# Perturbation Theory in Homological Algebra

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Let $A_1, A_2$ be (graded, augmented, connected) algebras over a ring with unit $R$. (In the present exposition we assume characteristic 2. This is merely to avoid signs.) $A_1 \otimes A_2$ has a canonical multiplication; in the theory of Hopf algebras a twisted multiplicative structure frequently occurs which has the following properties:

$$a_2 \circ a_2' = a_2 a_2'$$
$$a_1 \circ a_2 = a_2 \circ a_1 = a_1 \otimes a_2$$
$$a_1 \circ a_1' = a_1 a_1' + \text{terms in } A_1 \otimes I(A_2)$$

where $a_1 a_1' \in A_1$, $a_2$, $a_2' \in A_2$., $\circ$ denotes the "twisted" multiplication, juxtaposition the canonical one and we have identified $a_1$ with $a_1 \otimes 1$, $a_2$ with $1 \otimes a_2$; $I(A_2)$ is the augmentation ideal of $A_2$, i.e. the terms of degree $> 0$.

When this structure is iterated, we are given algebras $A_i$ $(i = 1,2,\ldots)$ and again $\bigotimes_{i=1}^{\infty} A_i = A_1 \otimes A_2 \otimes \ldots$ has a canonical multiplication denoted by $\Phi_0(\omega \otimes \omega') = \omega\omega'$ as well as a twisted multiplication $\Phi(\omega \otimes \omega') = \omega \circ \omega'$. To express the appropriate condition which is roughly that $\omega \circ \omega' - \omega\omega'$ consists of terms "further to the right" it is convenient to introduce a "weighted filtration" which is fundamental to all that follows. If $A = \bigotimes_{i=1}^{\infty} A_i$ we define $F_p = F_p A$ as the submodule generated by all elements of the form

$$a_{1,n_1} \otimes \cdots \otimes a_{i,n_i}$$

where $a_{i,n_i} \in A_{i,n_i}$, $A_{i,n_i}$ denoting the n-dimensional component of $A_i$, and such that

$$n_1 + 2n_2 + \cdots + in_i \geq -p.$$

Clearly $F_p A = A$ if $p \geq 0$, $F_p \subset F_{p+1}$, $\bigcap_p F_p = 0$, $\Phi_o(F_p \otimes F_q) \subset F_{p+q}$.

We now introduce our assumptions:

A0) Each $A_i$ is a commutative algebra.

A1) (Convergence Assumption)

There is an increasing integer function $\gamma(i)$, $\gamma(i) \longrightarrow \infty$ as $i \longrightarrow \infty$, such that $A_{i,n} = 0$ if $0 < n < \gamma(i)$.

A2) $(\Phi - \Phi_o)(F_p \otimes F_q) \subset F_{p+q-1}$.

In the elementwise notation, if $\omega \in F_p$, $\omega' \in F_q$ then $\omega \cdot \omega' - \omega\omega' \in F_{p+q-1}$; this means that the difference is composed of terms "further to the right" -- i.e., with bigger subscripts. It can be verified, for instance, that the usual representation of the Steenrod algebra (due to Milnor) does satisfy this condition.

From now on, let $A$ denote the algebra $\otimes A_i$ with the "twisted" multiplication $\Phi$. The problem we wish to consider is the computation of $\mathrm{Ext}_A(R,R)$.

For each $A_i$ we choose a resolution (e.g. the standard construction) $W_i = U_i \otimes A_i$ with differential $d_i$ and contracting homotopy $s_i$. We assume -- as in standard construction -- that $s_i W_i \subset U_i$.

It is well known that

$$W = \bigotimes_{i=1}^{\infty} W_i \approx \bigotimes_{i=1}^{\infty} U_i \otimes A$$

with differential

$$d = d_1 + d_2 + d_3 + \ldots$$

(using an evident "abuse" of notation) is a resolution for the algebra with product $\Phi_o$. The appropriate chain-homotopy is $S = S_1 + S_2 + S_3 + \ldots$ with

$$S_i = A_1 \otimes \ldots \otimes A_{i-1} \otimes S_i \otimes \epsilon_{i+1} \otimes \epsilon_{i+1} \otimes \ldots$$

where $A_i$ denotes the identity and $\epsilon_i$ the augmentation of $A_i$.

Our aim is, by a sequence of "perturbations" to derive, from $d$, a differential $D$ which is also a differential of $\bigotimes_{i=1}^{\infty} U_i \otimes A$, but is compatible with $\Phi$ instead of $\Phi_o$.

As a first approximation we take $D_1$ defined by

$$D_1(\omega \otimes a) = (d\omega) \circ a$$

where $\omega \in \otimes U_i$, $a \in A$ and $\circ$ denotes the twisted multiplication. Here we must explain that, since the $A_i$ are commutative (cf. (AO) above), $W_i$ and $U_i$ are algebras. We give $\otimes U_i \otimes A$

the canonical algebra structure, using the product $\Phi$ for A.

