# The obstruction theory based on a particular quotient group of $\pi_*(V_n)$[1]

by

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We shall be interested in the obstruction theory problems associated with $V_n \to BSO_n \to BSO$. The Steenrod obstruction theory shows that the lifting question for a map $f:X \to BSO$ such that $f/X^k, X^k$ is the k-skeleton, reduces to calculating a cohomology class $\mathcal{O}^{k+1}(X) \in H^{k+1}(X;\pi_k(V_n))$. If $A \subset \pi_k(V_n)$ and $B = \pi_k(V_n)/A$, there are coefficient homomorphisms

$$\begin{cases} \lambda_A : H^{k+1}(X;A) \to H^{k+1}(X;\pi_k(V_n)) \\ \lambda_B : H^{k+1}(X;\pi_k(V_n)) \to H^{k+1}(X;B). \end{cases}$$

Our purpose here is to present some techniques which seem to be useful for calculating a class $\mathcal{O}'$ such that

$$\lambda_A \mathcal{O}' = \mathcal{O}^{k+1}(X) \text{ if } \lambda_B(\mathcal{O}^{k+1}(X)) = 0$$

in terms of other invariants for a particular subgroup.

The subgroup in question in each dimension is the one

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of highest filtration in the sense of Adams [2]. These are the elements near the edge in the Adams Spectral Sequence for $V_n$. The relevant homotopy theory is done in [5] and here we show how these calculations can be used to shed light on the obstruction theory.

The following device is possibly a useful way to illuminate this program. Let $Y_n = \Sigma^{-8k}(V_n \wedge BSO[8k])$ in the stable category where $BSO[8k]$ is the 8k-1 connected fiber space over BSO, $8k \geq n$. Clearly $H^*(Y_n) \simeq H^*(V_n) \otimes_{Z_2} Ⓐ/Ⓐ_i$ (3) through dimension 16k-2 and thus through the stable range of $V_n$. Also there is a map $\lambda : V_n \to Y_n$. Let $A_n$ be the fiber of $\lambda$. Then there is a factorization

$$\begin{array}{ccc} A_n & \xrightarrow[i_2]{} & B\,\mathrm{spin}_n \\ & & \downarrow p_2 \\ Y_n & \xrightarrow[i_1]{} & Q_n \\ & & \downarrow p_1 \\ & & B\,\mathrm{spin} \end{array}$$

through the stable range. The main results of this paper deals with $p_1$. The following seems likely:

<u>Conjecture</u>. Suppose X is a 2n- 2 dimension complex. If $f: X \to B$ spin lifts to $f': X \to Q_n$ then there is a lifting $f'': X \to B$ spin.

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(3) Ⓐ is the mod 2 Steenrod algebra and $Ⓐ_i$ is the subalgebra generated by $Sq^j$ $j \leq 2^i$.

This conjecture is false for BSF bundles but seems likely for BSO bundles.

Finally the importance of getting cohomology operations such as found in this paper can be seen when the bundle in question is the normal bundle to a manifold. Then the Thom complex is the Spanier-Whitehead deal of the Manifold with an external basepoint. Thus any cohomology operation on the Thom class corresponds to an operation in the manifold which reaches the top dimensional class. In other words, the obstruction question is given in terms of the homotopy structure of the original manifold. In [4] this notion is exploited.

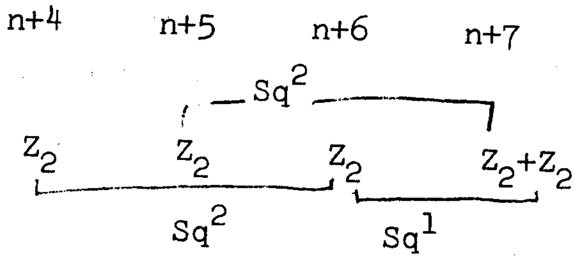
2. In this section we will give the necessary constructions and definitions which will enable us to state the form of the main theorem (2.7) about cohomology operation to which we alluded in the introduction. The proof of the main theorem and of several preliminary lemmas will be left to latter sections.

