LOOP OPERATIONS

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In this paper we introduce a new class of higher order "cohomology" operations of several variables from $H^*(X)$ to $H^*(\Omega X)$, the loop operations. The k^{th} order loop operation $< U_1,\ldots,U_k >_\Omega$ is defined as a subset of $H^{n-2}(\Omega X)$ if the k-fold matrix Massey product $< U_1,\ldots,U_k >$ contains 0 in $H^n(X)$. The primary loop operation is the loop suspension homomorphism σ. The secondary loop operation is related to the operation $<a,b,\Omega>$ defined by Spanier (p.522 [11]).

Loop Operations can be axiomatized, they can be described by universal examples or with the Eilenberg Moore spectral sequence, or they can be constructed from cochains using the main result of the author's paper [5]. We shall start with the last approach.

Throughout this paper we shall assume that X is a simply connected space whose singular cohomology over a fixed P.I.D. R has finite type. Let $C^*(X)$ be the normalized singular cochain algebra modified so that $C^0(X) \approx R$ and $C^1(X) = 0$. Finally let $QC^*(X)$ be the quotient complex of indecomposable cochains.

Theorem 1. With the above notation

$$\widetilde{H}^q(\Omega X) \approx \widetilde{H}^q(s^{-1}QC^*(X))$$

where $(s^{-1}A)^q = A^{q+1}$ for a complex A.

This is the main result of [5]. As a corollary of the proof of this theorem we have that the natural projection

$$\overline{B}C^*(X) \longrightarrow \overline{C}^*(X) \longrightarrow QC^*(X)$$

from the reduced bar construction of $C^*(X)$ to the indecomposable cochains is a cochain equivalence.

The projection $\tau : C^q(X) \longrightarrow (s^{-1}QC^*(X))^{q-1}$ is a cochain map of degree -1. If we identify $s^{-1}QC^*(X)$ with $C^*(\Omega X)$, then τ induces a homomorphism

$\sigma : \widetilde{H}^q(X) \longrightarrow \widetilde{H}^{q-1}(\Omega X)$ which is the ordinary loop suspension homomorphism.

Recall that the k-fold Massey product $\mathcal{U} = \langle u_1, \ldots, u_k \rangle$ is defined as a subset of $H^n(X)$ if there is a defining system of cochains $(a_{i,j})$, $1 \le i \le j \le k$, $(i,j) \neq (1,k)$, such that $a_{i,i}$ is a cocycle representative of u_i and

$$\delta a_{i,j} = \tilde{a}_{i,j} = \sum_{r=i}^{j-1} \overline{a}_{i,r}\, a_{r+1,j}$$

where $\overline{a} = (-1)^{\dim a} a$ (see [4]). This definition can be trivially extended to the case where $(U_1, \ldots, U_k)$ is a multipliable system of matrices with entries in $\widetilde{H}^*(X)$. That is U_i and U_{i+1} are multipliable and the matrix product $U_i \ldots U_j$ is a matrix whose entries are homogeneous classes and which is 1×1 if $(i,j) = (1,k)$ (see [7]).

If $0 \in \mathcal{U}$, then there is a defining system $(a_{i,j})$ for $\mathcal{U}$ and a cochain $a_{1,k}$ such that $\delta a_{1,k} = \tilde{a}_{1,k}$.

We will call the total set of cochains $A = (a_{i,j})$, for all $1 \le i \le j \le k$, a total defining system for $\mathcal{U}$.

Assume $0 \in \mathcal{U}$ and let A be a total defining system for $\mathcal{U}$. Then $\tau\, a_{1,k}$ is a cocycle of $QC^*(X)$ since $\delta a_{1,k}$ is decomposable.

Definition 2. Let $(U_1, \ldots, U_k)$ be a multipliable system of matrices with entries in $\tilde{H}^*(X)$. We say that the k^{th} order loop operation $\mathcal{U}_\Omega = \langle U_1, \ldots, U_k \rangle_\Omega$ is defined if $k = 1$ or if the matrix Massey product $\mathcal{U} = \langle U_1, \ldots, U_k \rangle$ contains $0 \in H^n(X)$. If it is defined, then $\mathcal{U}_\Omega$ is $\sigma\, u_1$ if $k = 1$ and otherwise it is the set of all classes $w \in H^{n-2}(\Omega X)$ for which there is a total defining system A with $\tau\, a_{1,k}$ a cocycle representative of w. In addition we set $\langle\,\rangle_\Omega = 1 \in H^0(\Omega X)$.

