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The Arf Invariant of a Manifold

By

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In [1], [2] and [3] the Arf invariant of a manifold is defined for various classes of manifolds. The aim of this lecture is to discribe a general technique for defining an Arf invariant which includes the above as special cases.

In the above papers the technique is roughly as follows: Let M be a smooth, compact, closed 2n-manifold. Although we will assume all manifolds are smooth, the techniques apply equally well to PL-manifolds. One assumes that the normal bundle ν of M embedded in R^{2n+k}, k large, has a special structure and one considers certain special values for n. In [1] one assume ν is framed and n is odd but $n \neq 1,3,7$. In [2] one assumes ν has a Spin structure and $n \equiv 1 \mod (4)$. Let $H^n(M) = H^n(M;Z_2)$. Using these special assumptions and various cohomology operations one obtains a function

$$\varphi : H^n(M) \longrightarrow Z_2$$

which has the following property:

(1) $\quad \varphi(u + v) = \varphi(u) + \varphi(v) + u \cup v(M)$

φ is thus a non-singular quadradic function over Z_2. Such functions are algebraically classified by their Arf invariant $A(\varphi) \in Z_2$ which is defined as follows: Let λ_i, μ_i, $i = 1,2,\ldots,\ell$ be a basis for $H^n(M)$ such that

$\lambda_i \lambda_j = \mu_i \mu_j = 0$, $\lambda_i \mu_j = 0$, $i \neq j$ and $\lambda_i \mu_i = 1$. Then

$$A(\varphi) = \Sigma\, \varphi(\lambda_i)\, \varphi(\mu_i)$$

The Arf invariant of $M, K(M)$, is then defined to be $A(\varphi)$. One then goes on to show that $K(M)$ is a cobordism invariant within the class of manifolds under consideration. The techniques in [3] are similar to this but technically more difficult to discribe. (see below)

We will follow a similar line of development, namely, we obtain quadradic functions on $H^n(M)$ from special structures on ν and obtain an invariant on M from an algebraic invariant of this function.

We first note the following easily proved lemma. All spaces will be assumed to have base points. $[\,,]$ and $\{\,,\}$ will denote homotopy classes of maps and homotopy classes of S-maps, respectively. Let $K_n = K(Z_2, n)$. Let $\lambda : M^+ \longrightarrow S^{2n}$ be a map which is degree one on M .
Let

$$\eta : H^n(M) = [M^+, K_n] \longrightarrow \{M^+, K_n\}$$

be the obvious map. Recall $\{S^{2n}, K_n\} \approx Z_2$. Let α be the generator.

Lemma 2. The function

$$\theta : H^n(M) \times Z_2 \longrightarrow \{M^+, K_n\}$$

defined by $\theta(u,t) = \eta(u) + t\, \lambda^* \alpha$ is an isomorphism if one defines addition on the product by

$$(u,t) + (v,s) = (u + v,\ u \cup v(M) + t + s) .$$

Note that this lemma shows that functions $\varphi : H^n(M) \longrightarrow Z_2$ satisfying (1) are in 1-1 correspondence with homomorphisms $\bar{\varphi} : \{M^+, K_n\} \longrightarrow Z_2$ such that

$\overline{\varphi}(\lambda^*\alpha) = 1$. The correspondence is $\overline{\varphi} \longrightarrow \varphi$ where $\varphi(u) = \overline{\varphi}\,\theta(u,0)$. Note also that if there is a $u \in H^n(M)$ such that $u^2 \neq 0$, no such $\overline{\varphi}$ will exist since $\lambda^*\alpha = \theta(0,1) = 2\,\theta(u,0)$. But there will always be a homomorphism $\overline{\varphi} : \{M^+, K_n\} \longrightarrow Z_4$ such that $\overline{\varphi}(\lambda^*\alpha) = 2$. For this reason we generalize the motion of a mod 2 quadradic function as follows:

Let $i : Z_2 \longrightarrow Z_4$ be the homomorphism sending 1 to 2. Let V be a finite dimensional vector space over Z_2.

<u>Definition 3</u>. A function $\varphi : V \longrightarrow Z_4$ is quadradic (non-singular) if there is a non-singular pairing $\mu : V \otimes V \longrightarrow Z_2$ such that

(4) $\quad \varphi(u + v) = \varphi(u) + \varphi(v) + i\mu(u \otimes v)$

Note that if $\mu(u \otimes u) = 0$ for all $u \in V$, $\varphi = i\varphi'$ where $\varphi' : V \longrightarrow Z_2$ satisfies

$$\varphi(u + v) = \varphi(u) + \varphi(v) + \mu(u \otimes v) .$$

<u>Lemma 5</u>. Quadradic functions $\varphi : H^n(M) \longrightarrow Z_4$ are in 1-1 correspondence with homomorphisms $\overline{\varphi} : \{M^+, K_n\} \longrightarrow Z_4$ such that $\overline{\varphi}(\lambda^*\alpha) = 1$.

