4.28}$$

Since by H3 the operator $(\mu+K) : C^{k,\alpha}(I_0) \to C^{k,\alpha}(I_0)$ has a bounded inverse, we have

$$|\chi_\psi^{n+1}|_{k,\alpha} \leqslant c_4(|\partial_s\phi^n|_{k,\alpha}^2+|\partial_s\eta^n|_{k,\alpha}^2+|\eta^n|_{k,\alpha}^2+|\chi_\psi^n|_{k,\alpha}^2+\|b\|). \tag{4.29}$$

In the process we have bounded all the operators and the right-hand side in the iteration corresponding to (4.16). Solving for ϕ^{n+1}, we find

$$|\partial_s\phi^{n+1}|_{k,\alpha} \leqslant c_6(|\partial_s\phi^n|_{k,\alpha}^2+|\partial_s\eta^n|_{k,\alpha}^2+|\eta^n|_{k,\alpha}^2+|\chi_\psi^n|_{k,\alpha}^2+\|b\|). \tag{4.30}$$

As in the channel case, boundedness of the iteration follows from (4.23), (4.29) and (4.30), provided we take $\|b\|$ sufficiently small. The convergence is very similar and we shall not do it explicitly here.

Upon convergence, we have $u, v \in C^{k-1,\alpha}(D_0)$ and so $g^{-1} \in C^{k-1,\alpha}(D, D_0)$. Since $\omega_0 \in C^{k-1,\alpha}(D_0)$, this and (3.22) imply that $\Omega = \omega_0 \circ g^{-1} \in C^{k-1,\alpha}(D)$, which in turn implies that $\Psi = \Delta^{-1}\Omega \in C^{k+1,\alpha}(D)$. □

Remarks. It is clear from (4.29) that χ is of the order of $\|b\|$, as is the linear solution χ^1; this is to be contrasted with the channel case, where χ is quadratic in $\|b\|$ (and where $\chi^1 = 0$). As in the channel case, $\phi \sim \mathcal{O}(\|b\|)$ (cf (4.30)) and $\eta \sim \mathcal{O}(\|b\|^2)$ (cf (4.23)).

5. Persistence of stability

If a steady flow persists when its domain boundary is deformed as established in the previous sections, it is natural to ask whether its stability properties also persist. Of interest here is the nonlinear stability of steady flows; it holds for a flow with vorticity Ω and streamfunction $\Psi = F(\Omega)$ if, in the evolution of the perturbed flow $\Psi+\hat\Psi(t)$, a suitable norm of the perturbation streamfunction $|\hat\Psi(t)|$ is bounded by $|\hat\Psi(0)|$ for all t.

For two-dimensional incompressible and inviscid flows in general domains, useful stability results of this type have been derived using the Arnold's energy–Casimir method (Arnold (1965, 1966); see also Holm *et al* (1985), Arnold and Khesin (1998)). The application of this method provides two sufficient conditions for stability: the steady flow $\Psi = F(\Omega)$ is nonlinearly stable if there are two constants c_1 and c_2 such that either

(i) $0 < c_1 \leqslant F' \leqslant c_2 < \infty$, or
(ii) $0 < \lambda_1 < c_1 \leqslant -F' \leqslant c_2 < \infty$,

where $\lambda_1 > 0$ is the smallest eigenvalue of $-\Delta$ in D. Recalling hypothesis H1 for the existence of a deformed flow, we note that this condition is implied by Arnold stability of the initial flow (assuming sufficient smoothness, etc). The proof of nonlinear stability relies on the invariance of the energy–Casimir functional

$$\mathcal{A} = \int_D \left[\frac{1}{2}|\nabla(\hat{\Psi}+\Psi)|^2 + G(\hat{\Omega}+\Omega)\right] \mathrm{d}X\,\mathrm{d}Y - \int_D \left[\frac{1}{2}|\nabla\Psi|^2 + G(\Omega)\right] \mathrm{d}X\,\mathrm{d}Y, \tag{5.1}$$

where $G' = F$ and on the choice of a norm of $\hat{\Psi}$ which bounds $\mathcal{A}$ above and below (cf, e.g., Holm *et al* (1985)). We note that since hypothesis H1 is weaker than the stability conditions (i) and (ii) (H1 only rules out a countable set of functions), our method works in many cases where the steady flow is not energy–Casimir stable, in contrast with the energy-minimization argument which can only yield steady flows which are also stable.

