EVALUATING SECONDARY OPERATIONS ON LOW DIMENSIONAL CLASSES

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1. Introduction.

We let A denote the Steenrod algebra $\mathcal{G}(2)$ [3] and suppose $\alpha\beta = 0$ is a relation of degree $r + 1$ in A,

1.1 $$\alpha\beta \equiv \sum_{k=0}^{r} \alpha_k\beta_k = 0$$

with α_k, $\beta_k \in A$ and degree α_k, degree β_k are both positive. In [1] Adams established the following result: associated with the relation

$$Sq^1Sq^4 + Sq^2Sq^3 + Sq^4Sq^1 = 0$$

there is a secondary operation Φ so that if $x \in H^2(x)$ satisfies $Sq^1(x) = 0$ then $\Phi(x)$ is defined and

$$x^3 \in \Phi(x).$$

Later, L. Kristensen [6] using cochain operations was able to evaluate certain kinds of operations Φ associated with relations of the form 1.1. More recently, Mahowald and Peterson [8], Mahowald [7] and Hughes-Thomas [5] have been able to evaluate further Φ's but using "invariant" methods.

Our object here is to prove the Theorem 1.5 which includes all these previous results and whose proof turns out to be surprisingly simple.

We assume the relation 1.1 (of degree $r + 1$) satisfies $r < 2q$. Let $B(q)$ be the left ideal in A generated by those elements which are zero on every cohomology class of dimension $\leq q$ (see [3]). Also, we suppose that

1.2 $\beta_o \in B(q)$, $\beta_o \notin B(q+1)$ so that
$\beta_o(x_{q+1}) = (\beta_o' x_{q+1})^2$ for some $\beta_o' \in A$,

1.3 $\beta_i \in B(q+1)$ or $\alpha_i \in B(q + \dim \beta_i)$ for $i = 1,2,\ldots,n$.

1.4 Let $\psi : A \longrightarrow A \otimes A$ be the diagonal map so that
$\psi(\alpha_o) = \Sigma(\alpha_o^i \otimes \alpha_o^{i'} + \alpha_o^{i'} \otimes \alpha_o^i) + \Sigma\, \bar{\alpha}_o^j \otimes \bar{\alpha}_o^j$,

then we require that $\bar{\alpha}_o^j \beta_o'$ is contained in the left ideal generated by $\beta_1, \ldots, \beta_n$. Under these conditions we have

<u>Theorem</u> 1.5: <u>There is a secondary operation</u> Φ <u>associated with</u> (1.1) <u>so that for any class</u> $x \in H^q(X)$ <u>with</u> $\Phi(x)$ <u>defined we have</u>

$$\Sigma\, \alpha_o^i \beta_o' x \cup \alpha_o^{i'} \beta_o' x \in \Phi(x).$$

Remark 1.6. We can remove the condition that $\bar{\alpha}_0^j \beta_0' \in I(\beta_1, \ldots, \beta_n)$ but then we obtain a possibly different value for $\Phi(x)$.

As a particular case of (1.5) let

1.7 $$Sq^a Sq^b = \binom{b-1}{a} Sq^{a+b} + \sum_{o=i}^{[a/2]} \binom{b-1-i}{a-2i} Sq^{a+b-i} Sq^i$$

be an Adém relation with $a < b$. We denote by $\Phi_{a,b}$ the operation associated to 1.7 ($\Phi_{a,b}$ is unstable if $\binom{b-1}{a} \neq 0$) and we have

Corollary 1.8. *There is an operation* $\Phi_{a,b}$ *associated with* (1.7) *so that for any class* $x \in H^{b-1}(X)$ *that satisifes* $Sq^i(x) = 0 \quad i = 1, \ldots, [a/2]$ *we have*

$$Sq^a(x) \cup x \in \Phi_{a,b}(x).$$

Note that under the hypothesis of (1.8) if $2^r < a < 2^r + 2^{r-1}$ then Sq^a is in the ideal generated by $Sq^1, \ldots, Sq^{[a/2]}$ so $Sq^a(x) = 0$.