In effect, we will work in the stable category. $K(Z,0)$ is the stable object whose cohomology mod 2, for example, is $\mathcal{A}//\mathcal{A}_0$ where $\mathcal{A}$ is the Steenrod Algebra and $\mathcal{A}_0$ is the sub-algebra generated by $Sq^0$ and $Sq^1$. $\Sigma_S K(Z,n)$ is the stable object such that

$$H^i(\Sigma_S K(Z,n)) = H^{i+n}(K(Z,n)).$$

Clearly there is a map $p:\Sigma_S K(Z,n) \to K(Z,0)$. Let $\tilde{K}(Z,0)$ be the universal stable object all of whose cohomology between 1 and 3 inclusive vanishes. Let $\Sigma_S\tilde{K}(Z,n)$ be the analgously defined complex. Let $F_n \xrightarrow[i]{} \Sigma_S \bar{K}(Z,n) \xrightarrow[p]{} K(Z,0)$ be a fibration in the stable category with $F_n$ as the fiber. This defines $F_n$.

Proposition 2.1 a) $F_n$ is $(n-1)$ connected.
b) There is a map $:RP_n \to F_n$ which induces an epimorphism in cohomology. c) $(\ker \quad *)^{n+5} = Z_8$ and $(\ker \quad *)^{n+i} = 0$ for all other $i \leq 7$ with integers for coefficients. d) With $Z_2$ for coefficients we have the following table for the kernel




The proof is a straight forward check of the relative cohomology of the map p.

The next thing we need to do is to define a resolution of $F_n$ which will be useful for us. Let $G_o$ be a graded group such that

$$G_o^t = \mathrm{Ext}^{o,t}(\tilde{H}^*(P_n);Z_2), \quad t \leq n+7, \ t \neq n+4_*;$$

$$= \mathrm{Ext}^{o,t}(\tilde{H}^*(P_n);Z_2) \oplus Z_8, \quad t = n+4;$$

$$= \mathrm{Ext}^{o,t}(\tilde{H}^*(P_n),Z_2) \oplus H_t(F_n,P_n;Z) \quad n+8 \leq t \leq 2n-2$$

$$= 0 \qquad t \geq 2n-1$$

By $K(G)$ where $G$ is any graded group we mean $\Pi_t K(G^t,t)$. There is a map $f_o:F_n \to K(G_o)$ such that $f_o{}^*$ is onto with integers for coefficients through dimension $2n-1$. Let $p^1:F_n{}^1 \to F_n$ be the fiber space induced by $f_o$ and let

$$\begin{array}{ccccccc}
\Omega K(G_p) & & \Omega K(G_2) & & & & \Omega^2 K(G_s) \\
\uparrow & & \uparrow & & & & \uparrow \\
F_n^{\;1} & \xleftarrow{p^2} & F_n^{\;2} & \xleftarrow{p^3} & \cdots & \leftarrow & F_n^{\;s} \leftarrow \cdots
\end{array}$$

be a minimal $Z_2$ decomposition of the space $F_n^{\;1}$. By this we mean a tower of fiber spaces such that

1) the Fiber in a product of Eidenberg MacLane spaces, $K(Z_2;k)$;

2) the maps $p^{s*}$ are zero; and

3) the composit maps $\Omega^{s+1}K(G_s^p) \to F_n^{2+1} \to \Omega^{s+1}K(G_{s+1})$ are zero in homotopy.

Thus we have defined a resolution of $F_n$.

2.2
$$\begin{array}{ccccccccc}
K(G_o) & & \Omega K(G_1) & & & & \Omega^s K(G_s) & & \\
\uparrow f_o & & \uparrow & & & & \uparrow & & \\
F_n & \longleftarrow & F_n^{\;1} & \longleftarrow & \cdots & \xleftarrow[p^s]{} & F_n^{\;s} & \xleftarrow[p^{s+1}]{} & \cdots
\end{array}$$

Let $E^{s,t} = \Pi_t(K(G_s))$. Clearly the map $\lambda : P_n \to F_n$ induces a map $\lambda_{\#} : \mathrm{Ext}_a^{s,t}(\tilde{H}^*(P_n), Z_2) \to E^{s,t}$ (which may fail to be uniquely defined is $s = 0$ or $1$). A key result leading to the main theorem of this section is the following.

<u>Proposition</u> 2.3 The map $\lambda_{\#}$ is an isomorphism for s and t satisfying:

| $s =$ | $4k, k \neq 0$ | $4k + 1$ | $4k + 2$ | $4k + 3$ | $0$ |
|---|---|---|---|---|---|
| $t - n <$ | $12k + 8$ | $12k + 10$ | $12k + 12$ | $12k + 17$ | $4$ |

The proof is given in section 3.

