ON SECONDARY COHOMOLOGY OPERATIONS II

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Introduction. The purpose of this paper is to give a detailed study of cochain operations giving rise to secondary cohomology operations. The main theorem is concerned with the problem of expanding $\Theta(xy)$, where Θ is such a cochain operation and x,y a pair of cochains. Such an expansion was studied in Kristensen [1965] in order to obtain a Cartan formula for secondary operations.

A primary term $\sum \beta'(x)\beta''(y)$, $\beta',\beta'' \in \mathcal{C}$, was left undetermined in the mentioned paper. The primary term in the expansion on cochain levels is determined in this paper (see Theorem 2.1 and Lemma 3.2).

As a consequence of this, the question of a Cartan formula for secondary operations is completely solved (see Section 3). The formulation of the Cartan formula in Kristensen [1965] was not very good. In fact, as it is formulated the theorem is not very useful. With a slight modification of the formulation, the theorem can be used. However, the Cartan formula for secondary operations will be considered in detail in the complete version of the present paper. The proof of the Cartan formula for secondary operations is also very much simplified.

As another application of Theorem 2.1 below, we have computed Massey products $\langle a,b,c \rangle \subset \mathcal{G}$, $a,b,c \in \mathcal{G}$, in the Steenrod algebra. This has as a consequence that the action of $\mathcal{G}$ in the cohomology of two-stage spaces with stable k-invariant now is known. The details about this is explained in a forthcoming paper. There are applications of more geometric nature, e.g. to the problem of sections in the tangent-bundle of a manifold.

In Kristensen and Madsen [to appear] we studied the algebra of operations in various cohomology theories. We gave a complete description of this algebra in case the spectrum for the cohomology theory is a two-stage space with k-invariant $Sq^3 + Sq^2Sq^1$. The multiplicative structure of this algebra is related to Massey products in $\mathcal{G}$. With the improved technique for computing Massey products, many other cases are within reach.

1. Cochain Operations. We shall use the theory of cochain operations as developed in Kristensen [1963], [1965] and in Kock, Kristensen and Madsen [1967]. In particular, we shall use the exact sequence.

$$(1.)\quad \mathcal{O}(U_1,\ldots,U_p;Z_2) \xrightarrow{\nabla} Z\mathcal{O}(U_1,\ldots,U_p;Z_2) \xrightarrow{\epsilon} G(U_1,\ldots,U_p;Z_2) \longrightarrow 0.$$

Here, $U_i (1 \leq i \leq p)$ is a finite dimensional graded vector-space over Z_2. An element of $\mathcal{O}(U_1,\ldots U_p;Z_2)$ is a family

$$\theta = \{\theta < n_1,\ldots,n_p>\}$$

of natural transformations (non-additive)

$$\theta< n_1,\ldots,n_p >: C^{n_1}(-,U_1)x\ldots x\, C^{n_p}(-,U_p) \longrightarrow C^{n+k}(-,Z_2)$$

where $n = \sum n_i$. The integer k is the degree of θ. We assume further

$$\theta(0,\ldots,0, x_i,0,\ldots,0) = 0$$

for $1 \leq i \leq p$ and $x_i \in C(X,U_i)$. The differential is defined by

$$(\nabla\theta)(x_1,\ldots,x_p) = \delta\theta(x_1,\ldots,x_p) + \sum_{i=1}^{p} \theta(x_1,\ldots,\delta x_i,\ldots,x_p).$$

The term $G(U_1,\ldots,U_p;Z_2)$ is the module of "multistable" cohomology operations:

$$\lambda : \bigotimes_{i=1}^{p} H(-;U_i) \longrightarrow H(-;Z_2)$$

we note that

$$\mathcal{G}(U_1,\ldots,U_p;Z_2) \cong \mathcal{G}(U_1;Z_2)\otimes\ldots\otimes\mathcal{G}(U_p;Z_2)$$

