The Borsuk–Ulam Theorem

Anthony Carbery

University of Edinburgh &
Maxwell Institute for Mathematical Sciences

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Outline

Brouwer fixed point theorem

The Brouwer fixed point theorem states that every continuous map $f : \mathbb{D}^n \to \mathbb{D}^n$ has a fixed point.

When $n = 1$ this is a trivial consequence of the intermediate value theorem.

In higher dimensions, if not, then for some f and all $x \in \mathbb{D}^n$, $f(x) \neq x$.

So the map $\tilde{f} : \mathbb{D}^n \to \mathbb{S}^{n-1}$ obtained by sending x to the unique point on $\mathbb{S}^{n-1}$ on the line segment starting at $f(x)$ and passing through x is continuous, and when restricted to the boundary $\partial\mathbb{D}^n = \mathbb{S}^{n-1}$ is the identity.

So to prove the Brouwer fixed point theorem it suffices to show there is **no** map $g : \mathbb{D}^n \to \mathbb{S}^{n-1}$ which restricted to the boundary $\mathbb{S}^{n-1}$ is the identity. (In fact, this is an equivalent formulation.)

Proof of Brouwer's theorem

It is enough, by a standard approximation argument, to prove that there is no **smooth** (C^1) map $g : \mathbb{D}^n \to \mathbb{S}^{n-1}$ which restricted to the boundary $\mathbb{S}^{n-1}$ is the identity.

Consider, for Dg the derivative matrix of g,

$$\int_{\mathbb{D}^n} \det Dg.$$

This is zero as Dg has less than full rank at each $x \in \mathbb{D}^n$.

So

$$0 = \int_{\mathbb{D}^n} \det Dg = \int_{\mathbb{D}^n} dg_1 \wedge dg_2 \wedge \cdots \wedge dg_n,$$

which, by Stokes' theorem equals

$$\int_{\mathbb{S}^{n-1}} g_1 \, dg_2 \wedge \cdots \wedge dg_n.$$

This quantity depends only the behaviour of g_1 on $\mathbb{S}^{n-1}$, and, by symmetry, likewise depends only on the restrictions of $g_2, \ldots, g_n$ to $\mathbb{S}^{n-1}$.

But on $\mathbb{S}^{n-1}$, g is the identity I, so that reversing the argument, this quantity also equals

$$\int_{\mathbb{D}^n} \det DI = |\mathbb{D}^n|.$$

This argument is essentially due to E. Lima. Is there a similarly simple proof of the Borsuk–Ulam theorem via Stokes' theorem?

Borsuk–Ulam theorem

The Borsuk–Ulam theorem states that for every continuous map $f : \mathbb{S}^n \to \mathbb{R}^n$ there is some x with $f(x) = f(-x)$. When $n = 1$ this is a trivial consequence of the intermediate value theorem.

In higher dimensions, it again suffices to prove it for smooth f.

So assume f is smooth and $f(x) \neq f(-x)$ for all x. Then

$$\tilde{f}(x) := \frac{f(x) - f(-x)}{|f(x) - f(-x)|}$$

is a smooth map $\tilde{f} : \mathbb{S}^n \to \mathbb{S}^{n-1}$ such that $\tilde{f}(-x) = -\tilde{f}(x)$ for all x, i.e. $\tilde{f}$ is *odd, antipodal* or *equivariant* with respect to the map $x \mapsto -x$.

So it's ETS there is no equivariant smooth map $h : \mathbb{S}^n \to \mathbb{S}^{n-1}$, or, equivalently, there is no smooth map $g : \mathbb{D}^n \to \mathbb{S}^{n-1}$ which is equivariant on the boundary.

Equivalent to Borsuk–Ulam theorem; BU generalises Brouwer fixed point thorem (since the identity map is equivariant).

WTS there does not exist a smooth $g : \mathbb{D}^2 \to \mathbb{S}^1$ such that $g(-x) = -g(x)$ for $x \in \mathbb{S}^1$. If there did exist such a g, consider

$$\int_{\mathbb{D}^2} \det Dg = \int_{\mathbb{D}^2} dg_1 \wedge dg_2.$$

This is zero as Dg has less than full rank at each x, and it equals, by Stokes' theorem,

$$\int_{\mathbb{S}^1} g_1 dg_2 = -\int_{\mathbb{S}^1} g_2 dg_1.$$

So it's enough to show that

$$\int_0^1 (g_1(t)g_2'(t) - g_2(t)g_1'(t))dt \neq 0$$

for $g = (g_1, g_2) : \mathbb{R}/\mathbb{Z} \to \mathbb{S}^1$ satisfying $g(t + 1/2) = -g(t)$ for all $0 \leq t \leq 1$.