Note that if $\mathcal{U}_\Omega$ is defined, so is $(\mathcal{U}_{i,j})_\Omega = \langle U_i, \ldots, U_j \rangle_\Omega$ for $1 \le i \le j \le k$ and that the latter is in general not zero. Also the loop operations are clearly natural. These operations are not cohomology operations in the usual sense since they go from $H^*(X)$ to $H^*(\Omega X)$, but $\mathcal{U}_\Omega$ is a k^{th} order operation in the sense of Spanier [11].

The indeterminacy of $\mathcal{U}_\Omega$ can be described in terms of lower order matrix loop operations. More precisely, if $x, y \in \mathcal{U}_\Omega$, then either $x - y \in \operatorname{Im} \sigma$ or there are matrices $V_1, \ldots, V_{k-r}$ for some $r \ge 1$ such that

$$\langle (U_1\ V_1) \begin{pmatrix} U_2 & V_2 \\ 0 & U_{r+2} \end{pmatrix}, \ldots, \begin{pmatrix} V_{k-r} \\ U_k \end{pmatrix} \rangle_\Omega$$

is defined and contains $x - y$. For example the indeterminacy of $< u,v,w >_\Omega$ is Im σ together with classes in $<(u\,x)\binom{y}{w}>_\Omega$. These latter loop operations are defined if $uy + xw = 0$, that is an element of the indeterminacy of the Massey triple product $< u,v,w >$ vanishes.

Example 3. Let $u \in H^m(X)$ and $v \in H^n(X)$ and assume $uv = 0$. Then there is a cochain $c \in C^{m+n-1}(X)$ such that $\delta c = \bar{a}b$ where a and b represent u and v respectively. Clearly $< u,v >_\Omega \subset H^{m+n-2}(\Omega X)$ is the set of all classes $\{\tau(c+z)\}$ where z ranges over all cocycles in $C^{m+n-1}(X)$. But $\{\tau z\} = \sigma\{z\}$ and so $< u,v >_\Omega$ is a coset of $\sigma H^{m+n-1}(X)$ in $H^{m+n-2}(\Omega X)$.

Example 4. Let u and v be as above, but do not assume $uv = 0$. By commutativity

$$< (u\ v)\ \binom{\pm v}{u} >_\Omega \text{ is defined and}$$

is represented by $\pm\tau(a \smile_1 b)$. It will be seen in Example 17 that $\pm\sigma u\,\sigma v$ is in this coset.

As an application of the above let $X = S^m \vee S^n$ and let u and v be the generators of $H^*(X)$ in dimensions m and n respectively. Then we may express the low dimensional classes of $H^*(\Omega X)$ without indeterminacy as follows:

dimension	generator
0	$< >_\Omega = 1$
$m-1$	$< u >_\Omega = \sigma u$
$n-1$	$< v >_\Omega = \sigma v$
$2m-2$	$< u,u >_\Omega$
$2n-2$	$< v,v >_\Omega$
$m+n-2$	$< u,v >_\Omega$ and $< v,u >_\Omega$.

Furthermore we have the relation

$$< u,v >_\Omega \pm < v,u >_\Omega = \pm < u >_\Omega \ < v >_\Omega .$$

We now list some elementary properties of the loop operations.

Property 5. if $r \in R$, then $r < U_1,\dots,U_k >_\Omega \subset < U_1,\dots,rU_i,\dots,U_k >_\Omega$

Property 6. $< U_1,\dots,U_k >_\Omega + < V_1,\dots,V_k >_\Omega \subset$

$$< (U_1 \ V_1), \begin{pmatrix} U_2 & 0 \\ 0 & V_2 \end{pmatrix}, \dots, \begin{pmatrix} U_k \\ V_k \end{pmatrix} >_\Omega .$$

Property 7. If W is a matrix with entries in $H^*(X)$ and if both loop operations are defined, then

$$< U_1,\dots,U_iW,\dots,U_k >_\Omega = < U_1,\dots,\bar{W}U_{i+1},\dots U_k >_\Omega .$$

