Secondary Compositions and the Adams Spectral Sequence

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Let $H: \mathcal{S}_{h*} \to \mathcal{A}$ be a cohomology functor from Boardman's stable homotopy category [1] to a graded abelian category; so H is contravariant and takes exact triangles to exact triangles. A spectrum X has an Adams system for H if there exist maps

$$X = X_o \leftarrow X_1 \longleftarrow \cdots \longleftarrow X_{n-1} \overset{x}{\longleftarrow} X_n \longleftarrow$$

such that $H(x) = 0$ for all $n \geq 1$ and if there are exact triangles

$$\begin{array}{ccc} X_{n-1} & \overset{x}{\longleftarrow} & X_n \\ & \searrow \quad \nearrow & \\ & K & \end{array}$$

such that (i) H(K) is projective

(ii) the natural transformation

$$\{X,K\} \longrightarrow \mathrm{Hom}_{\mathcal{A}}(H(K),H(X))$$

is an isomorphism.

We shall assume that $X_n \subset X_{n-1}$, for all $n \geq 1$, and that x is the inclusion.

If X has an Adams system for H and X' is any spectrum then there is an Adams spectral sequence

$$\mathrm{Ext}_{\mathcal{A}}^{*,*}(H(X),H(X')) \Longrightarrow \{X',X\}_* \ .$$

Moreover there are pairings of Adams sequences [3],

$$E(X',X) \otimes E(X'',X') \to E(X'',X)$$

$$E(X'',X') \otimes E(X''',X'') \to E(X''',X')$$

and so Massey products can be introduced. If $a \in E_r^{s,t}(X',X)$, $a' \in E_r^{s',t'}(X'',X')$ and $a'' \in E_r^{s'',t''}(X''',X'')$ are such that $aa' = 0$ and $a'a'' = 0$, then the Massey product $\langle a,a',a''\rangle$ can be defined as in [2] and belongs to a certain quotient group of $E^{p,q+1}(X''',X)$, where $p = s + s' + s'' - r+1$ and $q = t + t' + t'' - r+1$. Similarly matric Massey products can be formed. (In E_2 Massey products are defined by using the Yoneda product in $\mathrm{Ext}_{\mathcal{A}}$ and the isomorphism $E_2 \cong \mathrm{Ext}_{\mathcal{A}}$; as H is contravariant a sign is involved in this definition.)

<u>Theorem 1</u> (i) Let $a,a',a'' \in E_r$ be such that $aa' = 0$ and $a'a'' = 0$. Then

$$d_r\langle a,a',a''\rangle \subset - \left\langle \begin{matrix} b & a \end{matrix}, \begin{matrix} a' & 0 & a'' \\ b' & a' & b'' \end{matrix} \right\rangle$$

where $b = d_r a$, $b' = (-)^i d_r a'$, $b'' = (-)^{i+i'} d_r a''$ and $i = t-s$, $i' = t' - s'$.

(ii) If a,a',a'' also satisfy $a d_r a' = 0$ and $a' d_r a'' = 0$ then

$$d_r\langle a,a',a''\rangle \subset - \langle d_r a,a',a''\rangle - (-)^i\langle a,d_r a',a''\rangle - (-)^{i+i'}\langle a,a',d_r a''\rangle.$$

In the next theorem we must suppose that all Adams sequences are weakly convergent i.e. $E_\infty^{s,t} = \bigcap_{r>s} E_r^{s,t}$ for all s,t.