It will be convenient to retain the symbols $\Phi$ and $\circ$ for this multiplication. Clearly, now, $D_1$ is "$\Phi$-linear", as required; but $D_1^2 \neq 0$ and

$$D_1\{(\omega\otimes a) \circ (\omega'\otimes a')\} - \{D_1(\omega\otimes a)\} \circ (\omega'\otimes a') - (\omega\otimes a) \circ \{D_1(\omega'\otimes a')\} \qquad \ldots\ldots\ldots(1)$$

is, in general, not equal to zero. The following, however, follows from (A2):

If $\omega \otimes a$, $\omega' \otimes a'$ have filtrations $p,q$ respectively, then (1) above has filtration $\leq p + q - 2$ ........(2).

The filtration used here is the following: An element of $W = \otimes U_i \otimes A$ is given as filtration the sum of the "internal filtration" derived from that of A + the homological degree. Thus, if $a_i \in A_{t,n_i}$ and we use the standard construction, then

$$[a_1,a_2,\ldots,a_k]$$

has filtration $\leq -t(n_1+n_2+\ldots+n_k) + k$. $D_1$ thus decreases filtration by at least 1, and S increases it by at most 1.

We now describe the mechanism by which, starting from $D_1$, by a sequence of perturbations $D_2,D_3,\ldots,$ we derive the required differential D. First, some notations and definitions:

$I(W_i) = I(U_i) \otimes A_i + U_i \otimes I(A_i)$ is the augmentation ideal of the construction $W_i$, $\Pi_i : \overset{\infty}{\underset{i=1}{\otimes}} W_i \circlearrowleft$ is the projection onto the summand

$$W_1 \otimes W_2 \otimes \ldots \otimes W_{i-1} \otimes I(W_i).$$

Thus $\Pi_o + \Pi_1 + \Pi_2 + \ldots = \text{identity}$.

In the remaining formulas we assume that $\overset{\infty}{\underset{i=1}{\otimes}} W_i$ is always written in the form $\underset{i=1}{\otimes} U_i \otimes A = W$.

$$\iota_i : U_1 \otimes \ldots \otimes U_i \longrightarrow W$$

$$\iota^{i+1} : U_{i+1} \otimes U_{i+2} \otimes \ldots \otimes A \longrightarrow W$$

are the injections.

Suppose now $G : W \longrightarrow$ is a map of $R$ modules. We define $L(G) : W \longrightarrow W$ and $\theta(G) : W \longrightarrow W$ by

$$L(G) = \sum_{i=1}^{\infty} \Phi\{\Pi_{i+1} G \iota_i \otimes \iota^{i+1}\}$$

$$O(G) = \sum_{i=1}^{\infty} \Phi\{S_{i+1} G \iota_i \otimes \iota^{i+1}\}$$

$$= L(SG).$$

Observe that if $G$ decreases filtration by at least $p$, then so does $LG$; while $\theta G$ decreases it by at least $p - 1$. The following can now be proved; the verifications are quite tedious:

1) $L(L(G)) = L(G)$

2) If $G = L(G)$ and $H = L(H)$ then $GH = L(GH)$

3) We denote by $[G,H]$ the commutator $GH - HG$.

$$[D_1, \theta G] = \theta[D_1, G] + L(G) + X(G)$$

where $X(G): W \longrightarrow W$ has the following properties:

(i) $L(X(G)) = X(G)$

(ii) If $G$ decreases filtration by $p$, then $X(G)$ decreases it by at least $p + 1$.

The "error term" $X(G)$ arises because the expression (1) above is, in general, not zero. It owes property (ii) to (2) above.

4) $D_1^2 = L(D_1^2)$

5) $D_1^2$ decreases filtration by at least 3.

It is clear that $D_1$ decreases filtrations by 1, hence $D_1^2$ by at least 2; the fact that it is "one better" is due to $d^2 = 0$ and the relationship (A2) between $\Phi$ and $\Phi_o$. We now define, inductively,

$$D_{n+1} = D_n + \theta(D_n^2) \qquad (n \geq 1)$$

and prove

6) $D_n^2 = L(D_n^2)$

7) $D_n^2$ decreases filtration by at least $n + 2$.