The last two properties imply that

$$< u_1,\dots,u_i',\dots,u_k >_\Omega + < u_1,\dots,u_i'',\dots,u_k >_\Omega \subset < u_1,\dots,u_i'+u_i'',\dots,u_k$$

since $(u \ u) = u\,(1,1)$ and $(1,1)\begin{pmatrix} u & o \\ o & u \end{pmatrix} = (u \ u)$ (See Lemma 1[6]). Note that the entries of W in Property 7 are allowed to be from $H^o(X)$. In fact if

$< U_1,\dots,U_k >_\Omega$ is defined and W is a square matrix with entries only in $\tilde{H}^*(X)$, such that $(U_1,\dots,U_iW,\dots,U_k)$ is a multipliable system, then a generalization of the proof of Theorem 6 [4] shows that $< U_1,\dots,U_iW,\dots,U_k >_\Omega$ is defined and contains 0.

From this point on we shall assume that R is a prime field. This condition could be weakened, but then the results become more technical.

Under this condition the Eilenberg Moore spectral sequence [3] has

$E_2 = \mathrm{Tor}_{H^*(X)}(R,R)$. J. P. May [6] has

described the elements of this module as equivalence classes of slide cycles. These equivalences classes are written.

$\{U_1,\ldots,U_k\}$, where $(U_1,\ldots,U_k)$ is a multipliable system and $U_i U_{i+1} = 0$ for $i = 1,\ldots,k-1$. The equivalence relation is essentially Property 7.

Theorem 8. If $\mathcal{U}_\Omega = \langle U_1,\ldots,U_k\rangle_\Omega$ is defined in $H^*(\Omega X)$, then the slide cycle $\{U_1,\ldots,U_k\}$ lives to an element θ in E_∞. Furthermore if we consider θ as a coset of a subgroup of $H^*(\Omega X)$, then $\mathcal{U}_\Omega \subset \theta$. Conversely if $\theta \in E_\infty$, then θ is represented by a slide cycle $\{U_1,\ldots,U_k\}$ in E_2 such that $\mathcal{U}_\Omega = \langle U_1,\ldots,U_k\rangle_\Omega$ is defined. Finally for every $w \in H^*(\Omega X)$, there is a loop operation $\mathcal{U}_\Omega$ with $w \in \mathcal{U}_\Omega$.

The proof of this theorem uses Theorem 10 and Corollary 18 of [6], which characterizes the differentials in the Eilenberg Moore spectral sequence, together with the cochain equivalence $\overline{B}\, C^*(X) \longrightarrow QC^*(X)$. This theorem implies that the loop operations are related to the operations determined by the differentials in the Eilenberg Moore spectral sequence. But the loop operations have, in general, considerably less indeterminacy. Thus theorems about loop operations give results about the differentials.

The reduced bar construction has a coalgebra structure induced by the diagonal map on $\overline{B}C^*(X)$ determined by

$$\psi[a_1,\ldots,a_k] = \sum_{r=o}^{k} [a_1,\ldots,a_r]\otimes[a_{r+1},\ldots,a_k].$$

$H^*(\Omega X)$ has a Hopf algebra structure and the map

$$\mathrm{Tor}_{C^*(X)}(R,R) \longrightarrow H^*(\Omega X)$$

is a coalgebra isomorphism (see [3]). In fact for $R = Z_2$, B. Drachman [2] has introduced a product structure on $\overline{B}C^*(X)$ so that the map is a Hopf algebra isomorphism.

Theorem 9. Assume $\mathcal{U}_\Omega = < u_1,\ldots,u_k >_\Omega$ is defined in $H^{n-2}(\Omega X)$. Then

$$(\mathcal{U}_{i,j})_\Omega = < u_i,\ldots,u_j >_\Omega \text{ is defined}$$

and $\psi(\mathcal{U}_\Omega) \subset \sum_{r=o}^{k} (\mathcal{U}_{1,r})_\Omega \otimes (\mathcal{U}_{r+1,k})_\Omega$

The generalization of this theorem to matrix loop operations is more difficult to state but as easy to prove.

