On the structure of the operation algebra for certain cohomology theories.

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Let h denote a cohomology theory. By $\mathcal{A}(h,h)$ we denote the algebra of all stable cohomology operations of h into itself. We shall study the (Steenrod) algebra $\mathcal{A}(h,h)$ in the particular case when h is a "two-stage" cohomology theory $H_{(2)}$ associated with a finite collection $\{k_i\}$ of elements in the Steenrod algebra $\mathcal{A}$, i.e.

$$H^n_{(2)} = [\quad , K_{(2)}(n)] \ ,$$

where $\{K_{(2)}(n)\}$ is an Ω-spectrum with $K_{(2)}(n)$ the following pullback

$$\begin{array}{ccc} K_{(2)}(n) & \longrightarrow & \prod_i L(Z_2, n+\deg k_i - 1) \\ \downarrow & & \downarrow \\ K(Z_2,n) & \xrightarrow{\{k_i\}} & \prod_i K(Z_2, n+\deg k_i) \end{array} \quad .$$

The mapping $L(Z_2,m-1) \longrightarrow K(Z_2,m)$ denotes the standard fibration of Eilenberg-MacLane complexes. The additive structure of the operation algebra, which in this case is denoted $\mathcal{A}_{(2,2)}$, is fairly easy to describe. The multiplicative structure, however, is far more difficult. We describe a connection between the multiplicative structure of $\mathcal{A}_{(2,2)}$ and Massey products in $\mathcal{A}$. In a particular case with a single element in $\mathcal{A}$ as k-invariant $\{k_i\} = \{Sq^{(0,1)}\}$, we give a complete description of the operation algebra. This amounts to evaluation of a certain set of Massey products. In this case the spectrum $K_{(2)}$ is in an obvious way related to mod 2 connective K-theory. The operation algebra for connective K-theory has been determined by D.Anderson and M. Meiselman. The relationship between these two algebras seems worthwhile studying.

It is possible to determine $\mathcal{A}_{(2,2)}$ also in other cases. The method works equally well in the mod p case.

1. Cochain functors. In the papers Koch, Kristensen and Madsen [1967] and [1967a] we studied cochain functors for general cohomology theories. These cochain functors are basic for the work in the present paper. In this section we shall review certain parts of the theory of cochain operations and describe cochain functors needed in this paper.

If U is a graded Z_2-vectorspace of finite dimension, let $C(-,U)$ denote the usual cochain functor with coefficients in U. (In case $U = Z_2$ we stick to the notation C). Let $U_1,\ldots,U_p$ and V be graded Z_2-vectorspaces as above. A family $\theta = \theta\langle n_1,\ldots,n_p\rangle$ of natural transformations (non-additive)

$$\theta\langle n_1,\ldots,n_p\rangle : C^{n_1}(-;U_1)\times\ldots\times C^{n_p}(-;U_p)\longrightarrow C^{n+i}(-;V),$$

where $n = \sum n_i$, is called a cochain operation of degree i, provided

$$\theta(0,\ldots,0,x_i,0,\ldots,0) = 0$$

for $1 \leq i \leq p$ and $x_i \in C(X;U_i)$. The set of all cochain operations is denoted

$$\mathcal{C}(U_1,\ldots,U_p;V) ,$$

and has a differential ∇

$$(\nabla\theta)(x_1,\ldots,x_p) = \delta\theta(x_1,\ldots,x_p) + \sum_{i=1}^{p} \theta(x_1,\ldots,\delta x_i,\ldots,x_p) .$$

The cycles under ∇ are denoted $Z\mathcal{C}(U_1,\ldots,U_p;V)$.

For $V = Z_2$ the cohomology of $\mathcal{C}(U_1,\ldots,U_p;Z_2)$ is given by the exact sequence

$$(1.1)\ \mathcal{C}(U_1,\ldots,U_p;Z_2) \xrightarrow{\nabla} Z\mathcal{C}(U_1,\ldots,U_p;Z_2) \xrightarrow{\varepsilon} \mathcal{A}(U_1,\ldots,U_p;Z_2) \rightarrow 0$$

The term $\mathcal{A}(U_1,\ldots,U_p;Z_2)$ is the module of "multistable" cohomology operations:

$$\lambda : \bigotimes_{i=1}^{p} H(-;U_i) \longrightarrow H(-;Z_2) .$$

We note that

$$\mathcal{A}(U_1,\ldots,U_p;Z_2) \cong \mathcal{A}(U_1;Z_2) \otimes \ldots \otimes \mathcal{A}(U_p;Z_2)$$

and that

$$\mathcal{A}(U;V) \cong \mathcal{A} \otimes \mathrm{Hom}(U,V) .$$

The proof of (1.1) is given in Koch, Kristensen and Madsen [1967], see also Kristensen[1965]. Usually, we shall make no effort in notation to distinguish between an element of $Z\mathcal{C}(U_1,\ldots,U_p;V)$ and its associated element of $\mathcal{A}(U_1,\ldots,U_p;V)$. In each case it will be clear from the context whether a given letter denotes a cochain operation or a cohomology operation.