ETS

$$\int_0^1 (g_1(t)g_2'(t) - g_2(t)g_1'(t))dt \neq 0$$

for $g = (g_1, g_2) : \mathbb{R}/\mathbb{Z} \to \mathbb{S}^1$ satisfying $g(t + 1/2) = -g(t)$ for all $0 \leq t \leq 1$.

Clearly

$$(g_1(t)g_2'(t) - g_2(t)g_1'(t))dt$$

represents the element of net arclength for the curve $(g_1(t), g_2(t))$ measured in the anticlockwise direction. (Indeed, $|g| = 1$ implies $\langle g, g' \rangle = \frac{1}{2}\frac{d}{dt}|g|^2 = 0$, so that $\det(g, g') = \pm|g||g'| = \pm|g'|$, with the plus sign occuring when g is moving anticlockwise.) By equivariance, $(g_1(1/2), g_2(1/2)) = -(g_1(0), g_2(0))$, and

$$\int_0^1 g_1(t)g_2'(t)dt = 2\int_0^{1/2} g_1(t)g_2'(t)dt.$$

In passing from $(g_1(0), g_2(0))$ to $(g_1(1/2), g_2(1/2))$ the total net arclength traversed is clearly an odd multiple of π, and so we're done.

Theorem (Shchepin)

Suppose $n \geq 4$ and there exists a smooth equivariant map $f : \mathbb{S}^n \to \mathbb{S}^{n-1}$. Then there exists a smooth equivariant map $\tilde{f} : \mathbb{S}^{n-1} \to \mathbb{S}^{n-2}$.

Once this is proved, only the case $n = 3$ of the Borsuk–Ulam theorem remains outstanding.

Starting with f, we shall identify suitable equators $E_{n-1} \subseteq \mathbb{S}^n$ and $E_{n-2} \subseteq \mathbb{S}^{n-1}$, and build a smooth equivariant map $\tilde{f} : E_{n-1} \to E_{n-2}$.

We first need to know that there is some pair of antipodal points $\{\pm A\}$ in the target $\mathbb{S}^{n-1}$ whose preimages under f are covered by finitely many diffeomorphic copies of $(-1, 1)$. This is intuitively clear by dimension counting (WMA f is onto!) but for rigour we can appeal to Sard's theorem.

For $x \in f^{-1}(A)$ and $y \in f^{-1}(-A) = -f^{-1}(A)$ with $y \neq -x$, consider the unique geodesic great circle joining x to y. The family of such is clearly indexed by the two-parameter family of points of

$$f^{-1}(A) \times f^{-1}(-A) \setminus \{(x, -x) \ : \ f(x) = A\}.$$

Their union is therefore a manifold in $\mathbb{S}^n$ of dimension at most three. Since $n \geq 4$ there must be points $\pm B \in \mathbb{S}^n$ *outside* this union (and necessarily outside $f^{-1}(A) \cup f^{-1}(-A)$). Such a point has the property that *no* geodesic great circle passing through it meets points of both $f^{-1}(A)$ and $f^{-1}(-A)$ other than possibly at antipodes. In particular, no *meridian* joining $\pm B$ meets both $f^{-1}(A)$ and $f^{-1}(-A)$.

We now identify E_{n-2} as the equator of $\mathbb{S}^{n-1}$ whose equatorial plane is perpendicular to the axis joining A to $-A$; and we identify E_{n-1} as the equator of $\mathbb{S}^n$ whose equatorial plane is perpendicular to the axis joining B to $-B$. We assume for simplicity that B is the north pole $(0, 0, \ldots, 0, 1)$.

Lemma (Lemma 1)

Suppose $B = (0, 0, \ldots, 0, 1) \in \mathbb{S}^n$ and that $X \subseteq \mathbb{S}^n$ is a closed subset such that no meridian joining $\pm B$ meets both X and $-X$. Let $\mathbb{S}^n_\pm$ denote the open upper and lower hemispheres respectively. Then there is an equivariant diffeomorphism $\psi : \mathbb{S}^n \to \mathbb{S}^n$ such that

$$X \subseteq \psi(\mathbb{S}^n_+).$$

Remark 1. It is clear that we may assume that ψ fixes meridians and acts as the identity on small neighbourhoods of $\pm B$.