Theorem 4.4 of [3] implies that there is a resolution of $F_n \to \Sigma_S \tilde{K}(Z,n) \to \tilde{K}(Z,0)$ based on the resolution 2.2.

2.4 Let

$$\begin{array}{ccccccc} BK(G_0) & & K(G_1) & & & \Omega^{s-1}K(G_s) & \\ \uparrow g_0 & & \uparrow & & & \uparrow & \\ \tilde{K}(Z,0) & \leftarrow & A_n^1 & \leftarrow \ldots & \leftarrow & A_n^2 & \leftarrow \ldots \end{array}$$

represent this tower. Let $\alpha \in E^{s,t}$ and let $\iota_\alpha \in H^{t-s+1}(\Omega^{s-1}K(G_s))$ be a cohomology class such that $\alpha^*(\iota_\alpha) \neq 0$. (Recall $\alpha \in \Pi_t(K(G_s))$. The cohomology operation determined by $\alpha$, $\varphi(\alpha)$, has as its universal example in the sense of Adams [1] the triple $(A_n^s, , g_s^*(\varkappa_\alpha))$. Where $\varkappa$ is the fundamental class of $\tilde{K}(Z,0)$ projected to $A_n^s$.

In addition we have a MPT for $V_n \to BO_n[4] \to BO[4]$ as given by [3]. Let

2.5

$$\begin{array}{ccccccc} BK(L_0) & & K(L_1) & & & \Omega^{s-1}(K(L_s) & \\ \uparrow & & \uparrow & & & \uparrow & \\ BO[4] & \leftarrow & E_n^1 & \leftarrow \ldots & \leftarrow & E_n^2 & \leftarrow \ldots \end{array}$$

where

$$\begin{array}{ccccc} K(L_0) & & & \Omega^s K(L_s) & \\ \uparrow & & & \uparrow & \\ V_n & \leftarrow \ldots \leftarrow & & V_n^2 & \leftarrow \ldots \end{array}$$

is a minimal resolution of $V_n$.

Recall that $V_n$ is homotopically equivalent to $P_n$ through the $2n-2$ skeleton. Thus the map $\lambda$ induces a map $\mathrm{Ext}_{\mathcal{A}}^{s,t}(\tilde{H}^*(V_n), Z_2) \to E^{s,t}$. In [5] extensive calculations of the left side are made and we need only the following partial results.

<u>Proposition</u> 2.6 The following tables represent a portion of a subset of $Ext_{\mathcal{A}}^{s,t}(\tilde{H}^*(V_n),Z_2)$.

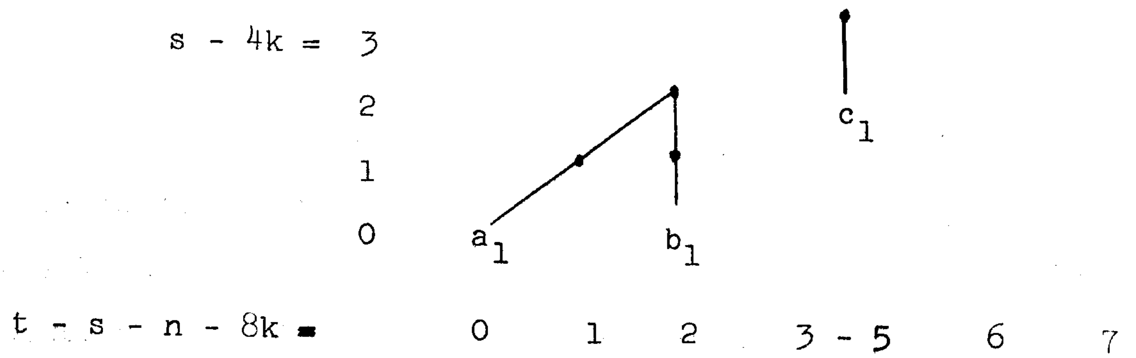



Table 2.6.1

A subset of a portion of $Ext_{\mathcal{A}}^{s,t}(\tilde{H}^*(V_n),Z_2)$ for $n \equiv 1 \bmod 4$.