An important consequence of (1.1) is

Proposition 1.2. Let

$$a \in \mathcal{C}(U;Z_2) ,$$

where U is a graded Z_2-vectorspace of dimension n. We consider a as a function in n variables and assume

(i) $\delta a(x_1,\ldots,x_n) + a(\delta x_1,\ldots,\delta x_n) = 0$

(ii) $a(0,\ldots,0,x_i,0,\ldots,0) = 0 , 1 \leq i \leq n .$

Then there exists a cochain operation $A(x_1,\ldots,x_n)$ with

(i) $\delta A(x_1,\ldots,x_n) + A(\delta x_1,\ldots,\delta x_n) = a(x_1,\ldots,x_n)$

(ii) $A(0,\ldots,0,x_i,0,\ldots,0) = 0 , 1 \leq i \leq n .$

Let $a \in Z\mathcal{C}$. By Proposition 1.1 there is a two variable cochain operation $d(a;x,y)$ with

(1.2) $\delta d(a;x,y) + d(a;\delta x,\delta y) = a(x+y) + a(x) + a(y)$,

$$d(a;x,0) = 0 \text{ and } d(a;0,y) = 0 .$$

Further, to each $\theta \in \mathcal{C}$ there is an operation $d(\theta;x,y)$ with

$$\delta d(\theta;x,y) + d(\theta;\delta x,\delta y) = \theta(x+y) + \theta(x) + \theta(y) + d(\nabla\theta;x,y)$$

(1.3)

$$d(\theta;x,0) = 0 \text{ and } d(\theta;0,y) = 0 .$$

If a and b are in $Z\mathcal{C}$ then a possible choice of $d(ab;x,y)$ is

$$(1.4) \quad d(ab;x,y) = ad(b;x,y) + d(a;bx,by) + d(a;\nabla d(b;x,y),\ bx + by) + d(a;\delta d(b;x,y),\ d(b;\delta x,\delta y)) .$$

Now, we are ready to describe the cochain functor for a "two stage" cohomology theory. For convenience let us consider the case with only one k-invariant $k \in \mathcal{A}_m$, $\{k_i\} = \{k\}$. We construct a new "cochain-functor" $C_{(2)} = C[k]$, $k \in Z\mathcal{A}$,

$$(1.5) \quad \begin{aligned} C^n_{(2)}(X) &= C^n(X) \times C^{n+m-1}(X) \\ \delta(x,w) &= (\delta x, \delta w + k(x)) \\ (x,w) + (y,v) &= (x+y, w+v+d(k;x,y)) . \end{aligned}$$

Notice that $C_{(2)}(X)$ is not an abelian group but only a loop with some further structure on it; in fact the functor $C_{(2)}$ is a c.g. functor (Definition 2.3 in Koch, Kristensen and Madsen [1967a]). The associated cohomology functor $H_{(2)} = H[k]$ has as spectrum the two stage Postnikov system, $\{K_{(2)}(n)\}$, with k-invariant k.

The short exact sequence of c.g. functors

$$0 \longrightarrow C^{n+m-1} \xrightarrow{\alpha} C^n_{(2)} \xrightarrow{j} C^n \longrightarrow 0$$

has an associated long exact sequence on cohomology level

$$(1.6) \quad \cdots \longrightarrow H^{n+m-1} \xrightarrow{\alpha} H^n_{(2)} \xrightarrow{j} H^n \xrightarrow{k} H^{n+m} \longrightarrow \cdots$$

which is the same as the mapping sequence associated with the fibration $K(Z_2,n+m-1) \dashrightarrow K_{(2)}(n) \longrightarrow K(Z_2,n)$.

According to (1.2)

$$(1.7) \qquad \mathcal{H}(k)(x) = d(k;x,x)$$

is an element of $Z\mathcal{C}$. The associated element in $\mathcal{Q}$ depends only on $\xi(k)$. The mapping

$$(1.8) \qquad \mathcal{H} : \mathcal{Q} \longrightarrow \mathcal{Q}$$

is a derivation of degree -1 (formula (1.4)). The dual mapping $\mathcal{H}^*: \mathcal{Q}^* \longrightarrow \mathcal{Q}^*$ was determined in Kristensen [1965]. It is multiplication with $\xi_1 \in \mathcal{Q}$.

In general $H_{(2)}(X)$ is not a Z_2-vectorspace. We have

$$(1.9) \qquad 2(x,w) = (0, \mathcal{H}(k)(x)) .$$

which is ~ 0 if $\mathcal{H}(k)(\{x\}) \in \mathrm{Im}(k) \subset H(X)$. If $\mathcal{H}(k) = 0$ in $\mathcal{Q}$ then this is the case for all elements in $H_{(2)}(X)$.

Let $\mathcal{C}_{(r,s)}$, $r,s = 1$ or 2, denote the set of (non-additive) natural transformations, again called cochain operations, from $C_{(r)}$ to $C_{(s)}$, ($C_{(1)} = C$). There is a differential ∇ on $\mathcal{C}_{(r,s)}$ (Koch, Kristensen and Madsen [1967a] section 3) and a short exact sequence

$$(1.10) \qquad \mathcal{C}_{(r,s)} \xrightarrow{\nabla} Z\mathcal{C}_{(r,s)} \xrightarrow{\varepsilon} \mathcal{Q}_{(r,s)} \longrightarrow 0 ,$$

giving the cohomology of $\mathcal{C}_{(r,s)}$. The term $\mathcal{Q}_{(r,s)}$ is the graded group of stable operations from $H_{(r)}$ to $H_{(s)}$. Using (1.9) one can prove that the functors $\mathcal{A}(-,H)$ and $\mathcal{A}(H_{(2)},-)$ are "exact" in the sense that applied to (1.6) they give exact sequences

$$(1.11)\quad \begin{array}{l} \cdots \to \mathcal{A} \xrightarrow{\circ k} \mathcal{A} \xrightarrow{j^*} \mathcal{A}_{(2,1)} \xrightarrow{\alpha^*} \mathcal{A} \longrightarrow \cdots \\ \cdots \to \mathcal{A}_{(2,1)} \xrightarrow{k\circ} \mathcal{A}_{(2,1)} \xrightarrow{\alpha_*} \mathcal{A}_{(2,2)} \xrightarrow{j_*} \mathcal{A}_{(2,1)} \to \cdots \end{array}$$

Hence we get short exact sequences

$$(1.12)\quad \begin{array}{l} 0 \longrightarrow \mathrm{Coker}(\circ k) \xrightarrow{j^*} \mathcal{A}_{(2,1)} \xrightarrow{\alpha^*} \mathrm{Ker}(\circ k) \longrightarrow 0 \\ 0 \longrightarrow \mathrm{Coker}(k\circ) \xrightarrow{\alpha_*} \mathcal{A}_{(2,2)} \xrightarrow{j_*} \mathrm{Ker}(k\circ) \longrightarrow 0 . \end{array}$$

Together with (1.7) they essentially determine the additive structure of $\mathcal{A}_{(2,1)}$ and $\mathcal{A}_{(2,2)}$.