and that

$$\mathcal{G}(U;V) \cong \mathcal{G}\otimes \mathrm{Hom}(U,V).$$

An important consequence of (1.1) is

Proposition 1.2. Let

$$a\in\mathcal{O}(U;Z_2)$$

where U is concentrated in degree zero, $U_0 = Z_2\otimes\ldots\otimes Z_2$ (n summands). We consider a as a function in n-variables and assume

$$\text{(i)}\quad \delta a(x_1,\ldots,x_n) + a(\delta x_1,\ldots,\delta x_n) = 0$$

$$\text{(ii)}\quad a(0,\ldots,0,x_i,0,\ldots,0) = 0 \quad 1 \leq i \leq n.$$

Then there exists a cochain operation $A(x_1,\ldots,x_n)$ with

$$\text{(i)}\quad \delta A(x_1,\ldots,x_n) + A(\delta x_1,\ldots,\delta x_n) = a(x_1,\ldots,x_n)$$

$$\text{(ii)}\quad A(0,\ldots,0,x_i,0,\ldots,0) = 0 \quad 1 \leq i \leq n.$$

Let $a\in Z\mathcal{O}$. By Proposition 1.2 there is a two variable cochain operation $d(a;x,y)$ with

$$(1.2)\quad \delta a(a;x,y) + d(a;\delta x,\delta y) = a(x+y) + a(x) + a(y)$$

$$d(a;x,0) = 0 \quad\text{and}\quad d(a;0,y) = 0.$$

Furthermore let us define

$$d(a;\{x_i\}) = d(a;x_1,\ldots,x_n)$$
$$= \sum_{i=1}^{n-1} d(a;x_i,x_{i+1}+\ldots+x_n)$$

Then

$$\delta d(a;\{x_i\}) + d(a;\{\delta x_i\}) = a(\sum x_i) + \sum a(x_i).$$

To each $\Theta \in \mathcal{O}$ there is an operation $d(\Theta;x,y)$ with

(1.3) $$\delta d(\Theta;z,y) + d(\Theta;\delta x,\delta y)$$
$$= \Theta(x+y) + \Theta(x) + \Theta(y) + d(\nabla\Theta;x,y)$$

$$d(\Theta;x,0) = 0 \quad \text{and} \quad d(\Theta;0,y) = 0.$$

Next, we consider the cochain Cartan formula. With definition of the cochain operations sq^i, and $d(sq^i)$ as in Kristensen [1963] using Steenrod's $\smile_i$-product we have

Proposition 1.3. There are cochain operations

$$T_n \in \mathcal{O}(Z_2,Z_2;Z_2)$$

satisfying

(i) $$T_n(x,y) = sq^n(xy) + \sum sq^i(x)sq^{n-i}(y)$$
$$+ d(sq^n;\delta x\cdot y,x\delta y) + \deg(x)d(sq^n;x\delta y,x\delta y)$$

(ii) $$T_n(x,1) = 0 \quad \text{and} \quad T_n(1,y) = 0$$

(iii) $$T_n(x,y) = 0 \text{ if } \delta x = \delta y = 0, \quad \dim(xy) < n-1.$$

(iv) $T_n(x,y) = \sum i\deg(y)sq^i(x)sq^{n-i-1}(y)$

if $\delta x = \delta y = 0$ $\dim(x,y) = n-1$.

For cocycles x,y and z

(v) $T_n(xy,z) + T_n(x,yz) + \sum sq^i(x)T_{n-i}(y,z)$
$+ \sum T_i(x,y)sq^{n-i}(z) \sim 0.$

The cochain operations T_n associated with sq^n or more general T_a associated with $a \epsilon Z$ are crucial in this paper. We list some more properties.