Proof: by induction, $n = 1$ are (4) and (5) above. Now:

$$D_{n+1}^2 = D_n^2 + [D_n, \theta(D_n^2)] + \{\theta(D_n^2)\}^2$$

$$[D_n, \theta(D_n^2)] = [D_1, \theta(D_n^2)] + \sum_{i=1}^{n-1} [\theta(D_i^2), \theta(D_n^2)]$$

and

$$[D_1, \theta(D_n^2)] = \theta[D_1, D_n^2] + L(D_n^2) + X(D_n^2).$$

Hence

$$D_{n+1}^2 = D_n^2 + L(D_n^2) + \theta[D_1, D_n^2] + X(D_n^2)$$

$$+ \sum_{i=1}^{n-1} [\theta(D_i^2), \theta(D_n^2)] + \{\theta(D_n^2)\}^2$$

$$= X(D_n^2) + \theta[D_1, D_n^2] + \sum_{i=1}^{n-1} [\theta(D_i^2), \theta(D_n^2)]$$

$$+ \{\theta(D_n^2)\}^2$$

by the inductive hypothesis and (6). Now, due to $L(X(G)) = X(G)$, $D_{n+1}^2 = L(D_{n+1}^2)$ follows from (2). Since $D_n^2$ decreases filtration by $n + 2$,

$X(D_n^2)$ decreases it by $n + 3$;

$$\theta[D_1, D_n^2] = \theta[D_n - \theta(D_1^2) - \ldots - \theta(D_{n.1}^2), D_n^2]$$

$$= - \sum_{i=1}^{n-1} \theta[\theta(D_i^2), D_n^2]$$

decreases filtration by at least $n + 3$; and the same is true of $\sum_{i=1}^{n-1} [\theta(D_i^2), \theta(D_n^2)]$. This completes the inductive proof of (7).

Since

$$D_n = D_1 + \theta\{D_1^2 + D_2^2 + \ldots + (D_{n-1})^2\}$$

it follows from the "converge assumption" that if $\omega \in W$ there is an $n$ such that

$$D_m\omega = D_n\omega \quad \text{for all} \quad m \geq n.$$

We can thus write in a well defined sense

$$D = \lim_{n \to \infty} D_n = D_1 + \sum_{i=1}^{\infty} \theta(D_i^2)$$

and it follows from the convergence assumption and (7) that $D^2 = 0$.

To prove that $D$ is the required differential we must prove that $W$ with differential $D$ is acyclic. To prove this we set up the spectral sequence induced by our filtration. In this spectral sequence, $\{E^r, d^r\}$ say, since $D_1 - d$ decreases filtration by 2, it is immediate that $d^1 = 0$, $d^1 = d$. Hence $E^2$ is trivial. Hence, by convergence, so are $E^\infty$ and $H(W)$.

This completes the proof of the fact that $\{W,D\}$ is a resolution for $A$.

Suppose we are given two algebras $A$, $B$ each with the sort of twisted structure discussed above, and constructions $\otimes U_i \otimes A = W$ (for $A$, as above) and $\otimes V_i \otimes B = W'$ (for $B$) together with a map $f: W \longrightarrow W'$ compatible with the untwisted multiplications of $A$, $B$ and the untwisted differential $d$; in particular, then, $df = fd$. By an entirely similar method of successive perturbations we can derive from $f$ a map $F = W \longrightarrow W'$ which is compatible with the twisted multiplications and such that $DF = FD$. This can be applied, in particular, to the diagonal map $W \longrightarrow W \otimes W$ which induces the multiplicative structure of Ext. Applying this, now, to the spectral sequence which we just used we obtain the following

Theorem: There is a convergent bigraded spectral sequence $(E_r^{p,q,t}, d_r)$ with $d_r : E_r^{p,q,t} \longrightarrow E_r^{p+r,q-r+1,t}$ such that

(a) $E_2^{p,q,t} = H^{p+q-t,t-q}(E^0A)$

(b) $E_\infty^{p,q,t} = E^0(H^{p+q-t,t}A)$

(c) $(E_r, d_r)$ has the structure of a differential algebra

(d) The isomorphisms (a) and (b) are isomorphisms of algebras $E_2 \longrightarrow H^*(E^0A)$, $E_\infty \longrightarrow E^0(H^*(A))$ respectively.

Throughout this statement, $p + q - t$ is the homological dimension, $t$ the "internal dimension" from $A$. Note that due to condition A(iv) on twisted multiplication, $E^0A$ is isomorphic as an algebra to $A$ with the untwisted multiplication $\Phi_0$; except that the grading of $E^0A$ is the <u>filtration</u> of $A$.

It is readily seen that this spectral sequence is analogous to a well-known spectral sequence of P. May. It is not, however, the same spectral sequence as we have used a different filtration.

If certain conditions are satisfied (and they are in the case of the Steenrod algebra), our method is, however, also compatible with the filtration used by May; and we obtain his spectral sequence.

The combination of the perturbation method with the May spectral-sequence appears to be an exceedingly powerful method of computation.

Complete details, computations and other applications will be given elsewhere.