The $\mathcal{A}$-module structure of $\mathcal{A}_{(2,1)}$ and the algebra structure of $\mathcal{A}_{(2,2)}$ is more difficult to determine. It requires determination of Massey products in $\mathcal{A}$. These Massey products we study in section 2.

It is not hard to see that $H_{(2)} = H[k]$ has a cupproduct structure if and only if k is primitive in the Hopf algebra $\mathcal{A}$. If k is primitive there are usually more products than one. Using cochain expressions for such a product one can show that at least one is associative. This has been carried out by H.A. Salomonsen.

2. Massey products in $\mathcal{A}$. The Steenrod algebra is the cohomology of $\mathcal{C} = \mathcal{C}(Z_2;Z_2)$. Since $\mathcal{C}$ is almost a graded algebra over Z_2(left distributively is missing; $a(b+c) \neq ab + ac$) we can define Massey products in $\mathcal{A}$.

Let

(2.1) $\qquad a \in \mathcal{A}(V_3,V_4),\ b \in \mathcal{A}(V_2,V_3),\ c \in \mathcal{A}(V_1,V_2)$,

where V_i, $i = 1,2,3,4$, is a finite dimensional graded vector-space. Let $ab = 0$ and $bc = 0$ and let on cochain level

$$\nabla R = ab\ ,\ \nabla S = bc\ .$$

then the cochain operation

(2.2) $\qquad m = Rc + aS + d(a;\delta S,S\delta) \in \mathcal{C}(V_1,V_4)$,

is ∇-cycle. Hence it determines an element in $\mathcal{A}(V_1;V_4)$, which represents the Massey product

$$\mathcal{E}(m) \in \langle a,b,c\rangle \subset \mathcal{A}(V_1,V_4)\ .$$

The indeterminacy is given by

$$a\mathcal{A}(V_1,V_3) + \mathcal{A}(V_2,V_4)\,c\ .$$

We use the following properties of the Massey product in $\mathcal{A}$.

Proposition 2.1. If $ab = 0$, $bc = 0$ and $cd = 0$ then

$$a\langle b,c,d\rangle = \langle a,b,c\rangle d\ .$$

Let a,b and c be as above. We consider the matrices

$$(2.3) \qquad A = [a, \mathcal{H}(a)], \quad B = \begin{bmatrix} \mathcal{H}(b) & b \\ b & 0 \end{bmatrix} \quad C = \begin{bmatrix} c \\ \mathcal{H}(c) \end{bmatrix},$$

where $\mathcal{H}$ is the mapping defined in (1.7).

Proposition 2.2. If a, b and c are as in (2.1) and A,B,C as in (2.3) then

$$\mathcal{H}(\langle a,b,c\rangle) \subset \langle A,B,C \rangle .$$

The proof of Proposition 2.1 is straightforward whereas the proof of Proposition 2.2 is more involved. It uses expansions of the type (1.4) for the compositions Rc and aS (see (2.2)).

Next we consider the diagonal ψ' in the Steenrod algebra. Let U,V and V_i, $i = 1,2,3,4$, be finite graded Boolean algebras. Then

$$\psi' : \mathcal{A}(U,V) \longrightarrow \hat{\mathcal{A}}(U,V) \otimes_V \mathcal{A}(U,V) .$$

Let us use the notation

$$\psi'(a) = \sum a' \otimes a''$$

for the diagonal of a, and likewise for b and c. Let $r = a \otimes b$ be a relation, i.e. $ab = 0$ in $\mathcal{A}(V_2,V_4)$. In what follows we omit this tensor product sign. We get (again omitting some tensorproduct signs)

$$\begin{aligned}\psi'(r) &= \sum a'b' \otimes a''b'' \\ &= \sum r' \otimes \alpha'' + \sum \alpha' \otimes r'' ,\end{aligned}$$

where r' and r'' are relations. Also $(s = bc)$

$$\psi(s) = \sum b'c' \otimes b''c''$$

$$= \sum s' \otimes \beta'' + \sum \beta' \otimes s'' ,$$

where $s' = \sum \gamma_1{}' \gamma_2{}'$, $s'' = \sum \gamma_1{}'' \gamma_2{}''$.

Hence we have the identity

(2.4) $\sum b'c' \otimes b''c'' \equiv \sum \gamma_1{}' \gamma_2{}' \otimes \beta'' + \sum \beta' \otimes \gamma_1{}'' \gamma_2{}''$, i.e.

each term $b'c' \otimes b''c''$ on the left cancels one of the terms on the right. The remaining terms cancel in pairs. We pick an element $\gamma' \otimes \gamma''$ from each such pair.