In Kock-Kristensen [1965] we considered the problem of expressing T_{ab} corresponding to a composition of two cochain operations in terms of T_a and T_b. For the cochain operations sq^i the result is

Proposition 1.4. There is a cochain operation $T_{k-j,j}$ associated with $sq^{k-j}sq^j$ such that for cocycles x and y

$$\begin{aligned} T_{k-j,j}(x,y) &= sq^{k-j}T_j(x,y) + \sum T_{k-j}(sq^i(x),sq^{j-i}(y)) \\ &+ d(sq^{k-j};sq^j(xy), \{sq^ix.sq^{j-i}(y)\}) \\ &+ \sum i\deg(y)\mathrm{H}(sq^{k-j})(sq^ix\cdot sq^{j-i}y) \\ &+ \mathrm{H}(sq^{k-j})sq^j(xy), \end{aligned}$$

and if $r = \sum C(j)sq^{k-j}sq^j, C(j)\epsilon Z_2,$ then

$$T_r = \sum C(j)T_{k-j,j}.$$

Let us consider the relations $(a,b,k \in Z)$

(1.4) $r(a,b;k) = \sum[\binom{b-1-j}{k+b-a-2j} + \binom{b-1-j}{j+b-a}]Sq^{k-j}Sq^{j}$,

in Steenrod's algebra G. We consider $r(a,b;k)$ as an element in $\mathcal{F}$ - the free associative algebra with unit generated by Sq^{i}, > 0. There is a diagonal

$$\psi : \mathcal{F} \longrightarrow \mathcal{F}$$

inducing the diagonal in G. An easy computation gives

(1.5) $\psi(r(a,b;k)) = \sum r(a-2i,b-i;k-i-j) \otimes Sq^{j}Sq^{i}$

$$+ \sum Sq^{j}Sq^{i} \otimes r(a-j,b-i;k-i-j).$$

In (1.4) we replace Sq^{i} by the cochain operation sq^{i} and consider $r(a,b;k)$ as a cochain operation.

Lemma 1.5. With T_r as in Proposition 1.4, we have for cocycles x,y and z

$$T(a,b;k)(x,y,z) \sim 0$$

where

$$\begin{aligned} T(a,b;k)(x,y,z) &= T_{r(a,b;k)}(x,yz) + T_{r(a,b;k)}(xy,z) \\ &+ \sum sq^{j}sq^{i}(x) \cdot T_{r(a-j,b-i;k-i-j)}(y,z) \\ &+ \sum T_{r(a-2i,b-i;k-i-j)}(x,y)sq^{j}sq^{i}(z) \\ &+ \sum r(a-2i,b-i;k-i-j)(x) \cdot T_{j,i}(y,z) \\ &+ \sum T_{j,i}(x,y) \cdot r(a-j,b-i;k-i-j)(z). \end{aligned}$$

Let $\hat{a} \in G$ have diagonal

$$(1.6) \qquad \psi(\hat{a}) = \sum \hat{a}' \otimes \hat{a}'' + \hat{\bar{a}} \otimes \hat{\bar{a}} + \sum \hat{a}'' \otimes \hat{a}' .$$

If a, a', a'' and $\bar{a}$ are cochain operations representing these operations and if T_a is associated with (1.6) then (by Kock-Kristensen [1965] p. 135-136) there are λ', λ'' and $\eta(a) \in Z\mathcal{G}$ such that for cocycles x and y of dimension p and q

$$\begin{aligned}(1.7) \quad & T_a(x,y) + T_a(y,x) + d(a;xy,yx) + a(x \smile_1 y) \\ & + \sum a'(x) \smile_1 a''(y) + \bar{a}(x) \smile_1 \bar{a}(y) + \sum a''(x) \smile_1 a'(y) \\ & + pq\, \kappa(a)(xy) + \eta(a)(x) \cdot \eta(a)(y) \\ & \sim \sum \lambda'(x) \cdot \lambda''(y) + \sum \lambda''(x) \cdot \lambda'(y).\end{aligned}$$

This can be improved in case $a = sq^n$. First however we examine $\eta(a)$ a little closer. The element $\bar{a}$ (see(1.6), we use simplified notation) is the image of a under the mapping (algebra morphisme)

$$\zeta : G \longrightarrow G$$

dual to the Frobenius mapping $\varphi : G^* \longrightarrow G^*$, $\varphi(x) = x^2$. The equation (1.7) can be used to determine ζ on Massey products in the Steenrod algebra. This is done in a forthcoming paper. We consider another mapping