Remark 2. It is also clear from the proof that we can find a smooth family of diffeomorphisms ψ_t such that ψ_0 is the identity and $\psi_1 = \psi$.

We shall need this later.

Lemma

Suppose $B = (0, 0, \ldots, 0, 1) \in \mathbb{S}^n$ and that $X \subseteq \mathbb{S}^n$ is a closed subset such that no meridian joining $\pm B$ meets both X and $-X$. Let $\mathbb{S}^n_\pm$ denote the open upper and lower hemispheres respectively. Then there is an equivariant diffeomorphism $\psi : \mathbb{S}^n \to \mathbb{S}^n$ such that

$$X \subseteq \psi(\mathbb{S}^n_+).$$

Continuing with the proof of the theorem, we apply the lemma with $X = f^{-1}(A)$. Let ϕ be restriction of ψ to $E = E_{n-1}$. Consider the restriction $\widehat{f}$ of f to $\phi(E)$: it has the property that $\widehat{f}(\phi(E))$ does not contain $\pm A$. Let r be the standard retraction of $\mathbb{S}^{n-1} \setminus \{\pm A\}$ onto its equator E_{n-2}; finally let

$$\tilde{f} = r \circ \widehat{f} \circ \phi,$$

which is clearly smooth and equivariant.

Proof of Lemma

See the pictures on the blackboard!

The Sard argument

WTS there is some pair of antipodal points $\{\pm A\}$ in the target $\mathbb{S}^{n-1}$ whose preimages under f are at most "one-dimensional", i.e. covered by finitely many diffeomorphic copies of $(-1, 1)$.

Sard's theorem tells us that the image under f of the set $\{x \in \mathbb{S}^n : \text{ rank } Df(x) < n - 1\}$ is of Lebesgue measure zero: so there are plenty of points $A \in \mathbb{S}^{n-1}$ at all of whose preimages x – if there are any at all – $Df(x)$ has full rank $n - 1$. By the implicit function theorem, for each such x there is a neighbourhood $B(x, r)$ such that $B(x, r) \cap f^{-1}(A)$ is diffeomorphic to the interval $(-1, 1)$. The whole of the compact set $f^{-1}(A)$ is covered by such balls, from which we can extract a finite subcover: so indeed $f^{-1}(A)$ is covered by finitely many diffeomorphic copies of $(-1, 1)$.

Proposition (Shchepin)

Suppose that $f : \mathbb{S}^3 \to \mathbb{S}^2$ is a smooth equivariant map. Then there exists a smooth $f^\dagger : \mathbb{D}^3 \to \mathbb{S}^2$ which is equivariant on $\partial\mathbb{D}^3 = \mathbb{S}^2$, and moreover maps $\mathbb{S}^2_\pm$ to itself.

Discussion: By identifying the closed upper hemisphere of $\mathbb{S}^3$ with the closed disc $\mathbb{D}^3$ we obtain a smooth map

$$\widehat{f} : \mathbb{D}^3 \to \mathbb{S}^2$$

which is equivariant on $\partial\mathbb{D}^3 = \mathbb{S}^2$.

Then the restriction of $\widehat{f}$ to $\partial\mathbb{D}^3 = \mathbb{S}^2$ gives a smooth equivariant map $g : \mathbb{S}^2 \to \mathbb{S}^2$. If we could take g to be the identity, we would be finished – the argument given for the Brouwer fixed point theorem showed that no such $\widehat{f}$ exists.

We cannot hope for this, but we can hope to “improve” the properties of g so that a similar argument will work. What we need more precisely is that g maps $\mathbb{S}^2_\pm$ to itself.

Proposition implies BU

Suppose there existed a smooth map $f^\dagger : \mathbb{D}^3 \to \mathbb{S}^2 \subseteq \mathbb{R}^3$ which was equivariant on $\partial\mathbb{D}^3 = \mathbb{S}^2$, and mapped $\mathbb{S}^2_\pm$ to itself.

Let C be the cylinder $\mathbb{D}^2 \times [-1, 1]$ in $\mathbb{R}^3$ with top and bottom faces $D_\pm$ and curved vertical boundary $V = \mathbb{S}^1 \times [-1, 1]$. Let $S_\pm$ be the upper and lower halves of $S = \partial C$. Let E be the equator of S.