Example 10. Assume that $X = SY$. Then all Massey products are defined and vanish in $H^*(X)$. Define a coalgebra map from $T\widetilde{H}^*(Y)$ to $H^*(\Omega SY)$ by sending $v_1\otimes\ldots\otimes v_k$ to $< sv_1,\ldots,sv_k >_\Omega$ where $s : H^q(Y) \longrightarrow H^{q+1}(SY)$ is the suspension isomorphism.

Note the loop operation is always defined. Using the fact that σ is a monomorphism together with Theorem 9, this map is seen to be a monomorphism. Using Properties 6 and 7 the map is seen to be an epimorphism. This gives an easy proof that $H^*(\Omega SY) \approx T\widetilde{H}^*(Y)$ as coalgebras.

Later in the paper we shall see that this map is an isomorphism of algebras over the Steenrod algebra modulo the indeterminacy of loop operations. More precisely there are formulas mod 2

$$Sq^t < u_1,\ldots,u_k >_\Omega \subset \sum_{\Sigma a_i = t} < Sq^{a_1} u_1,\ldots,Sq^{a_k} u_k >_\Omega$$

and

$$< u_1,\ldots,u_k >_\Omega < v_1,\ldots,v_j >_\Omega \subset \sum < x_1,\ldots,x_{k+j} >_\Omega$$

where the sum is taken over all shuffles $(x_1,\ldots,x_{k+j})$ of $(u_1,\ldots,u_k)$ and $(v_1,\ldots,v_j)$.

Example 11. Let X be the n dimensional complex projective space CP^n. Then $H^*(X)$ is a truncated polynomial algebra of height n on a 2 dimensional class u. Also $H^*(\Omega X) \approx E(x) \otimes \Gamma(y)$ where $E(x)$ is the exterior algebra on a 1 dimensional class x and $\Gamma(y)$ is the divided power algebra on a 2n dimensional class y. Then with no indeterminacy.

$$x = < u >_\Omega$$

$$y = y_1 = < u^{n-j}, u^{j+1} >_\Omega \text{ for all } j = 1,\ldots,n-1.$$

$$y_i = < u, u^n, \ldots, u, u^n >_\Omega \text{ (2i fold)}$$

$$xy_i = < u, u^n, \ldots, u^n, u >_\Omega \text{ (2 i + 1 fold).}$$

If $X' = CP^{\infty} \rtimes S^{2n+1}$ then $\Omega X'$ has the same homotopy type as ΩX and so $H^*(\Omega X') \approx E(x') \otimes \Gamma(y')$ where $\dim x' = 1$ and $\dim y' = 2n$. However the classes have quite different expressions in terms of loop operations. We still have $x' = \langle u' \rangle_\Omega$ where $u' \in H^2(X')$, but now $y'_i = \langle v', \ldots, v' \rangle_\Omega$ (i fold) where $v' \in H^{2n+1}(X')$. Note that in both cases the divided powers are neatly expressable in terms of loop operations.

Now assume that $f : Y \longrightarrow X$ and $u, v \in H^*(X)$ satisfy $f^* u = 0$ and $u\, v = 0$. Then the functional cup product $\bar{u} \smile_f v$ is defined as a coset of $f^* H^{m+n-1}(X) + H^{m-1}(Y) f^*(v)$ in $H^{m+n-1}(Y)$.

Theorem 12. $\sigma(\bar{u} \smile_f v) = (\Omega f)^* \langle u, v \rangle_\Omega$ as cosets of $\sigma f^* H^{m+n-1}(X)$ in $H^{m+n-2}(\Omega Y)$.

Proof. Recall that $\bar{u} \smile_f v$ is represented by the cocycle $\tilde{c}_2 = (f^\#(a_{1,2}) - \bar{c}_1\, f^\#(a_2))$ where $a_{1,2} = \bar{a}_1 a_2$ and $\delta c_1 = f^\# a_1$. Since $\tau\, c_2 = \tau\, f^\#(a_{1,2}) = (\Omega f)^\#\, \tau a_{1,2}$, the result clearly follows.

This Peterson Stein type formula can be generalized to arbitrary loop operations. The generalization, together with Theorem 9, can be used to axiomitize the loop operations. Uniqueness can be shown using universal examples, which we now describe.