The diagonal of our third order relation is as follows

$$\psi(rc + as) = \sum r'c' \otimes \alpha''c'' + \sum \alpha'c' \otimes r''c''$$
$$+ \sum a's' \otimes a''\beta'' + \sum a'\beta' \otimes a''s'' .$$

We assume that

(2.5) $$\psi(rc + as) \sim \sum \sigma' \otimes d'' + \sum d' \otimes \sigma''$$

where $d', d'' \in \mathfrak{A}$ and σ', σ'' are third order relations. The additive relation $\sim$ is given as follows: Terms of the form

$$D(a_1 r_1 \otimes a_2 r_2) , D(a_1 r_1 \otimes r_2 c_2)$$

$$D(r_1 c_1 \otimes a_2 r_2) , D(r_1 c_1 \otimes r_2 c_2)$$

are equivalent to zero, where

$$D(a_1r_1 \otimes a_2r_2) = \sum a_1\beta_1\gamma_1 \otimes a_2r_2$$

$$+ \sum a_1r_1 \otimes a_2\beta_2\gamma_2 ,$$

$$r_1 = \sum \beta_1\gamma_1 , \quad r_2 = \sum \beta_2\gamma_2 ,$$

and similarly for the others.

Proposition 2.3. Let a, b and c be as above. Then for $x,y \in H^*(X;V_1)$

$$\langle a,b,c\rangle(xy) = \sum m(\sigma')(x)\ d''(y) + \sum d'(x)\ m(\sigma'')(y)$$

$$+ \sum \alpha_0'c'(x)\cdot\alpha_0''c''(y) + a(\sum \beta_0'(x)\cdot\beta_0''(y))$$

$$+ \sum \deg(b')\deg(c'')\mathcal{K}(a)(b'c'(x)\cdot b''c''(y))$$

$$+ \sum \mathcal{K}(a)(\gamma'(x)\cdot\gamma''(y)) ,$$

where $m(\sigma)$ denotes the Massey product associated with the third order relation σ. The terms

$$\sum \alpha_0' \otimes \alpha_0'' \in \mathcal{Q}(V_2,V_4) \otimes_{V_4} \mathcal{Q}(V_2,V_4)$$

$$\sum \beta_0' \otimes \beta_0'' \in \mathcal{Q}(V_1,V_3) \otimes_{V_3} \mathcal{Q}(V_1,V_3) ,$$

are computed in Kristensen [to appear]. and

$$\sum \gamma' \otimes \gamma'' \in \mathcal{Q}(V_1,V_3) \otimes_{V_3} \mathcal{Q}(V_1,V_3)$$

is described in connection with (2.4).

As a corollary to Proposition 2.2 we mention

Corollary 2.4. If $a,b,c \in \mathrm{Ker}(\mathcal{K} : \mathcal{A} \to \mathcal{A})$ then

$$\mathcal{K}(\langle a,b,c\rangle) \subset a\mathcal{A} + \mathcal{A}c$$

and

$$\langle a,b,c\rangle \cap \mathrm{Ker}(\mathcal{K}) \neq \emptyset .$$

Massey products of length 4 and 5 can be defined in a similar fashion. The lack of distributivity in $\mathcal{O}$ makes the defining formulas rather involved. Because of this complication we are not able to define Massey products of arbitrary length. We notice that the obvious generalization of Proposition 2.1 is valid also in the case of Massey products of length 4 and 5.

3. <u>$\mathcal{A}$-module structure of $\mathcal{A}_{(2,1)}$.</u> Let us assume

$$\mathrm{Ker}(\circ k) = \mathcal{A}(a_1,\dots,a_s) .$$

If $a_j k = \nabla \theta_j$ then

$$Q_j(x,w) = \theta_j(x) + a_j(w) + d(a_j\delta w, \delta w + kx)$$

defines an element in $\mathcal{A}_{(2,1)}$ also denoted by Q_j. The submodule $B \subset \mathcal{A}_{(2,1)}$ generated by $Q_1,\dots,Q_s$ maps onto $\mathcal{A}$ (see (1.12)). The $\mathcal{A}$-module structure of $\mathcal{A}_{(2.1)}$ is determined if we can determine the element

$$\sum b_j Q_j \in \mathrm{im}(\mathrm{coker}(\circ k))$$

whenever $\sum b_j a_j = 0$. An easy computation gives

Proposition 3.1. If $\sum b_j a_j = 0$ then

$$j^*(\langle [b_1,\ldots,b_s], \begin{bmatrix} a_1 \\ \vdots \\ a_s \end{bmatrix}, k\rangle) = \sum b_j Q_j \ .$$

If $k = q_i = Sq^{(0,\ldots,0,1)}$ then $\ker(\circ q_i) = \mathcal{Q}(q_i)$. An easy application of section 2 gives

$$\langle q_i, q_i, q_i \rangle = 0 \ .$$

Remark: In fact one can prove that even the symmetric Massey product represented by $q_i\theta + \theta q_i + d(q_i;\delta\theta,\theta\delta)$ ($\nabla\theta = q_i^2$) is zero for suitable choice of θ. All third order relations considered in this paper have a symmetric form, and all Massey products considered are symmetric, thus have less indeterminacy.

Hence we have

Proposition 3.2. If $k = q_i$ then as an $\mathcal{Q}$-module $\mathcal{Q}_{(2,1)}$ is generated by 1 and Q with relations

$$q_i \cdot 1 = 0 \ , \ q_i \cdot Q = 0 \ .$$

4. Some lemmas in the Steenrod algebra.

Let $D_j : \mathcal{Q} \longrightarrow \mathcal{Q}$ $(j = 1,2)$ be the mappings

(4.1) $\qquad D_1 a = q_i a \ , \ D_2 a = a q_i \ ,$

where $q_i \in \mathcal{Q}$ as above is dual to $\xi_i \in \mathcal{Q}^*$. Since $q_i^2 = 0$, D_1 and D_2 are differentials. The element $q_i \in \mathcal{Q}$ is primitive; thus D_1 and D_2 are derivations of the coalgebra structure. We define $D: \mathcal{Q} \longrightarrow \mathcal{Q}$ to be the sum of the above differentials, i.e. $Da = q_i a + a q_i$. This differential is a derivation of the algebra structure as well as the coalgebra structure. We notice the dual mappings on a generator $\xi_n \in \mathcal{A}^*$ take the values

$$
\begin{aligned}
&D_1^*(\xi_n) = 0 \quad \text{if } n \neq i, \quad D_1^*(\xi_i) = 1, \\
(4.2)\qquad &D_2^*(\xi_n) = \xi_{n-i}^{2^i} \quad \text{if } n \geq i, \quad D_2^*(\xi_n) = 0 \quad \text{if } n < i, \\
&D^*(\xi_n) = \xi_{n-i}^{2^i} \quad \text{if } n > i, \quad D^*(\xi_n) = 0 \quad \text{if } n \leq i.
\end{aligned}
$$

It is now very easy, using Künneth's formula, to establish the following proposition. We use the notation $\hat{\xi}$ for a cohomology class determined by the cycle ξ.