Now C, with the all points on each vertical line of V identified, is diffeomorphic to $\mathbb{D}^3$, and S is also diffeomorphic to $\mathbb{S}^2$.

Lemma

There is no smooth map $f : C \to S$ which is equivariant on ∂C, which is constant on vertical lines in V and which maps $D_\pm$ into $S_\pm$.

Proposition

There is no smooth map $f : C \to S$ which is equivariant on ∂C, which is constant on vertical lines in V and which maps $D_\pm$ into $S_\pm$.

If such an f existed, then

$$\int_C \det Df = \int_C df_1 \wedge df_2 \wedge df_3$$

where Df is the derivative matrix of f. On the one hand this is zero as Df has less than full rank at almost every $x \in C$, and on the other hand it equals, by Stokes' theorem,

$$\int_{\partial C} f_3 \, df_1 \wedge df_2 = \int_V f_3 \, df_1 \wedge df_2 + 2 \int_{D_+} f_3 \, df_1 \wedge df_2$$

by equivariance.

$$0 = \int_V f_3 \, df_1 \wedge df_2 + 2 \int_{D_+} f_3 \, df_1 \wedge df_2$$

Now f maps V into E, so that $f_3 = 0$ on V, and the first term on the right vanishes.

As for the second term,

$$\int_{D_+} f_3 \, df_1 \wedge df_2 = \int_{D_+ \cap \{x \,:\, f_3(x)=1\}} f_3 \, df_1 \wedge df_2 + \int_{D_+ \cap \{x \,:\, f_3(x)<1\}} f_3 \, df_1 \wedge df_2.$$

The region of D_+ on which $f_3(x) < 1$ consists of patches on which $f_1^2(x) + f_2^2(x) = 1$, and so $2f_1 \, df_1 + 2f_2 \, df_2 = 0$. Taking exterior products with df_1 and df_2 tells us that on such patches we have $f_1 \, df_1 \wedge df_2 = f_2 \, df_1 \wedge df_2 = 0$. Multiplying by f_1 and f_2 and adding we get that $h \, df_1 \wedge df_2 = 0$ for all h supported on a patch on which $f_3(x) < 1$. So for any h we have

$$\int_{D_+ \cap \{x \,:\, f_3(x)<1\}} h \, df_1 \wedge df_2 = 0.$$

Hence

$$\int_{D_+} f_3\, df_1 \wedge df_2 = \int_{D_+ \cap \{x \,:\, f_3(x)=1\}} df_1 \wedge df_2$$

$$= \int_{D_+ \cap \{x \,:\, f_3(x)=1\}} df_1 \wedge df_2 + \int_{D_+ \cap \{x \,:\, f_3(x)<1\}} df_1 \wedge df_2$$

$$= \int_{D_+} df_1 \wedge df_2.$$

By Stokes' theorem once again we have

$$\int_{D_+} df_1 \wedge df_2 = \int_{\partial D_+} f_1 df_2 = -\int_{\partial D_+} f_2 df_1,$$

and, since f restricted to ∂D_+ is equivariant, this quantity is nonzero (and indeed is an odd multiple of π), by the remarks in the proof of the case $n = 2$ above.

So no such f exists and we are done.

Proof of Proposition

Proposition (Shchepin)

Suppose that $f : \mathbb{S}^3 \to \mathbb{S}^2$ is a smooth equivariant map. Then there exists a smooth $f^\dagger : \mathbb{D}^3 \to \mathbb{S}^2$ which is equivariant on $\partial\mathbb{D}^3 = \mathbb{S}^2$, and moreover maps $\mathbb{S}^2_\pm$ to itself.

Recall that f induces first $\widehat{f} : \mathbb{D}^3 \to \mathbb{S}^2$ (identifying the upper closed hemisphere of $\mathbb{S}^3$ with $\mathbb{D}^3$), and then a smooth equivariant $g : \mathbb{S}^2 \to \mathbb{S}^2$ by restricting $\widehat{f}$ to the boundary of $\mathbb{D}^3$.

Lemma (Lemma 8)

If $g : \mathbb{S}^2 \to \mathbb{S}^2$ is a smooth equivariant map, then there exists a smooth equivariant $g^\dagger : \mathbb{S}^2 \to \mathbb{S}^2$ which preserves the upper and lower hemispheres of $\mathbb{S}^2$.