(1.8) $$\eta: \mathcal{a} \longrightarrow \mathcal{a}$$

related to ζ. Besides being additive it has the properties

(i) $$\eta(\mathcal{a}_{2n}) = 0, \quad \eta(\mathcal{a}_{2n+1}) \subset \mathcal{a}_n$$

(ii) $\eta(ab) = \eta(a)\zeta(b) + m\,\zeta(\chi(a))\zeta(b) + \zeta(a)\eta(b)$

where $2m = \deg(b)$,

(iii) $$\eta(Sq^{2n+1}) = (n+1)Sq^n \quad ,$$

(iv) $(p+1)\chi(a)Sq^p(u) + aSq^{p-1}(u) + Sq^{n+p}\eta(a)(u) = 0,$

for $a \in \mathcal{a}_{2n+1}$ and for any p-dimensional cohomology class u.

Also for ψ the diagonal in $\mathcal{a}$.

(v) $$\psi\eta = (\zeta\chi\otimes\zeta + \zeta\otimes\eta + \eta\otimes\zeta)\psi.$$

The dual

(1.9) $$\eta^*: \mathcal{a}^* \longrightarrow \mathcal{a}^*$$

of η has the properties

(i) $$\eta^*(\xi_i) = \xi_{i+1}$$

(ii) $\eta^*(\alpha\beta) = \xi_1\alpha^2\beta^2 + \eta^*(\alpha)\cdot\beta^2 + \alpha^2\eta^*(\beta)$

We return to (1.7)

Proposition 1.6. With T_n as in Proposition 1.3, we have for x and y cocycles of dimension p and q

$$T_n(x,y) + T_n(y,x) + d(sq^n;xy,yx) + sq^n(x \smile_1 y)$$
$$+ \sum sq^i(x) \smile_1 sq^{n-i}(y) + pq\,\chi(sq^n)(xy)$$
$$+ \sum_{i+j=n-1} (ij+1)sq^i x \cdot sq^j y \sim 0.$$

2. A System of Cochain Operations. A system

$$\{R(a,b;k)\}, \quad a,b,k\in Z,$$

of cochain operations is said to be permissible provided

(i) $$\nabla R(a,b;k) = r(a,b;k)$$

(see(1.4)).

(ii) If u is a 1-dimensional cocycle

$$R(a,b;k)(u) = 0$$

except when $k=3$ and $\text{excess}(r(a,b;k)) = 2$ in which case

$$R(a,b;k)(u) = u^3.$$

From Kristensen [1963] it follows that permissible systems exists.

Let us consider a system of cohomology operations

$$A(a,b;k) \in (G \otimes G)_{k-1}$$

given by

(2.1) $$A(a,b;k) = (Sq^1 \otimes Sq^{(0,1)})(\sum C(j)(Sq^{k-j-3}Sq^{j-2} + Sq^{k-j-2}Sq^{j-2}))$$

where $C(j)$ is the coefficient from (1.4), and where the right module structure of $G \otimes G$ is given via the diagonal ψ. Also, $Sq^{(0,1)} = Sq^2Sq^1 + Sq^3$. In the obvious fashion we also consider $A(a,b;k)$ a cochain operation in two variables.

Theorem 2.1. There exists a unique permissible system $\{R(a,b;k)\}$ of cochain operations satisfying

$$R(a,b;k)(xy) + T_{r(a,b;k)}(x,y) + A(a,b;k)(x,y)$$
$$+ \sum R(a-2i,b-i;k-i-j)(x)\cdot sq^j sq^i(y)$$
$$+ \sum sq^j sq^i(x)\cdot R(a-j,b-i;k-i-j)(y) \sim 0,$$

for each pair of cocycles x and y. The system is unique in the sense that if $\{R'(a,b;k)\}$ is another such system then

$$R(a,b;k) - R'(a,b;k) \in Z\mathcal{O}$$

determines the zero cohomology operation for all a,b and k.