Proposition 4.1. The homology of $\mathcal{A}^*$ under D^* is as an algebra,

$$
(4.3)\qquad H(\mathcal{A}^*, D^*) = \bigotimes_{\nu=1}^{i} \frac{Z_2[\hat{\xi}_\nu]}{(\hat{\xi}_\nu^{m})} \otimes \bigotimes_{\mu=i+1}^{\infty} \frac{Z_2[\hat{\xi}_\mu^2]}{(\hat{\xi}_\mu^{m})}, \quad m = 2^i .
$$

The coalgebra structure in $H(\mathcal{A}^*, D^*)$ is induced from that of $\mathcal{A}^*$.

Let $p_j \in \mathcal{A}$ be the element dual to $\xi_j^2 \in \mathcal{A}^*$. Then by dualizing the above proposition in the case $i = 2$ (which is the case of particular interest to us) we get

Corollary 4.2. If $i = 2$, then as algebras

$$
(4.4)\qquad H(\mathcal{A}, D) /\!/ \Lambda(q_2) = \Lambda(\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots) \quad (p_0 = Sq^1)
$$

Here as usual the $A//B$ denotes $A/\bar{B}A$ whenever B is a subalgebra of A.

Let Z and Z' be the kernel and cokernel of the differential D. Since D_1 and D_2 commute with D they induce differentials on Z and Z'.

Lemma 4.3. The vectorspaces Z and A' are acyclic with respect to D_1 as well as D_2.

This lemma implies exactness in the sequence below

(4.5) $0 \to Z/Zq_i \longrightarrow \mathcal{A}/\mathcal{A}q_i \xrightarrow{D} \mathcal{A}/\mathcal{A}q_i \longrightarrow Z'/Z'q_i \longrightarrow 0$

5. The multiplicative structure of $\mathcal{A}_{(2,2)}$ when $k = Sq^{(0.1)}$

In this and the following section we specialize our considerations to the particular simple two-stage cohomology theory $H_{(2)}$ with k-invariant $k = q_2$ $(= Sq^3 + Sq^2Sq^1)$.

The $\mathcal{A}$-module $\mathcal{A}_{(2,1)}$ is known in this case by Proposition 3.2,

$$\mathcal{A}_{(2,1)} = \mathcal{A}/\mathcal{A}k \otimes \Lambda(Q) ,$$

where $\Lambda(Q)$ is the exterior algebra in one generator Q (of degree 5). Using the sequences (1.12) and (4.5) we get a short exact sequence

(5.1) $0 \longrightarrow Z'/Z'k \otimes \Lambda(Q) \xrightarrow{\alpha_*} \mathcal{A}_{(2,2)} \xrightarrow{j_*} Z/Zk \otimes \Lambda(Q) \longrightarrow 0$,

($\deg \alpha_* = -2$, $\deg j_* = 0$). Suppose $T \in \mathcal{A}_{(2,2)}$ is an element with $j_*(T) = Q$ (In section 6 we give cochain expression for a specific T with this property). Then as Z_2-vectorspaces

$$(5.2) \qquad \mathcal{A}_{(2,2)} = (Z/Zk \oplus s^{-2}(Z'/Z'k)) \otimes \Lambda(T) \ ,$$

where $s^{-2}(Z'/Z'k)$ denote the graded vectorspace $Z'/Z'k$ with degree lowered by 2. Notice that $s^{-2}(Z'/Z'k)$ has the structure of a Z/Zk bimodule, inherited from the multiplication in $\mathcal{A}$. Any function

$$a : Z/Zk \otimes Z/Zk \longrightarrow s^{-2}(Z'/Z'k)$$

with the property

$$(5.3) \qquad xa(y,z) + a(xy,z) + a(x,yz) + a(x,y)z = 0$$

defines an associative algebra structure on $Z/Zk \oplus s^{-2}(Z'/Z'k)$, (see e.g. Cartan and Eilenberg 1956 , Chapter XIV). The map a is called an extension cocycle for the algebra.

The multiplication in $Z/Zk \oplus s^{-2}(Z'/Z'k)$ is

$$(5.4) \qquad (z_1,z'_1)\cdot(z_2,z'_2) = (z_1z_2, z_1z'_2 + z'_1z_2 + a(z_1,z_2)).$$

Two mappings a and b define the same structure iff they are homologous, i.e. if there exists a mapping c: $Z/Zk \otimes Z/Zk \longrightarrow Z'/Z'k$ with

$$(5.5) \qquad xc(y) + c(x)\, y = a(x,y) + b(x,y) \ .$$

We now make $Z/Zk \oplus s^{-2}(Z'/Z'k)$ an algebra by specifying an extension cocycle $a(k)$. Let $p_i \in Z/Zk$ be as in section 4 ($p_0 = Sq^1$, $p_i = (\xi_i{}^2)^*$). If $I = \{i_1, i_2, \ldots, i_r\}$ is a set of non-negative integers, let $p_I = p_{i_1} \cdot p_{i_2} \cdots p_{i_r}$.

$$(5.6) \quad a(k)(z_1,z_2) = \begin{cases} p_{I-\{\nu\}} \cdot p_{J-\{\nu\}} \cdot p_{\nu+1} & \text{when } z_1=p_I,\ z_2=p_J \text{ and } I \cap J = \{\nu\},\ \nu \neq 0. \\ 0 & \text{otherwise} \end{cases}$$

The resulting algebra will be denoted $X(k)$. Notice that (5.6) says that the extension is almost inessential (the only non-trivial 'extension' is $(p_i,0)\cdot(p_i,0) = (0,p_{i+1})$).