We shall then extend $g^\dagger$ to all of $\mathbb{D}^3$, "interpolating" between $g^\dagger$ on $\mathbb{S}^2$ and $\widehat{f}$ on a shrunken $\mathbb{D}^3$.

Proof of Lemma 8

(Equivariant $g : \mathbb{S}^2 \to \mathbb{S}^2 \implies$ hemisphere preserving $g^\dagger$.)

- Choose $\pm A$ in target $\mathbb{S}^2$ such that $g^{-1}(\pm A)$ are finite.
- Choose $\pm B$ (wlog N and S poles) in domain $\mathbb{S}^2$ s.t. merdinial projections of $g^{-1}(\pm A)$ on standard equator E are distinct.
- Apply Lemma 3 with $X = g^{-1}(A)$: $\exists$ equivariant diffeo $\psi : \mathbb{S}^2 \to \mathbb{S}^2$ s.t. $g^{-1}(A) \subseteq \psi(\mathbb{S}^2_+)$.
- So $g_* := g \circ \psi$ is a smooth equivariant selfmap of $\mathbb{S}^2$; $g_*^{-1}(A) \subseteq \mathbb{S}^2_+$.
- Let $\tilde{g} : E \to E$ be the meridinial projection of $g_*(x)$ on E. (Well-defined since $g_*^{-1}(\pm A) \cap E = \emptyset$.) Then $\tilde{g}$ smooth and equivariant.

We next extend $\tilde{g}$ to a small strip around E:

Extend $\tilde{g}$ to a small strip around E; and then to all of $\mathbb{S}^2$.

- For $x \in \mathbb{S}^2$ let $l(x) \in [-\pi/2, \pi/2]$ denote its latitude with respect to E.
- For $x \neq \pm B$, let $\overline{x}$ denote its meridinial projection on E.
- For $0 < r < \pi/2$ let $E_r = \{x \in \mathbb{S}^2 : |l(x)| \leq r\}$.
- Consider only r so small that $E_r \cap g_*^{-1}(\pm A) = \emptyset$. Let $d = \text{dist}\,(\pm A, g_*(E))$.
- Since g_* is uniformly continuous, there is an $r > 0$ such that for $x \in E_r$ we have $d(g_*(x), g_*(\overline{x})) < d/10$.
- Now extend $\tilde{g}$ to E_r by defining $\tilde{g}(x)$ to be the point with the same longitude (merdinial projection) as $\tilde{g}(\overline{x})$ and with latitude $\pi l(x)/2r$. This extension is still equivariant and smooth.
- Define $\tilde{g}(x) = A$ for $l(x) > r$ and $\tilde{g}(x) = -A$ for $l(x) < -r$. Then $\tilde{g}$ is continuous, equivariant, preserves the upper and lower hemispheres and – except possibly on the sets $\{x : l(x) = \pm r\}$ – is smooth.

Thus $\tilde{g}$ satisfies all the properties we needed for $g^\dagger$ except for smoothness.

We shall need to sort this out and to also simultaneously establish an auxiliary property of $\tilde{g}$ which we'll need for the interpolation step to work.

Claim: For all $x \in \mathbb{S}^2$, $\tilde{g}(x) \neq -g_*(x)$.

Claim: For all $x \in \mathbb{S}^2$, $\tilde{g}(x) \neq -g_*(x)$.

- Consider, for $x \in E_r$, the three points $g_*(x)$, $g_*(\overline{x})$ and $\tilde{g}(x)$.
- Now $g_*(\overline{x})$ and $\tilde{g}(x)$ are on the same meridian, and $g_*(\overline{x})$ is distant at least d from $\pm A$.
- On the other hand, $g_*(x)$ is at most $d/10$ from $g_*(\overline{x})$.
- Thus $g_*(x)$ is at least $9d/10$ from $\pm A$ and lives in a $d/10$-neighbourhood of the common meridian containing $g_*(\overline{x})$ and $\tilde{g}(x)$.
- So for $x \in E_r$, $g_*(x)$ cannot equal $-\tilde{g}(x)$.
- For $l(x) > r$ we have $\tilde{g}(x) = A$ and $g_*(x) \neq -A$ because $g_*^{-1}(-A)$ is contained in the lower hemisphere.
- Similarly for $l(x) < -r$, $\tilde{g}(x) \neq -g_*(x)$. Thus for all $x \in \mathbb{S}^2$ we have $\tilde{g}(x) \neq -g_*(x)$.