Theorem 2 Let a,a',a" be permanent cycles in E_r such that aa' = 0 and a'a" = 0. Let $\omega, \omega', \omega''$ be homotopy classes realizing a,a',a" in E_∞ and suppose that $\omega\omega' = 0$ and $\omega'\omega'' = 0$. Then $\langle a,a',a''\rangle$ contains a permanent cycle that is realized by an element of the Toda bracket $\langle\omega,\omega',\omega''\rangle$ provided all elements of the following groups are permanent cycles.

$$E^{n,i+i'-n+1}_{s+s'-n+1}(X'',X) \qquad \text{where } 0 \leq n \leq s + s' - r$$

$$E^{n,i'+i''-n+1}_{s'+s''-n+1}(X''',X') \qquad \text{where } 0 \leq n \leq s' + s'' - r$$

Theorem 2 is false without the additional conditions that have been imposed; in the mod-2 Adams spectral sequence of a sphere, the Massey product $\langle h_4^2,h_0^4,h_1\rangle$ is a permanent cycle but the corresponding Toda bracket $\langle\theta_4,16\iota,\eta\rangle$ is trivial. In this case $E_3^{4,35}$ contains the element $h_0^3h_5$, which is not a permanent cycle.

Recently R. Lawrence of Chicago University has developed methods which allow theorems 1 and 2 to be extended to higher Massey products and higher differentials.

The additional conditions of theorem 2 are needed to ensure that the elements a,a',a" can be represented by maps whose "composition" is a boundary of a map of the correct filtration. Thus let a,a',a" be represented by maps $h: X' \to X_s$, $h': X'' \to X'_{s'}$, $h'': X''' \to X''_{s''}$. We can assume that $h(X'_n) \subset X_{n+s}$ and that $h'(X''_n) \subset X'_{n+s'}$. Then theorem 2 is implied by theorem 3, whose proof generalizes to give theorem 1(i); theorem 1(ii) is a trivial consequence of theorem 1(i). In theorem 3 we shall write $u = u' + s'' = s + u'' = s + s' + s''$ and $p-u = p' - u' = p'' - u'' = r-1$.

<u>Theorem 3</u> Let the following compositions be nulhomotopic

$$X'' \xrightarrow{h'} X'_{s'} \xrightarrow{h} X_{u'} \xrightarrow{\subset} X_{p'}$$
$$X''' \xrightarrow{h''} X''_{s''} \xrightarrow{h'} X'_{u''} \xrightarrow{\subset} X'_{p''} \; .$$

Then $\langle a, a', a''\rangle$ contains a permanent cycle that is realized by an element of $\langle \omega, \omega', \omega''\rangle$.

<u>Proof</u> There exist maps $H\colon TX'' \to X_{p'}$ and $H'\colon TX''' \to X'_{p''}$ extending the above compositions and such that $H(TX''_n) \subset X_{n+u'}$. So the following commutative diagram can be formed.

$$\begin{array}{ccccccc}
TX''' & & \xrightarrow{\quad H' \quad} & & X'_{p''} & \xrightarrow{h} & X_p \\
\uparrow & & & & \uparrow & & \uparrow \\
X''' & \xrightarrow{h''} & X''_{s''} & \xrightarrow{h'} & X'_{u''} & \xrightarrow{h} & X_u \\
\downarrow & & \downarrow & & & & \downarrow \\
TX''' & \xrightarrow{Th''} & TX''_{s''} & & \xrightarrow{\quad H \quad} & & X_p
\end{array}$$

Let $\nu \in \{X''', X_p\}_*$ be the class defined by the difference element

$$d(hH', H(Th''))\colon SX''' \to X_p .$$

It is clear that, in $\{X''', X\}_*$, ν defines an element of $\langle \omega, \omega', \omega''\rangle$. On the other hand, in $\{X''', X_p \cup Tx_u\}_*$, $i_*\nu = \nu^+ - \nu^-$.

An examination of the elements that ν^+ and ν^- define in $\{X''', X_p \cup TX_{p+1}\}_*$ shows that ν does indeed define an element of the Massey product $\langle a,a',a''\rangle$.

References

1. J. M. Boardman: Stable homotopy theory, (mimeographed notes), Warwick University, 1965.

2. W. S. Massey: Some higher order cohomology operations, Symposium Internacional de Topologia Algebraica, Mexico City, 1958, 145-154.

3. R. M. F. Moss: On the composition pairing of Adams spectral sequences, Proc. London Math. Soc. 18 (1968), 179-192.

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