The algebra $X(k)$ has a differential on it,

$$(5.7) \quad D(z,z') = (D(z'),0), \text{ where } D(z') = kz' + z'k$$

or, with other words, $X(k)$ is an algebra over the Hopf algebra $\Lambda(T)$ $(T\cdot(z_1,z_2) = D(z_1,z_2))$. Using the notion of semi-tensor product, in sign $\odot$ (Massey and Peterson [1965]), we are now ready to state the main result of this paper

Theorem 5.1. For $k = Sq^3 + Sq^2Sq^1$,

$$\mathcal{Q}_{(2,2)} = \Lambda(T) \odot X(k) .$$

6. Proof of Theorem 5.1.

The general idea behind the proof is the use cochain representatives for elements in $\mathcal{Q}_{(2,2)}$ and by this being able to express the multiplication in terms of Massey products. The Massey products involved are evaluated using Propositon 2.1 and in addition the fact that the spectrum $K_{(2)}$, $k = Sq^3 + Sq^2Sq^1$

is the bottom stage of the spectrum for connective K-theory.

Our first aim is to define a mapping $w\colon Z/Zk \oplus s^{-2}Z'/Z'k$ $\mathcal{A}_{(2,2)}$ such that

(6.1)
(i) $\mathrm{im}(w)$ is a subalgebra
(ii) $w|Z/Zk$ is a section to

$$\mathcal{A}_{(2,2)} \xrightarrow{j_*} Z/Zk \otimes \Lambda(Q) \xrightarrow{1\otimes\varepsilon} Z/Zk$$

We first need some preparational work. Let P_j and θ be elements of $\mathcal{C}$ with

(6.2) $\nabla\theta = kk$, $\nabla P_j = p_jk + kp_j$ (p_j as in section 4)

The expressions

$$(6.3)\quad \begin{aligned} &m(k^3)(x) = k\theta(x) + \theta k(x) \\ &m(k^2p_j)(x) = kP_j(x) + P_jk(x) + p_j\theta(x) + \theta p_j(x) + d(k;kp_j(x),p_jk(x \end{aligned}$$

(see (1.2))

are cocycles if x itself is a cocycle. In fact they represent the Massey products

$$(6.4)\quad \begin{aligned} &\langle k,k,k\rangle \\ &\left\langle [k,p_j],\ \begin{bmatrix} k & p_j \\ 0 & k \end{bmatrix},\ \begin{bmatrix} p_j \\ k \end{bmatrix} \right\rangle. \end{aligned}$$

Lemma 6.1. There exist choices of θ and P_j such that the Massey products $m(k^3)$ and $m(k^2p_j)$ both are zero (in $\mathcal{A}$).

The proof of the above lemma is very typical for the arguments used in this paper. We therefore give an outline.

According to Proposition 3.2 we may choose θ such that $m(k^3) = 0$. Using Proposition 2.1 with $a = [k,p_j]$, $b = c = \begin{bmatrix} k & p_j \\ 0 & k \end{bmatrix}$, $d = \begin{bmatrix} p_i \\ k \end{bmatrix}$ we get

$$[k, m(k^2 p_j)] = [p_j, m(k^3)] = 0 .$$

Thus $m(k^2p_j)$ determines an element in $H(\mathcal{Q},D)$ (cf. section 4). If we vary the choice of p_j by a primary cochain operation a ($\nabla a = 0$) the Massey product will vary with $[k, a]$. It is therefore enough to prove that $m(k^2p_j)$ is zero in $H(\mathcal{Q},D)$. From Proposition 4.1 we get

$$m(k^2p_j) = \lambda \cdot p_1 p_j k \; , \quad \lambda \in Z_2 .$$

Finally we apply Proposition 2.3 to see that $\lambda = 0$.

We need one more element in $\mathcal{O}$. Let $\chi \in \mathcal{O}$ be such that

(6.5) $\quad \nabla \chi = k\theta + \theta k + d(k;\delta\theta,\theta\delta)$

The 4-fold Massey product $m(k^4) = \langle k,k,k,k \rangle$ is defined as follows on a cocycle x,

(6.6) $m(k^4)(x) = k\chi(x) + \chi k(x) + \theta\theta(x) + d(k;k\theta(x),\theta k(x)).$

Using essentially only Proposition 2.1. (generalized to 4-fold products) one may prove

Lemma 6.2. There is a choice of $\chi \in \mathcal{O}$ such that $m(k^4) = 0$ in $\mathcal{Q}$.

The expressions

$$w(p_j)(x,w) = (p_j(x), P_j(x) + p_j(w)),$$

(6.7) $$w([a,k])(x,w) = ([a,k](x), [a,\theta](x) + [a,k](w) + d(k;ak(x),ka(x)),$$

$$T(x,w) = (\theta(x)+k(w), \chi(x)+\theta(w)+d(k;\theta(x),\theta(x)+k(w))),$$

($a \in Z\mathcal{O}$, $[a,k] = ak + ka$) are cocycles in $C_{(2)}$ whenever (x,w) is a cocycle ($\delta x = 0$, $\delta w = k(x)$). A (non-trivial) computation shows that all three expressions define stable cohomology operations, i.e. elements $w(p_j)$, $w([a,k])$ and T in $\mathcal{A}_{(2,2)}$.