Mollify $\tilde{g}$ in small neighbourhoods of $\{x : l(x) = \pm r\}$ (and then renormalise to ensure that the target space remains $\mathbb{S}^2$!) to obtain $g^\dagger$ which is smooth, equivariant, preserves the upper and lower hemispheres and, being uniformly very close to $\tilde{g}$, is such that $g^\dagger(x) \neq -g_*(x)$ for all x.

Hence we also have, with ψ as above,

$$g^\dagger(x) \neq -(g \circ \psi)(x) \text{ for all } x.$$

Let ψ_t be a smooth family of diffeomorphisms interpolating between $\psi_0 = I$ and $\psi_1 = \psi$.

With this in hand, the proposition follows by defining, for $x \in \mathbb{S}^2$ and $0 \le t \le 1$, $\tilde{f} : \mathbb{D}^3 \to \mathbb{S}^2$ by

$$\tilde{f}(tx) = \frac{(3t-2)g^\dagger(x) + (3-3t)(g \circ \psi)(x)}{|(3t-2)g^\dagger(x) + (3-3t)(g \circ \psi)(x)|} \text{ when } 2/3 \le t \le 1,$$

$$\tilde{f}(tx) = g \circ \psi_{3t-1}(x) \text{ when } 1/3 \le t \le 2/3$$

and

$$\tilde{f}(tx) = \widehat{f}(3tx) \text{ when } 0 \le t \le 1/3.$$

This makes sense because for $2/3 \le t \le 1$ we have $(3t-2)g^\dagger(x) + (3-3t)(g \circ \psi)(x) \ne 0$; then $\tilde{f}$ has all the desired properties of $f^\dagger$ (including continuity) except possibly for smoothness at $t = 1/3$ and $2/3$. To rectify this we mollify $\tilde{f}$ in a small neighbourhood of $\{t = 1/3\}$ and $\{t = 2/3\}$ and renormalise once more to ensure that the target space is indeed still $\mathbb{S}^2$. The resulting $f^\dagger$ now has all the properties we need.

Borsuk–Ulam again

Theorem (Borsuk–Ulam)

If $F : \mathbb{S}^M \to \mathbb{R}^M$ is continuous, then there is some x with $F(x) = F(-x)$.

So if F is also odd, i.e. $F(-x) = -F(x)$ for all x, then there is some x with $F(x) = 0$.

Trivially the same applies to functions $F : \mathbb{S}^M \to \mathbb{R}^N$ with $M \geq N$ – just add extra zero components of F until there are M of them.

A typical application

Theorem (Ham Sandwich Theorem)

Suppose we have open sets $U_1, \ldots, U_n$ in $\mathbb{R}^n$. Then there exists a hyperplane bisecting each U_j.

If P is a hyperplane $\{x \ : \ a_0 + a_1 x_1 + \cdots + a_n x_n = 0\}$, let $P^+ = \{x \ : \ a_0 + a_1 x_1 + \cdots + a_n x_n > 0\}$ and similarly for P^-. Note that changing the signs of all the coefficients swaps $P^\pm$. We say P **bisects** U if $|U \cap P^+| = |U \cap P^-|$.

Every point $\mathbf{a} = (a_0, \ldots, a_n) \in \mathbb{S}^n \subseteq \mathbb{R}^{n+1}$ corresponds to some hyperplane $P_\mathbf{a}$.

Then the map

$$\mathbf{a} \mapsto \left\{ \int_{U_j \cap P_\mathbf{a}^+} 1 - \int_{U_j \cap P_\mathbf{a}^-} 1 \right\}_{j=1}^{n}$$

is a continuous odd map from $\mathbb{S}^n$ to $\mathbb{R}^n$ and so there is an **a** which maps to zero.

So, given n 1-separated unit balls $B_1, \ldots, B_n$ in $\mathbb{R}^n$, there is a degree 1 algebraic hypersurface (i.e. a hyperplane!) Z such that

$$\mathcal{H}_{n-1}(Z \cap B_j) \geq C_n.$$

Here, $\mathcal{H}_{n-1}$ denotes $n-1$-dimensional surface area.

More generally, if we have *many more* balls than the dimension n, can we find an algebraic hypersurface Z – not of degree 1 but of *controlled degree* – such that

$$\mathcal{H}_{n-1}(Z \cap B) \geq C_n$$

for all B?

Yes!