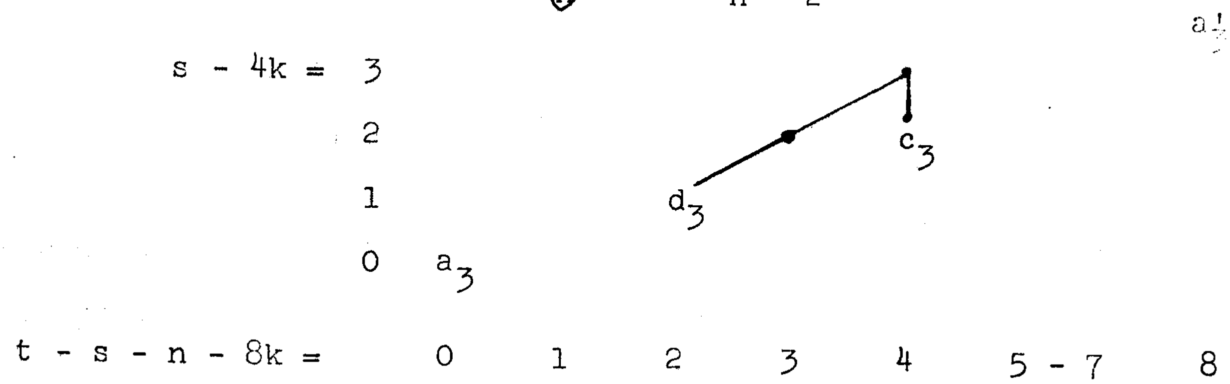



Table 2.6.2

A subset of a portion of $Ext_{\mathcal{A}}^{s,t}(\tilde{H}^*(V_n),Z_2)$ for $n = 3 \bmod 4$ with $h_0 a_3^!(k) = a_3(k+1)$.

Recall that associated with an element $\alpha \in Ext_{\mathcal{A}}^{s,t}(\tilde{H}^*(V_n),Z_2)$ there is a characteristic class $k_\alpha \in H^{t-s+1}(E_n^s;Z_2)$ which is called the k-invariant associated with $\alpha$.

The main result of this section is the following:

<u>Theorem</u> 2.7 Let $U_s$ be the Thom Class of the universal bundle over $E_n^s$. Then for each $\alpha$ given in the tables 2.6.1 and 2.6.2 $U_s \cup k_\alpha \in \wp_{\lambda_{\#}(\alpha)} U_s)$.

The theorem is proved in section 4. The special case of the theorem is due to Thom for $s = 0$ and for $s = 1$ is due Mahowald and Peterson [6]. A related theorem is also theorem A of [5].

## 3. Proof of Proposition 2.3

First observe that for $s = 0$ the proposition is obviously true. As a second step we will prove the following.

Proposition 3.1 The induced map $V_n^1 \to F_n^1$ produces an epimorphism in cohomology.

Proof. Consider the diagram

$$\begin{array}{ccccccc} \cdots \to H^{*-1}(F_n^1) & \xrightarrow{\delta} & H^*(K(G_o)) & \xrightarrow{f_o^*} & H^*(F_n) & \to \cdots \\ \downarrow \lambda_1^* & & \downarrow \lambda_o^* & & \downarrow \lambda^* & \\ \cdots \to H^{*-1}(V_n^1) & \xrightarrow{\delta} & H^*(K(L_o)) & \to & H^*(V_n) & \to \cdots \end{array}$$

Let $B \in H^{*-1}(V_n^1)$. Since $\lambda_o^*$ is onto there is a class $\overline{B}$ such that $\lambda_o^*\overline{B} = \delta B$. The group $G_o$ was constructed so that $f_o^*/\ker\lambda_o^*$ is onto $\ker \lambda^*$. Thus if $f_o^*\overline{B} \neq 0$ we can add to it a class $\gamma$ such that $f_o^*(\overline{B} + \gamma) = 0$ and $\lambda_o^*(\overline{B} + \gamma) = \delta B$. This proves that $\lambda_1^*$ is onto which is the proposition.

Now consider the cofibration $V_n{}^1 \to F_n{}^1 \to F_n{}^1/V_n{}^1$. Clearly 2.1 asserts that the cohomology of this sequence splits into short exact sequences.

$$0 \to H^j(F_n{}^1/V_n{}^1) \to H^j(F_n{}^1) \to H^j(V_n{}^1) \to 0.$$

Thus we have a long exact sequence in Ext

$$\cdots \to Ext_{Ⓐ}^{s,t}(\tilde{H}^*(V_n^1), Z_2) \to Ext_{Ⓐ}^{s,t}(\tilde{H}^*(F_n^1), Z_2) \to Ext_{Ⓐ}^{s,t}(\tilde{H}^*(F_n^1/V_n^1), Z_2) \xrightarrow{\delta} \cdots$$