Now suppose that the flow $\psi_0 = F_0(\omega_0)$ in D_0 is stable by either condition (i) or (ii) applied to F_0. Then it is clear that for a sufficiently small boundary deformation $|b|_{C^{k,\alpha}(\partial D_0)}$ the deformed flow $\Psi = F(\Omega)$ is also stable by (i) or (ii). This follows from the estimates on χ ((3.50) for parallel flow and its analogue in the more general case; note that (4.29) implies that the latter is of order $\|b\|$) and from the relation

$$F' = F_0' \cdot \left(1 + \frac{\nabla\chi}{\nabla\psi_0}\right),$$

which is easily deduced from $F \circ \omega_0 = \psi_0 + \chi$.

In symmetric domains (channels or discs), the stability of some symmetric flows which do not satisfy (i) or (ii) may be established using the energy–Casimir–momentum method (Holm *et al* 1985). This method takes advantage of the (translational or rotational) invariance of the system by adding to the energy–Casimir functional (5.1) the conserved quantity associated with the invariance, a multiple of the momentum

$$\mathcal{M} = \int_{D_0} y\hat{\omega}_0\,\mathrm{d}x\,\mathrm{d}y$$

in the case of a parallel flow in a channel. The stability conditions (i) and (ii) are then extended to give nonlinear extensions of the celebrated Rayleigh–Fjørtoft conditions: the stability of a parallel flow with streamfunction $\psi_0(y)$ and vorticity $\omega_0(y)$ is guaranteed if there exist constants c_1, c_2 and w such that either

(i′) $0 < c_1 \leqslant (\mathrm{d}\psi_0/\mathrm{d}y - w)/(\mathrm{d}\omega_0/\mathrm{d}y) \leqslant c_2 < \infty$, or
(ii′) $0 < \lambda_1 < c_1 \leqslant -(\mathrm{d}\psi_0/\mathrm{d}y - w)/(\mathrm{d}\omega_0/\mathrm{d}y) \leqslant c_2 < \infty$.

Since it relies on the symmetry of the domain, the energy–Casimir–momentum stability of a parallel flow $\psi_0(y)$ in a channel cannot be expected to persist in the deformed domain. However, one can consider the momentum-like quantity

$$\mathcal{M}_* = \int_{D_0} y\hat{\omega}_*\,\mathrm{d}x\,\mathrm{d}y, \tag{5.2}$$

where $\hat{\omega}_* = \hat{\Omega} \circ g$ is the pull-back of the perturbation vorticity $\hat{\Omega}$ from D to D_0, and show that its time derivative satisfies

$$\frac{\mathrm{d}\mathcal{M}_*}{\mathrm{d}t} = \frac{1}{2}\int_{D_0} \hat{\psi}_*(\Delta_g\partial_x - \partial_x\Delta_g)\hat{\psi}_*\,\mathrm{d}x\,\mathrm{d}y, \tag{5.3}$$

where $\hat{\psi}_* = \hat{\Psi} \circ g$. Since the operator $\Delta_g\partial_x - \partial_x\Delta_g$ may be bounded by a norm of the boundary deformation $|b|_{C^{k,\alpha}(\partial D_0)}$, a construction similar to that of the energy–Casimir–momentum invariant for parallel flows can be used to show that a norm of the disturbance streamfunction $\hat{\Psi}$ grows at most exponentially, with a growth rate bounded by $|b|_{C^{k,\alpha}(\partial D_0)}$.

6. A numerical example

As mentioned in the introduction, the results of sections 3 and 4 provide an approach for the calculation of new steady flows. To illustrate this, we now present an explicit numerical example of one such flow, obtained by deforming a simple parallel flow in a channel.