Define

(6.8) $$w(p_{j_1} \dots p_{j_n}) = w(p_{j_1}) \cdot w(p_{j_2}) \cdot \dots \cdot w(p_{j_n})$$

whenever $0 \leq j_1 < j_2 < \dots < j_n$. From Corollary 4.2 we have

(6.9) $$Z/Zk = \Lambda(p_0, p_1, \dots) \oplus \text{im } D, \quad (D(a) = [a,k])$$

Hence, all together, we get a map $w = Z/Zk \longrightarrow \mathcal{A}_{(2,2)}$. Taking w on $Z'/Z'k$ to be α_* (see (5.1)) we have obtained a mapping

$$w = Z/Zk \oplus s^{-2}(Z'/Z'k) \longrightarrow \mathcal{A}_{(2,2)} .$$

It is an easy consequence of the defining expressions (6.7) that w has the required properties (6.1).

Proposition 6.3. The mapping w is a homomorphism of algebras $w\colon X(k) \longrightarrow \mathcal{A}_{(2,2)}$, where $X(k)$ is the algebra defined in section 5.

Proof. The proof consists of checking all the relevant relations such as $[w(p_i,0),w(p_j,0)] = 0 \ (i \neq j)$, $w(p_j)^2 = w(0,p_{j+1})$ etc. We sketch the crucial fact, that is, $w(p_j)^2 = w(0,p_{j+1})$. A computation on cochain level gives

$$w(p_j)^2(x,w) = (0,m(kp_j^2)(x)) \quad (\delta x = 0, \ kx = \delta w)$$

where

$$m(kp_j^2) = P_j \cdot p_j + p_j \cdot P_j + kR_j + R_j k + d(p_j;p_jk(\ ),kp_j(\ )),$$

and $R_j \in \mathcal{C}$ has $\nabla R_j = p_j p_j$.

It is clear that $m(kp_j^2)$ represents the Massey product

$$\left\langle [p_j,k] , \begin{bmatrix} p_j & k \\ 0 & p_j \end{bmatrix} , \begin{bmatrix} k \\ p_j \end{bmatrix} \right\rangle .$$

As in Lemma 5.2 $m(kp_j^2) \in \mathcal{A}$ is a cycle under D and it is therefore enough to determine $m(kp_j^2)$ in $H(\mathcal{A},D)/H(\mathcal{A},D)\cdot k$. Using Corollary 4.2, $m(kp_j^2) = \lambda p_{j+1}$ $(\lambda \in Z_2)$. Note that p_{j+1} would be in the indeterminacy of $m(kp_j^2)$ if we were allowed to choose P_j freely. However, P_j has to be chosen such that $m(k^2p_j) = 0$ (cf. Lemma 5.1). We now apply Proposition 2.3. to get $\lambda = 1$.

Remark: In the above use of Proposition 2.3 it is essential to have some information about the terms $\Sigma\, \alpha_0' \otimes \alpha_0''$ and $\Sigma\, \beta_0' \otimes \beta_0''$ (see Kristensen [to appear]).

Proposition 6.4. The element $T \in \mathcal{A}_{(2,2)}$ commutes with the subalgebra $X(k) \subseteq \mathcal{A}_{(2,2)}$.

Proof. A computation on cochain level using (6.7) gives

$$[T, D(a)] = \alpha_*([a, m(k^4)]) ,$$
(6.8)
$$[T, p_i] = \alpha_*(m(k^3 p_i)) ,$$

where $m(k^4)$ and $m(k^3 p_i)$ are the Massey products

$$m(k^4) = \langle k,k,k,k \rangle ,$$
(6.6)
$$m(k^3 p_i) = \left\langle [k, p_i] , \begin{bmatrix} k & p_i \\ 0 & k \end{bmatrix}, \begin{bmatrix} k & p_i \\ 0 & k \end{bmatrix}, \begin{bmatrix} k \\ p_i \end{bmatrix} \right\rangle .$$

These Massey products are defined since we already proved that $m(k^3)$ and $m(k^2 p_i)$ are zero. The obvious generalization of Proposition 2.1 gives

(6.7) $D(m(k^4)) = 0$ and $D(m(k^3 p_i)) = [p_i, m(k^4)]$

The indeterminacy on $m(k^4)$ coming from choice of $\chi \in \mathcal{C}$ ($\nabla \chi = \langle k,k,k \rangle$) is $\mathrm{im}(D)$. Since $H^{10}(\mathcal{A}, D)/H^7(\mathcal{A}, D)\cdot k=0$ we can choose χ such that $m(k^4) = 0$ in $\mathcal{A}/\mathcal{A}k$.

Remark. By a heavy use of Proposition 2.1 (generalized to 4-fold Massey products) one can in fact prove that $m(k^4) = 0$ in $\mathcal{A}$. We will not give this rather technical proof here. However, the result is used below.

In order to prove that $\alpha_*(m(k^3 p_i)) = 0$ or, equivalently, that $m(k^3 p_i) = 0$ in $Z'/Z'k$ we use the well-known spectral sequence (Atiyah-Hirzebruch [1960])

$$(6.8)\qquad \begin{aligned} E_2 &= H(\underset{\sim}{X}, k(pt;Z_2)) , \\ E_\infty &= E_0(k(\underset{\sim}{X};Z_2)) . \end{aligned}$$

Here $\underset{\sim}{X}$ is a spectrum and k is the connective K-theory with coefficients in Z_2.