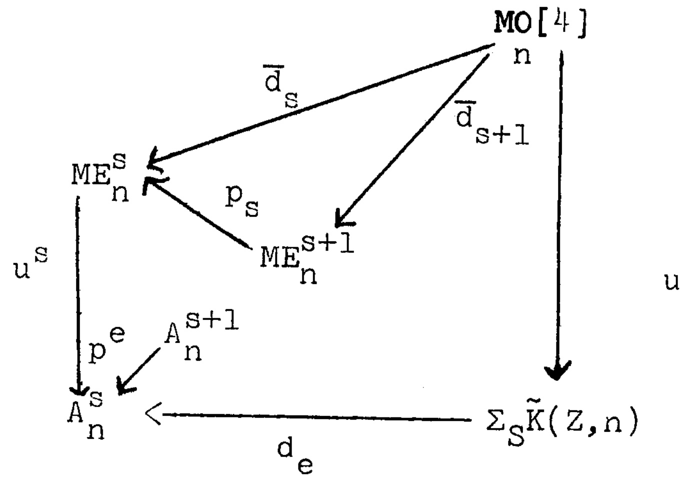
Where $\delta$ is a map of degree (1,0).

Next observe that $H^*(F_n^1)$ and $H^*(V_n^1)$ are both $A_o$ free. Clearly $H^*(F_n^1/V_n^1)$ is then $\mathcal{A}_o$ free too and thus the Adams edge theorem applies and gives the proposition immediately.

## 4. Proof of Theorem 2.7

As a first step we have:

Proposition 4.1 For each s we have a commutation diagram




Where $u^{s*}(\varkappa) = U_s$.

The proof is immediate.

The above proposition assures us that $\varphi_\alpha$ is defined on $U_s$ for each $\alpha \in E^{s,t}$.

Proposition 4.2 Suppose $n \equiv 1 \bmod 4$ and theorem 2.7 is true for $a_1(k)$, $b_1(k)$ and $c_1(k)$. Then the theorem holds for $h_1a_1(k)$, $h_1^2a_1(k)$, $h_ob_1(k)$, $h_oc_1(k)$ and $a_1(k+1)$.

Proof. Consider the diagram

$$\begin{array}{ccccc}
ME_n^{4k} & \leftarrow & ME_n^{4k+1} & \overset{\bar{i}}{\leftarrow} & \mathcal{F}^{4k} \\
\downarrow u^s & & \downarrow u^{s+1} & & \downarrow u'^{s+1} \\
A_n^{4k} & \leftarrow & A_n^{4k+1} & \leftarrow & \Omega^{4k}K(G_{4k})
\end{array}$$

Where $\textcircled{F}^{4k}$ is the fiber of the top row. Since the fiber of $E_n^{4k+1} \to E_n^{4k}$ is $\Omega^{4k}K(L_{4k})$ which is $(8k-1+n)$-connected as is easily seen from $\mathrm{Ext}_{\textcircled{A}}^{4k,*}(\tilde{H}^*(V_n),Z_2)$. Thus, through dimension $8k-3+n$, $\textcircled{F}^{4k} = K(Z_2,8k) \times K(Z_2,8k+2)$ and $u'^{s+1*}$ is an isomorphism. Now $u^{s*}\tau(\iota_{a_1}) = U \cup k_{a_1}$ for a suitable choice of $k_{a_1}$ by hypothesis. A similar statement holds for $\iota_{b_1}$. Now the universal example for $\varphi_{h_1a_1}$ and $\varphi_{h_ob_1}$, $v_1$ and $v_2$ respectively, satisfy:

$$i^*v_1 = Sq^2\iota_{a_1} \text{ and } i^*v_2 = Sq^2Sq^1\iota_{a_1} + Sq^1\iota_{b_1}.$$

Thus, $u^{s+1*}(v_j) \neq 0$ for $j = 1$ and $2$ since $\bar{i}^*u^{s+1*}v_j \neq 0$. Clearly, $d^*_{s+1}v_j = 0$ and so $\bar{d}^*_{s+1}u^{s+1*}v_j = 0$. Thus, the k-invariants $k_{h_1a_1}$ and $k_{h_ob_1}$ can be chosen to satisfy

$U \cup k_{h_1a_1} = u^{s+1*}v_1$ and $U \cup k_{h_ob_1} = u^{s+1*}v_2$.