We take the initial domain D_0 to be the channel $(x, y) \in [0, 2\pi] \times [0, 1]$ with periodicity in x assumed. For our illustration, the initial parallel flow is chosen to be $U_0(y) = (y - \frac{1}{2})^3 + \frac{1}{4}$. This flow is stable according to Fjørtoft's theorem (e.g. Drazin and Reid (1981)), however, numerical experiments not shown here with the possibly unstable flow $U_0(y) = (y - \frac{1}{2})^3 - (y - \frac{1}{2})/2 + \frac{1}{4}$ produce essentially similar results. The walls of the channel are then deformed by sinusoidal deformations: $b^0(x) = \varepsilon \cos 2x$ on $y = 0$ and $b^1(x) = \varepsilon \sin 3x$ on $y = 1$. Thus $D = \{(X, Y) : b^0(X) \leqslant Y \leqslant 1 + b^1(X) \text{ for } X \in [0, 2\pi]\}$. We use the iterative procedure described in section 3 to compute ϕ, η and χ in the original domain D_0 which, being a channel, allows for a simple discretization.

Expanding the unknowns in Fourier modes in x, $\eta(x, y) = \sum_k \mathrm{e}^{ikx} \eta_k(y)$, etc (3.24) becomes a family of decoupled one-dimensional two-point boundary-value problems for $\eta_k^{n+1}(y)$; these are then discretized using finite differences and solved using a relaxation algorithm (`solvde()` in Press *et al* (1992)). The nonlinear term $u_y^n v_x^n - u_x^n v_y^n$, which is known from the previous iteration, is computed using a pseudospectral method (e.g. Canuto *et al* (1988)). The treatment for (3.31) and (3.36) is completely analogous, with the nonlinear term $\Phi_{\mathrm{nl}}(u^n, v^n, \chi^n)$ also computed pseudospectrally.

We used 64 Fourier modes in x and 65 grid points in y for the computation presented here; the use of higher resolutions does not alter the result perceptibly. As may be expected, we found that the iterations converge for sufficiently small boundary deformations. The numerical iterations cease to converge for ε rather small—about $\varepsilon = 0.007$ or so.

Figure 1(*a*) shows the displacement 'streamfunction', ϕ, for $\varepsilon = 0.0025$. Its scale for ϕ can be appreciated by noting that, on the boundary, $\partial_x \phi \simeq b(x)$. It can be seen that ϕ possesses several fixed points, most of them near the $y = \frac{1}{2}$ line. The 'divergent' component, η, is about a factor of ε smaller than ϕ, and so we do not plot it here; related to this fact, we note that ϕ of the linearized problem is visually indistinguishable from that plotted in figure 1(*a*) since the nonlinear correction is $\mathcal{O}(\varepsilon)$ times smaller. In figure 1(*b*), the streamfunction $\psi_0(y)$ and velocity $U_0(y)$ are shown, along with the change in streamfunction $\chi(y)$.

The numerical scheme described above is relatively simple since the symmetric initial (i.e. computational) domain allows for a separation of variables for the linear problem. This is not possible in the more general case of non-symmetric initial domains D_0, for which a more sophisticated discretization such as the use of finite-element methods is necessary. Other issues here would include the numerical equivalent of the $C^{3,\alpha}$ boundary as required by theorem 2, and the need for a high-order accuracy (we found that the numerical iteration is very sensitive to the smoothness of the solution). Since each of these poses a significant numerical problem in itself, we defer this to a future work, likely in the context of a specific application.

Another simplifying factor is the fact that the linear operator DN in the iteration (2.19) is evaluated at w^0, which does not depend on x. A Newton–Raphson iteration, where DN is taken at w^n, would converge faster (when it does) but would be more difficult to implement since w^n depends on both x and y.