In particular the operations $\Phi_{2^i, 2^i}$ associated with the relations

$$Sq^{2^i} Sq^{2^i} = \sum_{j=0}^{i-1} Sq^{2^{i+1}-2^j} Sq^{2^j}$$

are unique, and if there is an $x \in H^{2^i-1}(X)$ on which

$\Phi_{2^i,2^i}$ is defined then

$$0 \in \Phi_{2^i,2^i}(x).$$

Remark 1.9. A Spanish version of this article has appeared in Acta Politecnica (in Mexico).

Remark 1.10. In a recent paper, Kristensen and Madsen prove a theorem very similar to 1.5 by using cochain operations.

2. A fibration of G. W. Whitehead

In [13] G. W. Whitehead introduced a fibration to study the suspension map on the cohomology primitives of the Eilenberg-MacLane spaces $K(\Pi,n)$ (see also [2]).

Let G be an associative H-space with unit; then G admits a classifying space B_G [11]. Let ΣG be the suspension of G and denote by $G*G$ the join of G with itself. We then have a fibration

2.1 $$G*G \xrightarrow[\bar{u}]{} \Sigma G \xrightarrow[q]{} B_G$$

which we obtain by considering the constructions of Milnor or Milgram [11], for the classifying space B_G. Hence we have the principal fibration

$$G \longrightarrow G*G \xrightarrow[\bar{u}]{} \Sigma G$$

so q is the classifying map for $\bar{u}$. It is known (and evident from the construction) that q is the adjoint of the identity $G \longrightarrow G$ and that if we identify $G*G$ with $\Sigma(G \wedge G)$, where $G \wedge G$ is the smash product of G with itself, then $\bar{u}$ is the suspension of the multiplication

$$u\colon G \times G \longrightarrow G$$

(see [2]). The fibration (2.1) is very useful, since if G is $q - 1$ connected it follows that $G*G$ is $2q$ connected and B_G is q-connected so the Serre exact sequence is valid in dimensions $\leq 3q$.

Since G is an H-space $H^*(G)$ is a Hopf algebra and we write $P(H^*(G))$ for the primitives. Given an arbitrary space X we write $Q(H^*(X))$ for the indecomposables in $H^*(X)$, (see [4]).

Theorem 2.2. *If* G *is an associative H-space with unit,* B_G *its classifying space and* G *is* q-1-*connected, then*

$$\sigma^*: Q(H^i(B_o)) \longrightarrow P(H^{i-1}(G))$$

is an epimorphism for $i \leq 3q$.

3. Massey-Peterson fiberings

Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibering. We say it is a Massey-Peterson fibering [9], [10], if the Serre sequence for the fibering satisfies

$$E_r = E_r^{*,0} \otimes E_r^{0,*} \qquad 2 \leq r \leq \infty$$

Now note that if $F' \xrightarrow{i'} E' \xrightarrow{p'} B'$ is a fibering so $F = \Omega F'$, $E = \Omega E'$, $B = \Omega B'$, $i = \Omega(i')$ and $p = \Omega(p')$, then the Serre spectral sequence for $F \longrightarrow E \longrightarrow B$ is a spectral sequence of Hopf algebras and we have (see [12] for a particular case):

Theorem 3.2. Let the Massey-Peterson fibering $F \longrightarrow E \longrightarrow B$ be the loop fibering (as above) of $F' \longrightarrow E' \longrightarrow B'$. Then for any $x \in P(H^*(E))$ one of the following holds

$$i^*(x) \neq 0 \quad \text{or} \quad x = p^*(b).$$

Proof: By assumption $H^*(E)$ is a Hopf algebra and E_∞ in the Serre spectral sequence for $F \longrightarrow E \longrightarrow B$ is the bi-graded Hopf algebra associated with the filtered algebra $H^*(E)$. Evidently if $x \in P(H^*(E))$ then the image of x in E_∞ is also primitive, but the only primitives in E_∞ belong to $E_\infty^{*,0} \oplus E_\infty^{0,*}$ and since

$p^*H^*(B)$ is $E_\infty^{*,0}$ while $i^*(H^*(E)) = E_\infty^{0,*}$ the result follows.

Remark: Theorem (3.2) is still true if we only suppose F, E, B in the Massey fibering $F \longrightarrow E \longrightarrow B$ are H-spaces with unit and p, i are H-maps.