In fact, we have:

Proposition (Stone-Tukey, Gromov)

Suppose we have $N \gg n$ 1-separated unit balls B in $\mathbb{R}^n$. Then there exists an algebraic hypersurface Z such that

$$(i) \quad deg\, Z \leq C_n N^{1/n}$$

and

$$(ii) \quad \mathcal{H}_{n-1}(Z \cap B) \geq C_n$$

for all B.

Proof of Proposition

Given 1-separated $\{x_1, \ldots, x_N\} \subseteq \mathbb{R}^n$. Then there is a p with deg $p \leq C_n N^{1/n}$ and zero set Z such that $\mathcal{H}^{n-1}(Z \cap B(x_j, 1)) \geq C_n$ for all j.

Consider the map

$$F : p \mapsto \left\{ \int_{\{p>0\}\cap B(x_j,1)} 1 - \int_{\{p<0\}\cap B(x_j,1)} 1 \right\}_j$$

defined on $X_{d,n} = \{$ polys of degree $\leq d$ in n real variables$\}$.

Clearly F is continuous, homogeneous of degree 0 and odd.

So we can think of F as

$$F : \mathbb{S}^M \to \mathbb{R}^N$$

where $\mathbb{S}^M$ – with $M + 1 = \binom{n+d}{n} \sim d^n$ – is the unit sphere of $X_{d,n}$.

So by Borsuk–Ulam, if $M \geq N$, then F vanishes at some p.

For such p (which we can choose with deg $p \leq C_n N^{1/n}$) we have $\mathcal{H}^{n-1}(Z \cap B(x_j, 1)) \geq C_n$ for all j.

We've seen that given $N \gg n$ 1-separated unit balls B in $\mathbb{R}^n$, then there exists an algebraic hypersurface Z such that

$$(i) \quad \deg Z \leq C_n N^{1/n}$$

and

$$(ii) \quad \mathcal{H}_{n-1}(Z \cap B) \geq C_n$$

for all B.

What about a version of this "with multiplicities"?

That is, given 1-separated unit balls B_j in $\mathbb{R}^n$ and given $M_j \geq 1$, can we find an algebraic hypersurface Z such that

$$(i) \quad \deg Z \leq C_n \left(\sum_j M_j^n \right)^{1/n}$$

and

$$(ii) \quad \mathcal{H}_{n-1}(Z \cap B_j) \geq C_n M_j$$

for all j?

Proposition

Given 1*-separated unit balls* B_j *in* $\mathbb{R}^n$ *and given* $M_j \geq 1$, *we can find an algebraic hypersurface* Z *such that*

$$(i) \quad deg\, Z \leq C_n \left(\sum_j M_j^n \right)^{1/n}$$

and

$$(ii) \quad \mathcal{H}_{n-1}(Z \cap B_j) \geq C_n M_j$$

for all j

Proof.

Chop each B_j into M_j^n equal sub-balls and apply the same strategy: we obtain M_j^n contributions of $M_j^{-(n-1)}$ to $Z \cap B_j$ and the total number of constraints is $\sum_j M_j^n$. □

A Qusetion

Let $S_e(Z)$ be the component of surface area of Z in the direction perpendicular to the unit vector e. Let $\{e_j(Q)\}$ be any approximate orthonormal basis and let $S_j(Q) = S_{e_j}(Z \cap Q)$.

By the G.M./A.M. inequality we have

$$\prod_{j=1}^{n} S_j(Q)^{1/n} \leq C_n \sum_{j=1}^{n} S_j(Q) \sim \mathcal{H}_{n-1}(Z \cap Q)$$

because the directions involved in the S_j are approximately orthonormal.

So a *harder* task is, given M, to find a polynomial p of degree at most $C_n \left(\sum_Q M(Q)^n\right)^{1/n}$, with zero set Z, such that

$$M(Q) \leq C_n \prod_{j=1}^{n} S_j(Q)^{1/n} \text{ for all } Q \in \operatorname{supp} M.$$

Algebraic geometry and algebraic topology

Something close to this is indeed true and is a consequence of work of Guth on the multilinear Kakeya problem.

However, the argument currently relies upon the whole machinery of algebraic topology. Some of the tools needed:

- $\mathbb{Z}_2$ -cohomology
- Covering spaces
- Cup products
- Lusternik–Schnirelmann theory
- Commutative diagrams and long exact sequences

Is there an "elementary" proof of this result appealing directly to Borsuk–Ulam?