The convergence of the numerical scheme appears limited to very small values of ε. We emphasize that this does not imply that any of H1–H3 is violated and that no isovortical steady flows exist for larger values. Rather, it shows the limited usefulness of the iteration with a symmetric initial domain as a means of finding new steady flows. To compute steady flows for larger domain deformations, one would need to proceed incrementally, increasing ε by

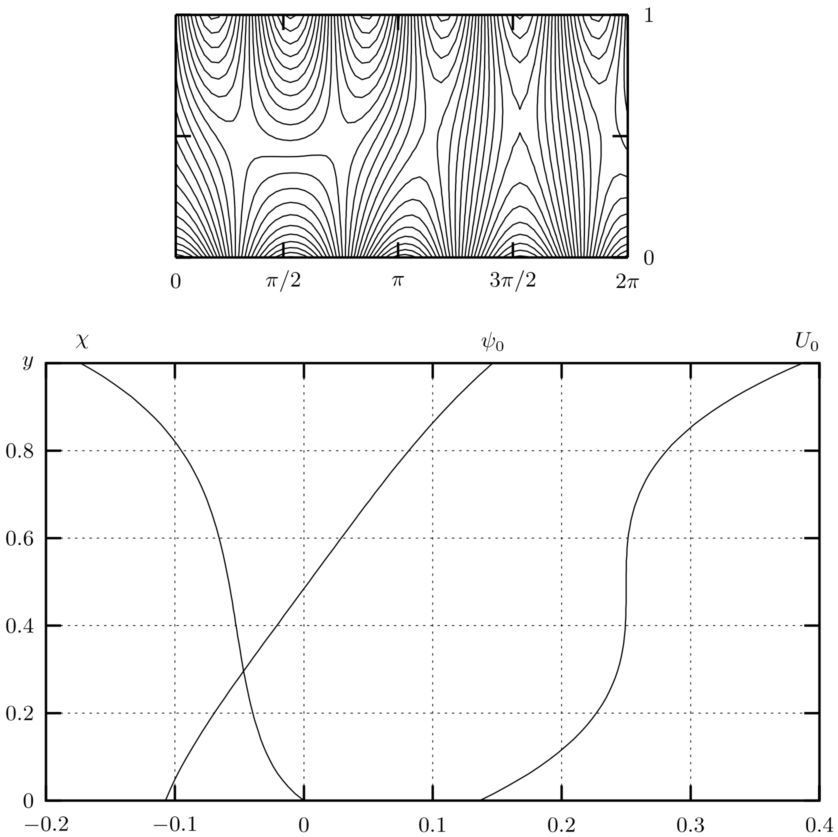

Figure 1. Deformation of a parallel flow. The basic flow has $U_0(y) = (y - \frac{1}{2})^3 + \frac{1}{4}$, and the boundary deformation is defined by $b^0(x) = \varepsilon \cos 2x$ and $b^1(x) = \varepsilon \sin 3x$ with $\varepsilon = 0.0025$. (*a*) Top panel: contour plot of the displacement streamfunction $\phi(x, y)$ in the channel. Contour level is 2×10^{-4}. (*b*) Bottom panel: change in the streamfunction $\chi(y)$, magnified by 10^4, initial streamfunction $\psi_0(y)$ and velocity $U_0(y)$.

small steps and using the flow computed at each step as the intial flow for the next step. This, of course, is numerically much more involved since it requires an implementation for non-symmetric initial domains.

A possible explanation for the smallness of ε required for the convergence of our iteration (analytical and numerical) is the following: suppose that a steady flow ω_0 in domain D_0 is continuously deformable to Ω in D without violating H1–H3. A given path γ connecting D_0 and D in the space of domains then defines a path Γ of steady flows connecting ω_0 and Ω in the space of isovortical flows (equivalently, one may take Γ to live in the space of area-preserving diffeomorphisms with $g_0 : D_0 \to D_0$ and $g : D_0 \to D$). Now there is a neighbourhood of Γ outside which our iteration fails to converge, and so if Γ is significantly 'curved', small steps (i.e. repeated applications of theorem 2) are needed in order to remain in this neighbourhood.

7. Discussion and future work

In section 3 we have shown that a parallel flow persists as a steady flow under finite deformations of its channel domain. Apart from sufficient smoothness, the only hypothesis required is that the velocity does not vanish anywhere. A similar computation shows that this result also holds for an axisymmetric flow, provided that the velocity vanishes only at the centre of the disc, where the vorticity cannot be zero (cf the comment following H2). For the more general case of a non-symmetric flow discussed in section 4, hypothesis H2 can be viewed as the natural extension of this constraint on the non-vanishing of the velocity; but two additional hypotheses, H1 and H3, are necessary for our proof of the existence and uniqueness of a steady deformed flow. There appears, therefore, to be a significant difference between the symmetric (parallel or axisymmetric) and non-symmetric cases.