Corollary 3.3. Let $F \longrightarrow E \longrightarrow B$ be a 2-stage Postnikov system with stable k invariant that is the loop space of another fibering; then $F \longrightarrow E \longrightarrow B$ satisfies the hypothesis of 3.2.

Proof: See [9] and [10].

4. The proof of 1.5.

Given the relation (1.1) we construct the operation Φ associated with it as follows. Let t be an integer much larger than n and consider the fiber E_t in the map f

$$4.1 \qquad E_t \xrightarrow{p} K_t \xrightarrow{f_t} \prod_{k=0}^{n} K_{t+\dim \beta_k}$$

defined by $f_t^*(\gamma_{t,t+\dim \beta_k}) = \beta_k \gamma_t$ where K_q is a $K(Z_2,q)$ and γ_q or $\gamma_{t,q}$ is the fundamental class in dimension q. As is well known, we have the fibering

$$4.2 \qquad \prod_{k=0}^{n} (K_{t+\dim \beta_k - 1}) \xrightarrow{i} E_t \xrightarrow{p} K_t$$

and (since $t \gg r$) in the Serre exact sequence $\delta(\gamma_{t,k}) = \beta_k \gamma_t$ so for the class $m = \Sigma\, \alpha_k \gamma_{t,k}$ we have $\delta(m) = 0$ and $m = i^*(v_t)$ for some $v \in H^*(E_t)$.

Now there is the suspension map σ_t of degree -1 $\sigma_t : H^*(E_t) \longrightarrow H^*(E_{t-1})$, and we say a sequence $(v_1, v_2, \ldots, v_t, \ldots)$ of elements $v_i \in H^*(E_i)$ is a representative for Φ if

(i) $\qquad \sigma_t(v_t) = v_{t-1}$ for all t

(ii) $\qquad i^*(v_t) = \Sigma\, \alpha_k \gamma_{t,k}$.

Clearly two such representatives for Φ differ by a primary operation $(v_t + v_t' = \theta\, \gamma_t)$.

Next we must examine the E_t for small t. Renumber the β_k so

$$\beta_o \in B(q) \quad \beta_o \notin B(q+1)$$
$$\beta_1 \ldots \beta_m \in B(q+1)$$

and the remainder are arbitrary. Then let $\overline{E}_t$ be the fiber in the map

$$\overline{E}_t \xrightarrow{\overline{p}} K_t \xrightarrow{g_t} \left(\prod_{k=m+1}^{n} K_{t + \dim \beta_k}\right) \times K_{t + \dim \beta_o}$$

where $g_t^*(\gamma_{t,t + \dim \beta_k}) = \beta_k(\gamma_t)$ and we have

Lemma 4.3:

(i) E_{q+1} is homeomorphic to

$$\overline{E}_{q+1} \times \prod_{k=1}^{m} K_{t + \dim \beta_k - 1},$$

(ii) $\overline{E}_q$ is homeomorphic to $\overline{\overline{E}}_q \times K_{q + \dim \beta_o - 1}$ (where $\overline{\overline{E}}_q$ is the fiber in the map

$$h_t \colon K_t \longrightarrow \prod_{k=m+1}^{n} K_{t + \dim \beta_k})$$

but not as an H-space. In fact

(iii) $\gamma_{q,q + \dim \beta_o - 1}$ in $H^*(\overline{E}_q)$ can be chosen so the diagonal on $(\gamma_{q,q + \dim \beta_o - 1})$ gives

$$\gamma_{q,q + \dim \beta_o - 1} \otimes 1 + \beta_o^!(\gamma_q) \otimes \beta_o^!(\gamma_q)$$
$$+ 1 \otimes \gamma_{q,q + \dim \beta_o - 1}.$$

Proof: (i) follows since the only nonzero k-invariants in 4.1 with $t = q + 1$ are when $k = 0$ or is greater than m, so up to homotopy f_{q+1} factors as $j \cdot g_{q+1}$ where $j : \left(\prod_{k=m+1}^{n} K_{t + \dim \beta_k}\right) \times K_{t + \dim \beta_o} \longrightarrow \prod_{k=0}^{n} K_{t + \dim \beta_k}$ is the evident inclusion.

(ii) now follows since $f_q^*(\gamma_{q,q + \dim \beta_o}) = 0$ as well.

