Operations and Two Cell Complexes

by

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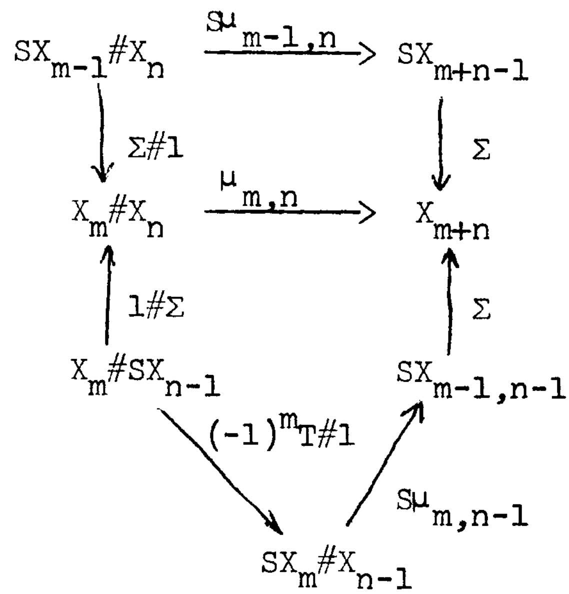
We are concerned with a spectrum $X(f) = \{X_n\}$ where $X_n = S^n \cup_f e^{n+k+1}$ for $n > k + 1$, and $X_n = S_n$ otherwise. This is constructed for any $f \in G_k$, the k-stem of the stable homotopy groups of spheres. We will obtain invariants of f by asking questions about $X(f)$; e.g., is $X(f)$ a ring spectrum? If so, what properties does it have as a ring spectrum?

<u>Definition 1</u>. A spectrum $\underline{X} = \{X_n\}$ is a ring spectrum if there is a map $\iota : \underline{S} \longrightarrow \underline{X}$ called the unit, where $\underline{S}$ is the sphere spectrum, and a multiplication:

$$\mu_{m,n} : X_m \# X_n \longrightarrow X_{m+n}$$

such that the diagrams:



commute, and $\mu_{0,0} = 1$.

In the case of $X(f)$, ι exists, and $\mu_{m,n}$ is easily defined if $m,n \leq k + 1$. $\mu_{m,n}$ can be defined for all m and n if $\mu_{k+2,k+2}$ can be defined.

<u>Definition 2</u>. A ring spectrum $\underline{X} = \{X_n\}$ is called homotopy associative if the diagram

$$\begin{array}{ccc} X_\ell \# X_m \# X_n & \xrightarrow{\mu_{\ell,m}\#1} & X_{\ell+m} \# X_n \\ \downarrow {\scriptstyle 1\#\mu_{m,n}} & & \downarrow {\scriptstyle \mu_{\ell+m,n}} \\ X_\ell \# X_{m+n} & \xrightarrow{\mu_{\ell,m+n}} & X_{\ell+m+n} \end{array}$$

commutes up to homotopy.

<u>Definition 3</u>. A ring spectrum $\underline{X} = \{X_n\}$ is called homotopy commutative if the diagram:

$$\begin{array}{ccc} X_m \# X_n & & \\ \downarrow {\scriptstyle (-1)^{mn}T} & \searrow^{\mu_{m,n}} & \\ & & X_{m+n} \\ X_n \# X_m & \nearrow_{\mu_{n,m}} & \end{array}$$

commutes up to homotopy.

It is obvious that one can define higher order associativity and commutativity as Stasheff does for H spaces [7].

We first ask for conditions on f which determine whether $X(f)$ is a ring spectrum. The following theorem is due essentially to Toda [8, p. 28], although its roots go back to Barratt [1].

<u>Theorem 4</u>. If $(1-(-1)^k)f \simeq 0$, we can define $f^* \in G_{2k+1}/I_k$ where $I_k = \{0\}$ if k is even and $I_k = f \circ G_{k+1}$ if k is odd. $X(f)$ is a ring spectrum iff $f^* \equiv 0$.

Furthermore, if $(1-(-1)^k)f \not\simeq 0$, $X(f)$ is not a ring spectrum. Define

$$a_1(f) = (1-(-1)^k)f,$$

and

$$a_2(f) = f^*.$$

Then $a_2(f) \neq \emptyset$ iff $a_1(f) = 0$, and $X(f)$ is a ring spectrum iff $0 \in a_2(f)$.