The proof of uniqueness is easy. In the proof of existence we make use of the Cobar resolution (see Adams [1960])

$$F(\mathcal{G}): \overline{\mathcal{G}} \longrightarrow \overline{\mathcal{G}} \otimes \overline{\mathcal{G}} \longrightarrow \overline{\mathcal{G}} \otimes \overline{\mathcal{G}} \otimes \overline{\mathcal{G}} \longrightarrow$$

with homology

$$H(F(\mathcal{G})) = \mathrm{Ext}_{\mathcal{G}*}(Z_2,Z_2)$$
$$= \Lambda\{Q_i; i \geq 0\}$$

where $Q_o = Sq^1, Q_1 = Sq^{(o.1)}, \ldots, Q_i = Sq^{(o,o,\ldots,o,1)}$.

The proof is by induction over the dimension k.

Proposition 2.2. Let $\{R(a,b;k)\}$ be as in Theorem 2.1. If

$$\sum_i r(a_i,b_i;k) = 0 \in \mathcal{F},$$

then

$$\sum_i R(a_i,b_i;k) \smile 0,$$

i.e. determines the zero element in G.

In Kristensen [1963] we determined the value of cochain operations similar to $R(a,b;k)$ in low dimensions. Here we have

Proposition 2.3. If

$$R(a,b;k) = Sq^{k-q}Sq^q + Sq^{k-t}Sq^t + \text{term of excess} > q,$$

where $2q > k-q$ and $t = k-q/2$, then if x is a cocycle of dimension $< q-1$

$$R(a,b;k)(x) \sim 0.$$

If $\dim(x) = q-1$

$$R(a,b;k) \sim \sum sq^i(x) \cdot sq^j(x) + \alpha(x)^2$$

where $i+j = k-q$, $i < j$, and $\alpha \in Z\mathcal{O}(Z_2,Z_2)$.

Remark 2.4. The unknown operation α gives rise to an element $\hat{\alpha} \in G$. The proof of Theorem 2.1 could be extended so as to yield the value of $\hat{\alpha}$. So far, I have not carried this out.

3. <u>A Secondary Cartan Formula.</u> Let $\sum \alpha_\nu a_\nu + b = 0$ be a relation in G. Let ϕ be a secondary operation associated with this relation. The operation ϕ is defined in dimensions less than the excess of b, and commutes with suspension in this region. It is additive except in the top dimension. The deviation from additivity is given in Kristensen [1963].

A Cartan formula for $\phi(xy)$ is not so easy to describe since it depends on the "reason" for that

$$a_\nu(xy) = 0.$$

Let the diagonal of a_ν be $\psi(a_\nu) = \sum a'_\nu \otimes a''_\nu$.

Definition 3.1. We say that x and y are complementary classes with respect to the relation $\sum \alpha_\nu a_\nu + b = 0$ provided

$$a'_\nu(x) = 0 \quad \text{or} \quad a''_\nu(y) = 0$$

for each term $a'_\nu \otimes a''_\nu$ in $\psi(a_\nu)$, all ν.

Here we shall only consider the complementary case. Let $\phi(a,b;k)$ be the secondary operation associated with relation $r(a,b;k)$ (1.4). These operations are unique in the sense described in Theorem 2.1. The Cartan formula can be derived from the following Lemma. If $Sq(k-t)Sq(t)$ appears in $r(a,b;k)$ let the set of integers $\{o,1,\ldots,t\}$

be divided into two disjoined subsets I' and I'' such that $Sq^i(x) = 0$ for $i \in I'$ and $Sq^j(y) = 0$ for $j \in I''$. Then (denoting by x and y cocycles representing the cohomology classes) if $\delta w_i = sq^i(x)$, $\delta w_j = sq^j(y)$, $i \in I'$, $j \in I''$,