(iii): Consider the map

$$\varphi : \bar{E}_{q+1} \longrightarrow \mathcal{E}_{q+1 + \dim(\beta_o^!)}$$

where $\mathcal{E}_{q+1 + \dim \beta_o^!}$ is the fiber in the map

$$\mathcal{E}_{q+1 + \dim \beta_o^!} \longrightarrow K_{q+1 + \dim \beta_o^!} \xrightarrow{s} K_{2(q+1+\dim \beta_o^!)}$$

where $s^*(\gamma) = \gamma \cup \gamma$ and $\varphi^*(\gamma_{q+1 + \dim \beta_o^!}) = \beta_o^! \gamma_{q+1}$.

Looping φ we have

$$\Omega\varphi : \bar{E}_q \longrightarrow \Omega\mathcal{E}_{q+1 \oplus \dim(\beta_o^!)} = K_{q+\dim \beta_o^!} \times K_{2(q+\dim \beta_o^!)}.$$

It is easy to see $i^*(\Omega\varphi)^* \gamma_{2(q+\dim \beta_o^!)} = \gamma_{q,q + \dim \beta_o^!}$ so we can choose $\gamma_{q,q + \dim \beta_o^!}$ in $H^*(E_q)$ as $(\Omega\varphi)^*\gamma_{2(q+\dim \beta_o^!)}$. On the other hand, $(\Omega\varphi)^*$ induces a map of Hopf algebras since it is a loop map and it is a result of [12] (lemma 3.1) that

$$\psi(\gamma_{2(q + \dim \beta_o^!)}) = \gamma_{2(q + \dim \beta_o^!)} \otimes 1 + \gamma_q \otimes \gamma_q + 1 \otimes \gamma_{2(q + \dim \beta_o^!)}.$$

(iii) now follows.

Returning to 1.5 we see that v_q is a primitive and can be written in the form

4.4 $$\alpha_o \gamma_{q,q + \dim \beta_o} + \sum_{k=m+1}^{n} \alpha_k (\gamma_{q + \dim \beta_k}) + w$$

and $i^*(w) = 0$.

On the other hand each of the $\gamma_{q + \dim \beta_k}$ $k = 1,\ldots,n$ are primitive since they are in the image under suspension of classes in $H^*(E_{q+1})$. Also, by 4.3 (iii), we have

4.5 $$\begin{aligned} \Psi(\alpha_o \gamma_{q,q + \dim \beta_o}) &= \Sigma \alpha_o^i \beta_o^!(\gamma_q) \otimes \alpha_o^{i'} \beta_o^!(\gamma_q) \\ &+ \Sigma \alpha_o^{i'} \beta_o^!(\gamma_q) \otimes \alpha_o^i \beta_o^!(\gamma_q) \\ &+ \Sigma \bar{\alpha}_o^j \beta_o^!(\gamma_q) \otimes \bar{\alpha}_o^j \beta_o^!(\gamma_q) \ . \end{aligned}$$

(By assumption this last sum is zero.) Now if y equals

$$\sum_i \alpha_o^{i'} \beta_o^!(\gamma_q) \cup \alpha_o^i \beta_o^!(\gamma_q)$$

then $\Psi(y)$ also gives the first two summands in 4.5,

$$x = w + y$$

is primitive and since $i^*(w) = 0$ as is $i^*(y)$ it follows that $x = p^*(b)$ for some b in $H^*(K)$.

On the other hand,

4.6 $$y + \sum_{k=m+1}^{n} \alpha_k(\gamma_q + \dim \beta_k) + \alpha_0 \gamma_{q,q + \dim \beta_0}$$

is $\sigma(A)$ for some A in $H^*(E_{q+1})$ by 2.2. Clearly $i^*(A) = i^*(v_{q+1})$ so $A + v_{q+1} = p^*(b')$ and b can be chosen as $\sigma(b')$.

This shows b is primitive. But the only primitives in $H^*(K_q)$ are stable operations on the fundamental class, thus $b = \theta(\gamma_q)$ and varying Φ by the primary operation θ we obtain Φ', a stable secondary operation associated with (1.1) and v'_q is exactly 4.6. Now since $\alpha_k(\gamma_q + \dim \beta_k)$ $(k = 0, m + 1,\ldots,17)$ are in the natural indeterminacy of Φ' the proof of 1.5 is complete.

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