<u>Theorem 5</u>. (Toda [8, p. 30]). $g \circ a_2(f) \subset \{f,g,f\}$.

<u>Theorem 6</u>. $a_2(f \circ g) \supseteq f^2 \circ a_2(g) + a_2(f) \circ g^2$.

<u>Theorem 7</u>. $0 \in 2a_2(2f)$.

<u>Theorem 8</u>. $a_2(m \quad) = \begin{cases} \eta & m \equiv 2 \pmod 4 \\ 0 & \text{otherwise} \end{cases}$ ([1,8]),

$a_2(\eta) \equiv \nu \pmod{\eta^3}$, $a_2(\eta^2) = 0$, $a_2(\eta^3) = 0$, $a_1(\nu) = 2\nu \neq 0$, $a_2(\nu^2) = 0$.

<u>Theorem 9</u>. If $0 \in a_2(f)$, we can define

$$C_2(f) \in G_{2k+2}/f \circ G_{k+2}$$

$0 \in C_2(f)$ iff $X(f)$ is homotopy commutative.

Theorem 10. $C_2(4\iota) = \eta^2$, $C_2(\eta^2) = \nu^2$, $C_2(\nu^2) = \sigma^2$.

C^2 is equivalent to the $\cup_2$ product considered by Adams, Barratt and Mahowald [2].

Theorem 11. If $0 \in a_2(f)$, we can define

$$a_3(f) \in G_{3k+3}/f \circ G_{2k+3}.$$

$0 \in a_3(f)$ iff $X(f)$ is homotopy associative.

Theorem 12. $a_3(f \circ g) \supseteq f^3 \circ a_3(g) + a_3(f) \circ g^3$.

Theorem 13. $a_3(m\iota) = \begin{cases} \alpha_1 & m \equiv 3 \pmod 9 \\ 0 & \text{otherwise} \end{cases}$

where $\alpha_1 \in G_3$ has order 3.

Modulo the vanishing of the above obstructions, we would like to generalize a theorem of Hoffman [5]. This has proven useful in making certain constructions (Gray [3], Hoffman [4]).

Let $G_m(f) = [X_{m+t}(f), X_t(f)]$, the group of stable homotopy classes.

Theorem 14. (Hoffman [5]). There is a differential in $G(t\iota)$ for t odd:

$$d\colon G_m(t\iota) \longrightarrow G_{m+1}(t\iota)$$

such that:

a) $d(\alpha \circ \beta) = d(\alpha) \circ \beta + (-1)^{\deg \alpha}\, \alpha \circ d(\beta)$

b) $d^2 = 0$ if $(t,3) = 1$

c) $d(\alpha) = d(\beta) = 0$ implies $\alpha\beta = (-1)^{(\deg \alpha)(\deg \beta)}\beta\alpha$.

<u>Lemma 15</u>. Suppose $X(f)$ is a ring spectrum with multiplication $\mu_{m,n}$. Then there is a unique (up to homotopy) collection of maps:

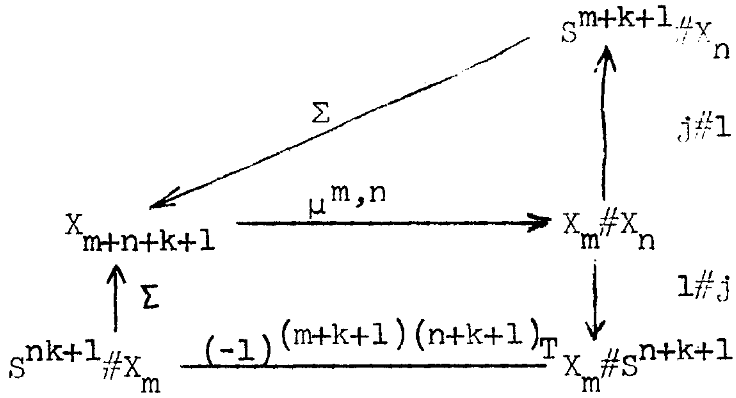
$$\mu^{m,n}: X_{m+n+k+1} \longrightarrow X_m \# X_n$$

such that

a) $\mu_{m,n} \circ \mu^{m,n} \simeq 0$

b) $\mu^{0,0} \simeq 0$

c) the diagrams:



commute up to homotopy.