$$\delta u_t = sq^t(xy),$$

where

$$u_t = T_t(x,y) + \sum_{I'} w_i sq^{t-i}(y) + \sum_{I''} sq^{t-j}(x)v_j.$$

Hence we can write down a cocycle representative for $\phi(xy)$

(3.1) $\quad \phi(a,b;k)(xy) = \{R(a,b;k)(xy) + \sum C(t)sq^{k-t}(u_t)\}.$

Lemma 3.2. The class (coset) $\phi(a,b;k)(xy)$ is represented by the cocycle

$$\begin{aligned} &\sum R(a-2i,b-i;k-i-j)(x)\cdot sq^j sq^i(y) \\ &+ \sum sq^j sq^i(x)\cdot R(a-j,b-i;k-i-j)(y) \\ &+ \sum_{I'} C(t)sq^{k-t-s} w_i sq^s sq^{t-i}(y) \\ &+ \sum_{I''} C(t)\cdot sq^{k-t-s} sq^{t-j}(x) sq^s v_j \\ &+ A(a,b;k)(x,y), \end{aligned}$$

where $A(a,b;k)$ is the primary term described in (2.1).

We remark that the Lemma could as well have been formulated for a linear combination of the relations $r(a,b;k)$.

In order to get a Cartan formula for the secondary operations we only have to assemble the cocycle given in Lemma 3.2 to a sum of products of operations on x and on y. In this paper we shall only give an example.

Example 3.3. We consider the relations

$$r^{1,n}\colon\ Sq^1Sq^n + \binom{n-1}{1}Sq^{n+1}$$

$$r^{2n}\colon\ Sq^2Sq^n + Sq^{n+1}Sq^1 + \binom{n-1}{2}Sq^{n+2}$$

The associated secondary operations we denote $\phi^{1.n}$ and $\phi^{2.n}$. An easy application of Lemma 3.2 gives:

if $n \equiv 1 (\mathrm{mod}\ 2)$ and $Sq^{2i+1} = 0$, $Sq^{2i+1}(y) = 0$, $2i+1 \leq n$, (i.e. x and y complementary) then

$$\phi^{1.n}(xy) = \sum\phi^{1,2i+1}(x)Sq^{n-2i-1}(y) + \sum Sq^{2i}(x)\phi^{1.n-2i}(y),$$

if $n \equiv 0 (\mathrm{mod}\ 2)$, $p+q = n$, where $p = \dim x$, $q = \dim y$, and $x^2 = 0$, then

$$\phi^{1.n}(xy) = xSq^1(x)Sq^{q-1}(y) + \phi^{1.p}(x)\cdot y^2 + Sq^{p-1}(x)\cdot y\cdot Sq^1(y),$$

if $n \equiv 1,2 (\mathrm{mod}\ 4)$ and $Sq^ix = 0$, $Sq^iy = 0$ provided $i \equiv 1,2,3 (\mathrm{mod}\ 4)$, $i \leq n$, then

$$\phi^{2.n}(xy) = \sum Sq^{4i}(x)\phi^{2.n-4i}(y) + \sum\phi^{2.n-4i}(x)Sq^{4i}(y),$$

if $n \equiv 0,3 \pmod 4$, $p+q = n+1$, and $x^2 = 0$, $y^2 = 0$, $Sq^1x = 0$, $Sq^1y = 0$, then

$$\phi^{2.n}(xy) = \phi^{2.p}(x)\cdot Sq^{q-1}(y) + Sq^{p-1}(x)\phi^{2,q}(y)$$
$$+ x\cdot Sq^2(x)Sq^{q-2}(y) + Sq^{p-2}(x)\cdot y\cdot Sq^2(y).$$

These operations have nice applications for instance in the study of H-spaces. We note that the primary term $A(x,y)$ (see Lemma 3.2) is zero in the case considered above.

BIBLIOGRAPHY

J. F. Adams, On the Non-Existence of Elements of Hopf Invariant One, Ann. of Math. 72 (1960), 20-104.

A. Kock, L. Kristensen and I. Madsen, Cochain Functors for General Cohomology Theories I-II, Math. Scand. 20 (1967), 131-150, 151-176.

L. Kristensen, On Secondary Cohomology Operations, Math. Scand. 12 (1963), 57-82.

L. Kristensen, On a Cartan Formula for Secondary Cohomology Operations, Math. Scand. 16 (1965), 97-115.

L. Kristensen and I. Madsen, On the Structure of the Operation Algebra for Certain Cohomology Theories (to appear).