Let $L_2 = S^1 \cup_2 e^2$ and let $BU(t,\ldots,\infty)$ be the $(t-1)$ connective covering of BU. We put $BU_2(t,\ldots,\infty) = BU(t+2,\ldots,\infty)^{L_2}$, where X^Y denotes the space of maps from Y to X (compact-open topology). The spaces $BU_2(t,\ldots,\infty)$ obviously form an Ω-spectrum. This is the spectrum for $k(-;Z_2)$. We shall study the spectral sequence (6.8) with $\underset{\sim}{X} = \{BU_2(t,\ldots,\infty)\}$. According to Adams [1961] (see also Maunder [1967]).

$$H^*(BU_2(2t+2,\ldots,\infty);Z_2) = \mathcal{Q}/\mathcal{Q}k\ (i_{2t})$$

in the stable range. Hence if $\underset{\sim}{X} = \{BU_2(t,\ldots,\infty)\}$ we get

$$E_2^{p,2q} = (\mathcal{Q}/\mathcal{Q}k)_p$$

We need the following lemma concerning the differentials in this spectral sequence

Lemma 6.3. The first few differentials are

(i) $d_3(a) = D(a)$

(ii) $d_5(a) = m(k^2a)$

(iii) $d_7(a) = m(k^3a)$

Proof. The first three k-invariants in $BU_2(t,\dots,\infty)$ are k, $Qu(k^2)$ and $Qu(k^3)$. Here $Qu(k^2)$ is the secondary operation associated with the relation $k^2 = 0$ while $Qu(k^3)$ is the tertiary operation associated with the relation $kQu(k^2) = 0$ (or $\langle k,k,k\rangle = 0$) (see e.g. Stong [1963]). If one uses the description of the above spectral sequence given in Koch, Kristensen and Madsen [1967a]. (20) p. 175) it is not hard to compute the differentials and obtain the claimed results.

It is well-known (and easy to prove) that the only non-vanishing differential in this spectral sequence is d_3 (Maunder [1967]). Thus in particular $m(k^3p_i) = 0$ in E_4, i.e. $\hat{m}(k^3p_i) = 0$ in $H(\mathcal{A},D)/H(\mathcal{A},D)k$. Since the indeterminacy on $m(k^3p_i)$ contains $\mathrm{im}(D)$ we have for an appropriate choice that $m(k^3p_i) = 0$ in $Z'/Z'k$.

We close this section by

Proposition 6.4. The element $T \in \mathcal{A}_{(2,2)}$ has $T^2 = 0$.

Proof. Computing on cochain level we get $T^2 = \alpha_*(m(k^5))$, where $m(k^5)$ is the Massey product

$$m(k^5) = \langle k,k,k,k,k\rangle .$$

This product is defined since $m(k^4) = 0$.
Now we use $Dm(k^5) = 0$ and $H^{12}(\mathcal{A},D)/H^9(\mathcal{A},D)k = 0$.

7. Adams spectral sequence.

Let X and Y be CW complexes of finite type and let $\{X,Y\}_{(2)}$ denote the 2-primary component of the track group $\{X,Y\}$. Let $H_{(2)}$ be an arbitrary two stage cohomology theory with k-invariant k and operation algebra $\mathcal{Q}_{(2,2)}$.

Proposition 7.1. There is a spectral sequence $(E_r(X,Y),d_r)$, natural in X and Y, with

$$E_2 = \operatorname{Ext}_{\mathcal{Q}_{(2,2)}}(H_{(2)}(Y),H_{(2)}(X))$$

$$E_\infty = E_0(\{X,Y\}_{(2)}) .$$

If Z is another CW-complex of finite type, there is a pairing of spectral sequences ,

$$E_r(X,Y) \otimes E_r(Y,Z) \longrightarrow E_r(X,Z) .$$

This pairing is compatible with the composition pairing in homotopy.

Proof. Except for the converging part which is treated in Lemma 7.3 below the theorem is proved by appealing to work of Moss [1968].

Definition 7.2. An element $\alpha \in \{X,Y\}_{(2)}$ is said to have Adams filtration $\geq i$ with respect to a cohomology theory h if there exists a sequence of spaces $Y_0, Y_1, \ldots, Y_i$ and maps $y_j = Y_j \longrightarrow Y_{j-1}$ such that

(i) $y_j^* = h(Y_{j-1}) \longrightarrow h(Y_j)$ is zero

(ii) $Y_0 = S^{\nu} Y$ for some ν

(iii) $\alpha \in \operatorname{im}(\{X, Y_i\}_{(2)} \longrightarrow \{X, Y\}_{(2)})$

Let m be the smallest integer for which $k^m = 0$, where k is the k-invariant of $H_{(2)}$.

Lemma 7.3. If an element $\alpha \in \{X, Y\}_{(2)}$ has Adams filtration less than or equal to i with respect to the usual cohomology functor H, then α has Adams filtration less than or equal to $m(i+1) - 1$ with respect to $H_{(2)}$.

Proof. Let $f\colon Z \to W$ be a map with $f^*\colon H_{(2)}(W) \to H_{(2)}(Z)$ the zero map. We use the functorial sequence (1.6) to conclude that $f^*(w) \in \operatorname{im}(k\colon H(Z) \longrightarrow H(Z))$ for any $w \in H(W)$. Thus if $Z_1 \xrightarrow{f_1} Z_2 \xrightarrow{f_2} Z_3 \longrightarrow \ldots \xrightarrow{f_{n-1}} Z_n$ is a string with $f_i^*\colon H_{(2)}(Z_{i+1}) \longrightarrow H_{(2)}(Z_i)$ the zero map then

$\operatorname{im}((f_{n-1} \ldots f_1)^*\colon H(Z_n) \to H(Z_1)) \subseteq \operatorname{im}(k^m\, H(Z_1) \to H(Z_1)) = 0$

and the lemma follows.

We owe this proof to D. Anderson.

Obviously there are serious problems in connection with the determination of $H^{**}(\mathcal{A}_{(2,2)})$. Low dimensional computations, however, seem to reveal interesting phenomena.

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