Now for $\alpha \in G_m(f)$, we define $d_L(\alpha)$ and $d_R(\alpha) \in G_{m+k+1}(f)$ as the compositions:

$$X_{x+t+k+m+1} \xrightarrow{\mu^{s,m+t}} X_s \# X_{m+t} \xrightarrow{1\#\alpha} X_s \# X_t \xrightarrow{\mu_{s,t}} X_{x+t}$$

$$X_{s+t+k+m+1} \xrightarrow{\mu^{s+m,t}} X_{s+m} \# X_t \xrightarrow{\alpha\#1} X_s \# X_t \xrightarrow{\mu_{s,t}} X_{x+t}$$

respectively. They are independent of s and t for s and t large.

Let $\overline{x} = ixj : X_{m-k-1} \longrightarrow X_n$ where $x : S^m \longrightarrow S^n$ and let $[\ ,\ ]$ denote the graded commutator.

Theorem 16. $d_L(\alpha) - d_R(\alpha) = [\overline{C_2(f)},\alpha]$.

Theorem 17. $d(\alpha\circ\beta) = d(\alpha)\circ\beta + (-1)^{\deg\alpha}\alpha\circ d(\beta)$.

Theorem 18. $d_L^2(\alpha) = d_R^2(\alpha) = [\overline{a_3(f)},\alpha]$.

Theorem 19. If $a_1(f) = a_2(f) = 0$, and the Toda bracket

$$\{f,g,f,h,f\}$$

is defined, $gha_3(f) \in \{f,g,f,h,f\}$. If $a_1(g) = a_2(g) = 0$, $\{f,g,f,g,f\}$ is defined.

Theorem 20. In $G(f)$

$$[\alpha,\beta] = d(\alpha)[\overline{I},\beta] + [\alpha,\overline{I}]d(\beta)$$

provided $0 \in C_2(f)$.

If $C_2(f) \neq 0$ the answer is more complicated.

Theorem 21. (Toda).

1) $X(3\iota)$ is not homotopy associative

2) $f \in G_{2k}, 3f \simeq 0$ implies $\alpha_1 f^3 \equiv 0 \pmod 3$

3) $\alpha_1\beta_1^3 = 0$ where $\alpha_1 \in G_3 \otimes Z_3$ and $\beta_1 \in G_{10} \otimes Z_3$ are nonzero generators.

These are easily seen to be equivalent:

1) $\Longrightarrow$ 2): 1) implies that $a_3(3\iota) = \pm\alpha_1$. Since $3f \simeq 0$ and $f \in G_{2k}$, $a_1(f) = 0$ and $0 \in a_2(f)$ by Theorem 7. Therefore, $0 = a_3(0) = a_3(3f) = 27a_3(f) + \alpha_1 f^3$. Hence, $\alpha_1 f^3 \equiv 0 \pmod{27}$.

2) $\Longrightarrow$ 3): This is clear since $3(G_{33} \otimes Z_3) = 0$, [8].

3) $\Longrightarrow$ 1): If $X(3\iota)$ is homotopy associative, Theorem 19 implies $0 \in \{3\iota, \beta_1, 3\iota, \beta_1, 3\iota\}$. Since $G_{33} \otimes Z_3 = 0$, $0 = \{\beta_1, 3\ , \beta_1, 3\ , \beta_1\}$; hence $\gamma = \{\beta_1, 3\iota, \beta_1, 3\iota, \beta_1, 3\iota\}$ is defined. It is detected by the cohomology operation $P^3 P^3 P^3$ + other decomposables = 0. Hence $\gamma \neq 0 \in G_{34} \otimes Z_3$; by [8] $b_1 \in E_2^{2,34}$ is a permanent cycle and $\alpha_1 \beta_1^3 \neq 0$, a contradiction.

Other results similar to Theorem 19 can be obtained. For example:

<u>Theorem 22</u>. If $fg \simeq 0$, $0 \in a_2(f)$, and $0 \in a_2(g)$, then

$$\{2f, f, g, 2g\} \cap 2\{f, g, f, g\} \neq \emptyset.$$

The general pattern of these results, however, is not yet clear.

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