Definition 13. Let $\mathcal{N} = (N_1, \ldots, N_k)$ be a set of matrices whose entries are integers. Replace each integer $n_{r,s}$ in N_i by an arbitrary cohomology class $v_{r,s}$ of that dimension. If the set of matrices $(V_1, \ldots, V_k)$ is a multipliable system, then we say that $\mathcal{N}$ is the dimension of $(V_1, \ldots, V_k)$.

Theorem 14. Let $\mathfrak{N}$ be the dimension of some multipliable system. Then there is a space $E = E(\mathfrak{N})$ with the following properties.

a) There is a multipliable system $(V_1, \ldots, V_k)$ in $H^*(E)$ with dimension $\mathfrak{N}$ such that $\mathcal{V}_\Omega = \langle V_1, \ldots, V_k \rangle_\Omega$ is defined and contains a canonical class β.

b) If $(U_1, \ldots, U_k)$ is a multipliable system in $H^*(X)$ with dimension N such that $\mathcal{U}_\Omega = \langle U_1, \ldots, U_k \rangle_\Omega$ is defined, then for every $w \in U_\Omega$ there is a map $f : X \longrightarrow E$ such that $f^* V_i = U_i$ and $(\Omega f)^* \beta = w$.

c) ΩE has the homotopy type of a product of Eilenberg MacLane spaces.

d) The submodule of primitive classes $PH^*(\Omega E) = \operatorname{Im} \sigma$.

Such universal examples have been constructed for the non-matrix case by G. Porter [9]. He shows that properties a), b) and c) are satisfied by these spaces. His methods can be extended to the matrix case or semi simplicial theory can be used. Property d) implies that although ΩE splits as a product of $K(\pi, n)$'s as a space, it definitely does not split that way as an H-space.

Using this theorem we show that the shuffle product and the Cartan formula on $E_1 = T\widetilde{H}^*(X)$ of the Eilenberg Moore spectral sequence pass to the limit and induce the correct formula on the associated graded module $E^0 H^*(\Omega X)$. These formulas will give us information about the structure of $H^*(\Omega X)$ as an algebra over the Steenrod algebra. V. Puppe [10] has proven the Cartan formula, mod 2, using entirely different techniques.

Property 15. Assume that $\mathcal{U}_\Omega = < u_1, \ldots, u_k >_\Omega$ and $\mathcal{V}_\Omega = < v_1, \ldots, v_j >_\Omega$ are defined and that $k \leq j$. Then $\mathcal{W}_\Omega = < W_1, \ldots, W_{j+k} >_\Omega$ is defined and

$$\mathcal{U}_\Omega \mathcal{V}_\Omega \subset \mathcal{W}_\Omega \text{ (Product formula)}$$

where, up to sign,

$$W_i = \begin{cases} \begin{pmatrix} u_i & v_1 & 0 & \cdots & 0 \\ 0 & u_{i-1} & v_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & v_i \end{pmatrix} & \text{if } i \leq k \\[2ex] \begin{pmatrix} v_{i-k} & \cdots & 0 & 0 \\ u_k & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & u_1 & v_i \end{pmatrix} & \text{if } k < i \leq j \\[2ex] \begin{pmatrix} v_{i-k} & \cdots & 0 \\ u_k & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & v_j \\ 0 & \cdots & u_{i-j} \end{pmatrix} & \text{if } j < i \end{cases}$$

For example, up to sign,

$$< u_1, u_2 >_\Omega < v_1, v_2, v_3 >_\Omega \subset$$

$$< (u_1, v_1) \begin{pmatrix} u_2 & v_1 & 0 \\ 0 & u_1 & v_2 \end{pmatrix} \begin{pmatrix} v_1 & 0 & 0 \\ u_2 & v_2 & 0 \\ 0 & u_1 & v_3 \end{pmatrix} \begin{pmatrix} v_2 & 0 \\ u_2 & v_3 \\ 0 & u_1 \end{pmatrix} \begin{pmatrix} v_3 \\ u_2 \end{pmatrix} >_\Omega .$$

This is nothing but the shuffle product disguised in matrix form. If all loop operations were defined, then by Property 6 and 7 the product formula reduces to the situation described in Example 10. The generalization to matrix loop operations is again more difficult to state but nearly as easy to prove.