It is easy to understand why H1, i.e. the invertibility of $\Delta - \omega_0'$, is not needed explicitly in the symmetric cases: it is a direct consequence of the non-vanishing of the velocity. As the following calculation shows for the parallel case, if $\Delta - \omega_0'$ has a nontrivial homogeneous solution, U_0 must vanish somewhere,

$$(\Delta - \omega_0')\phi = [\Delta - (\partial_{yy}U_0/U_0)]\phi = 0 \Leftrightarrow \nabla \cdot [U_0^2 \nabla(\phi/U_0)] = 0 \\ \Rightarrow \int U_0^2 |\nabla(\phi/U_0)|^2 \,\mathrm{d}y = 0 \tag{7.1}$$

(cf Howard (1961)). In view of this, it is natural to ask whether an analogous result holds in the general, non-symmetric case, that is, whether H2 generally implies H1. We have not been able to establish this. The precise role and physical significance of H3 still elude us at the moment.

Note that there is an important difference between the symmetric and non-symmetric flows: the change in the streamfunction χ is quadratic in $\|b\|$ in the symmetric case but is linear in $\|b\|$ in general. This shows that a symmetric flow is a critical point of the ω–ψ relationship in the sense that, using ε as a parameter for a domain deformation 'path', $\mathrm{d}F/\mathrm{d}\varepsilon = 0$ as the path passes through a symmetric domain.

As mentioned in the introduction, one of the interests of the isovortical steady flows considered in this paper is that they can be achieved at least approximately by an adiabatically slow deformation of the fluid domain. In this context, an intriguing question concerns the nature of the flow evolution when an isovortical steady flow does not exist. We might speculate that in such a case the adiabatic deformation of the domain leads to a complex transient flow, even at leading order. This is suggested by the behaviour of parallel flows whose velocity vanishes somewhere so that theorem 1 does not apply. When disturbed, these flows, even when Arnold stable, exhibit complicated transient dynamics associated with the formation of a critical layer (Stewartson 1978, Warn and Warn 1978). This phenomenon confirms the importance of assuming a non-vanishing velocity to ensure the persistence of steady parallel flows. Viewing H2 as the natural extension to non-parallel flows of this assumption, one might conjecture that a physical phenomenon similar to a critical layer occurs in flows for which H2 is violated, i.e. in flows for which

$$\oint \frac{\mathrm{d}l}{|\nabla\psi_0|} = \infty \tag{7.2}$$

for some streamline. The dynamics in such non-parallel flows is certainly worth investigating. It should be noted, moreover, that the presence of a stagnation point, where $\nabla\psi_0 = 0$ and H2 is violated, is known to lead to some form of instability (Friedlander and Vishik 1992, Vishik 1996).

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Appendix. Summary of notation

Let $g : D_0 \to D : (x, y) \mapsto (X, Y) = (x + u(x, y), y + v(x, y)) = (x, y) + \nabla\eta(x, y) + \nabla^{\perp}\phi(x, y)$. Then we have:

Object	Original domain D_0	Deformed domain D
Coordinates	(x, y)	(X, Y)
Vorticity	ω_0	$\Omega := \omega_0 \circ g^{-1}$
Streamfunction	$\psi_0 = F_0 \circ \omega_0$	$\Psi = F \circ \Omega$
Pulled-back streamfunction	$\psi_* := \Psi \circ g$	Ψ
	$=: \psi_0 + \chi$	
Pulled-back Δ (for $f : D_0 \to \mathbb{R}$)	$\Delta_g f := [\Delta(f \circ g^{-1})] \circ g$	$\Delta(f \circ g^{-1})$

Primes denote derivative with respect to ψ_0:

$$\omega_0' := \nabla\omega_0/\nabla\psi_0 = (F_0^{-1})' \circ \omega_0, \qquad \psi_*' := \nabla\psi_*/\nabla\psi_0 \qquad \text{and} \qquad \chi' := \nabla\chi/\nabla/\psi_0.$$

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