In an entirely similar way we use the defining relation for $h_1{}^2a_1$, $Sq^2\iota_{h_1a_1} + Sq^1\iota_{h_ob_1}$ to prove the theorem for $h_1{}^2a_1$; the defining relation for $h_oc_1$, $Sq^{2,1,2}\iota_{h_1{}^2a} + Sq^1\iota_{h_oc_1}$ to prove the theorem for $h_oc_1$; and finally the relation for $a_1(k+1)$, $Sq^{2,1}\iota_{h_oc_1}(k)$ to complete the proof.

<u>Proposition</u> 4.3 Suppose $n \equiv 3(4)$ and theorem 2.7 holds for $d_3(k), c_3(k)$ and $a_3^!(k)$. Then the theorem holds for $h_1d_3(k)$, $h_1^2d_3(k)$, $a_3(k+1)$ and $d_3(k+1)$.

<u>Proof</u> The proof is entirely similar to the proof of proposition 4.2.

The defining relations are

$h_1 d_3(k)$ $\quad Sq^2 \iota_{d_3}$

$h_1^2 d_3(k)$ $\quad Sq^2 \iota_{h_1 d_3} + Sq^1 \iota_{c_3}$

$a_3(k+1)$ $\quad Sq^{2,1,2} \iota_{h_o c_3} + Sq^1 \iota_{a_3^1}$

$d_3(k+1)$ $\quad Sq^{2,1} \iota_{a_3}$.

We will use 2 and 3 to give an induction proof of theorem 2.7. First observe that the theorem is clearly true for $a_1(0)$, $b_1(0)$ and $a_3(0)$. The key to the induction is the following result.

Proposition 4.4 Under the usual projection $p:V_n \to V_{n+2}$ we have: a) if $n \equiv 1 \bmod 4$, $p_{\#} b_1 = a_3$ and $p_{\#} c_1 = c_3$; b) if $n \equiv 3 \bmod 4$, $p_{\#} c_3 = h_o^2 b_1$ and $p_{\#} a_3^1 = h_o c_1$. This result can be rather easily read off of the table given in [5].

Now we can prove theorem 2.7. The argument is similar in each case. Thus we will do just one case. Consider $a_3^1(k)$ for $n \equiv 3 \bmod e$. Recall that this is a class with filtration $(4k+3, 12k+11+n)$ We start with $n + 8k + 7$. Clearly $a_1(0)$ and $b_1(0)$ for this case satisfy the theorem; thus 4.2 implies $h_o^2 b_1(0)$ does too.

Now consider mapping the tower for $n + 8k + 7$ into that for $n + 8k + 9$. Proposition 4.4 implies that if the theorem holds for $h_o^2 b_1(0)$ it holds for $c_3(0)$. Also clearly the theorem holds for $a_3(0)$.

Now map the tower for $n + 8k + 5$ into that for $n + 8k + 7$.

Proposition 4.4 implies that if the theorem holds for $c_3(0)$ it holds for $c_1(0)$ and if it holds for $a_3(0)$ it holds for $b_1(0)$.

Now suppose by induction the theorem holds for $c_1(i)$ and $b_1(i)$, $i \leq j < k$ and $n \neq 8(k-j) + 5$. Clearly it holds for $a_1(0)$ and thus 4.2 implies it holds for $h_o^2 b_1(i)$ and $h_o c_1(i)$, $i \leq j$. This implies the theorem for $n + 8(k-j) + 3$ and $a_3^1(i)$ and $c_3(i)$, $i \leq j$. Again the theorem is clearly true for $a_3(0)$ and thus 4.3 implies the theorem for $h_o c_3(i)$ and $a_3(i+1)$ for $i \leq j$.

Now this implies the theorem for $b_1(i+1)$ and $c_1(i)$, $i \leq j$ and $n + 8(k-j) + 1$. The theorem is clearly true for $a_1(0)$ and so 4.2 implies the theorem for $h_o^2 b_1(i+1)$ and $h_o c_1(i)$ $i \leq j$.

Finally this implies the theorem for $n + 8(k-j) - 1$ and $a_3^1(i)$ and $c_3(i+1)$, $i \leq j$. The theorem is clear for $a_3(0)$ and thus 4.3 implies the theorem for $h_o c_3(i+1)$ and $a_3(i+1)$ which for $n + 8(k-j-1) + 5$ implies the theorem for $c_1(i)$ and $a_1(i)$, $i \leq j + 1$.

This completes the induction and the proof of the theorem.

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