Property 16. Let $R = Z_p$ and let P^t be the Steenrod p^{th} power if p is odd or Sq^t if p = 2. Define

$$\mathcal{P}^t U = \begin{pmatrix} P^0U & \cdots & P^tU \\ & \ddots & \vdots \\ 0 & & P^0U \end{pmatrix}$$

and let $\mathcal{P}_R^t U$ be the top row and

$\mathcal{P}_C^t U$ be the last column of the above matrix. Then (compare Milgram [8])

$$(\mathcal{P}^t \mathcal{U})_\Omega = \langle \mathcal{P}_R^t U_1, \ldots, \mathcal{P}^t U_i, \ldots, \mathcal{P}_C^t U_k \rangle_\Omega$$

is defined and $P^t(\mathcal{U}_\Omega) \subset (\mathcal{P}^t \mathcal{U})_\Omega$ (Cartan Formula)

We prove both of these formulas in the universal example by induction on the order of the loop operations. By applying ψ to both sides of the above inclusions and using the inductive hypothesis, it is seen that the formulas are true up to a primitive. But $PH^*(\Omega E) = \operatorname{Im} \sigma$ which is contained in the indeterminacy of the right hand sides. The formulas then follow immediately.

Example 17. The product formula states that, up to signs, $\sigma u \sigma v = \langle u \rangle_\Omega \langle v \rangle_\Omega \subset \langle (u,v)\binom{v}{u} \rangle_\Omega$. If a and b represent u and v respectively then, up to sign,

$$\delta(a \smile b) = ab + ba = (a\ b)\binom{b}{a} \text{ and}$$

so $\tau(a \smile b)$ represents $\sigma u\ \sigma v$. Thus the $\smile$ product in $C^*(X)$ induces the cup product on $\operatorname{Im}\sigma \subset H^*(\Omega X)$. (See Example 4).

Note, however, if $R = Z_2$ and $u = v$ then $aa + aa = 0$ and so $0 \in \langle (u,u)\binom{u}{u} \rangle_\Omega$. On the other hand $(\sigma u)(\sigma u) = Sq^{n-1}\sigma u$ need not be 0. But it is contained in Im σ and thus in the indeterminacy of $\langle (u,u),\binom{u}{u} \rangle_\Omega$, so there is no contradiction.

This example does show, however, that the product formula does not give complete information about the algebra structure of $H^*(\Omega X)$. Similarly the Cartan formula cannot be used to detect the nontrivial Steenrod operations in $H^*(\Omega^2 S^k)$. Yet we have seen in Example 11 that the operations do detect the divided powers. We close this paper with two last examples.

Example 18. Let $X = S^{k-1}CP^2$, $k \geq 2$, $R = Z_2$, and let $u \in H^k(X)$ and $v \in H^{k+2}(X)$ be the generators. Then $H^*(\Omega X)$ is generated by loop operations $\langle w_1,\ldots,w_r \rangle_\Omega$ where w is u or v and r is arbitrary. In particular there are non zero classes with no indeterminacy

$$\alpha_r = \langle u,\ldots,u \rangle_\Omega \qquad (r\text{-fold})$$
$$\beta_r = \langle v,\ldots,v \rangle_\Omega$$

with $\dim \alpha_r = r(k-1)$ and $\dim \beta_r = r(k+1)$. Since $Sq^2 u = v$, the Cartan formula shows that $Sq^{2r}\alpha_r = \beta_r$ for all r.

Example 19. Borel [1] has shown that

$H^*(B\,Spin(10);\ Z_2) = Z_2[w_4, w_6, w_7, w_8, w_{10}, x_{32}]/(w_7 \cdot w_{10})$

where $(w_7\ w_{10})$ is the ideal generated by the product $w_7\ w_{10}$. It can be checked that $x_{32} = \langle w_7, w_{10}, w_7, w_{10} \rangle$. Also it is not hard to see that $H^*(Spin(10); Z_2)$ is an exterior algebra on primitive classes $\langle w_i \rangle_\Omega$ for $i = 4,6,7,8,10$ and the class $y_{15} = \langle w_7, w_{10} \rangle_\Omega$ where $\psi y_{15} = 1 \otimes y + \langle w_7 \rangle_\Omega \otimes \langle w_{10} \rangle_\Omega + y \otimes 1$.

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