To obtain algebraic invariants of these quadradic functions we compute their associated Grothendeck ring. Let $\varphi_i : V_i \longrightarrow Z_4$, $i = 1,2$, be quadradic functions. φ_1 and φ_2 are <u>isomorphism</u> if there is a linear isomorphism $n : V_1 \longrightarrow V_2$ such that $\varphi_2 = \varphi_1 n$. The sum $\varphi_1 + \varphi_2 : V_1 \otimes V_2 \longrightarrow Z_4$ is defined by $(\varphi_1 + \varphi_2)(u)\ \ \varphi_1(u) + \varphi_2(u)$.

$V_1 \times V_2 \longrightarrow Z_4$ given by $(u,v) \longrightarrow \varphi_1(u)\,\varphi_2(v)$, defines, via (4) a quadradic function $\varphi_1\varphi_2 : V_1 \otimes V_2 \longrightarrow Z_4$. Let Q_2 be the Grothendeck ring consisting of the free abelian group generated by the isomorphism classes of quadradic functions modulo the usual relations.

Let $\gamma^+, \gamma^- : Z_2 \longrightarrow Z_4$ be given by $\gamma^{\pm}(1) = \pm 1$.

Theorem 6.

$$Q_2 = Z\,[t]/_{(4t,\,t^2-2t)}$$

where $1 = \{\gamma^+\}$ and $t = \{\gamma^+\} - \{\gamma^-\}$.

We are interested in applying our results to cobordism theory so we need a notion of cobordism for quadradic functions. For a M with a certain structure, we will obtain a $\varphi : H^n(M) \longrightarrow Z_4$. If $M = \partial N$, we will see that $\varphi j^* = 0$, where $j^* : H^n(N) \longrightarrow H^n(M)$. Recall,

$$H^n(N) \xrightarrow{j^*} H^n(M) \xrightarrow{\delta} H^{n+1}(N,M)$$

is exact and $\delta(j^*u \cup v) = u \cup \delta v$.

Definition 7. A quadradic function $\varphi : V \longrightarrow Z_4$ is cobordant to zero if there is an exact sequence of vector spaces over Z_2,

$$V_1 \xrightarrow{a} V \xrightarrow{b} V_2$$

such that $\varphi a = 0$ and a non-singular pairing $\mu' : V_1 \otimes V_2 \longrightarrow Z_2$ such that $\mu'(u \otimes bv) = \mu(au \otimes v)$ where μ is the pairing associated to φ.

<u>Lemma 8</u>. The elements of Q_2 represented by functions cobordant to zero form an ideal generated by $t-2$.

Since $4t = 0$, $Q_2/(t-2)$ is cyclic of order 8. For a quadradic function φ, let $A(\varphi) \in Z_8$ be defined by

$$A(\varphi)\, 1 = \{\varphi\} \in Q_2/_{(t-2)}$$

We summarize the various properties of A in the following theorem:

<u>Theorem 9</u>. (i) $A(\varphi_1 + \varphi_2) = A(\varphi_1) + A(\varphi_2)$.

(ii) $A(\varphi_1 \varphi_2) = A(\varphi_1)\, A(\varphi_2)$.

(iii) If φ is cobordant to zero, $A(\varphi) = 0$.

(iv) $A(\varphi) = \dim V \bmod 2$ where $\varphi : V \longrightarrow Z_4$.

(v) If $\varphi = i\varphi'$, where $\varphi' : V \longrightarrow Z_2$,

$A(\varphi) = 4$ Arf invariant of φ'.

(vi) If U is a finitely generated free module over Z and $\psi : U \longrightarrow Z$ is a quadradic function with determinant ± 1, then ψ induces a quadradic function

$$\varphi : U/_{2U} \longrightarrow Z_4$$

and $A(\varphi) =$ signature of ψ, mod (8). (Any element of $Q_2/(t-2)$ may be represented by the reduction of an integer form but it is not true that any $\varphi : V \longrightarrow Z_4$ is the reduction of an integer form. For example

$\varphi : Z_2 + Z_2 \longrightarrow Z_4$, by $\varphi(0,1) = \varphi(1,0) = 2$ and $\varphi(0,0) = \varphi(1,1) = 0$, has rank 2 and $A(\varphi) = 4$.)

We next describe how quadradic functions may be obtained in cobordism theory. Suppose $\zeta = \{\zeta_k\}$, $k = 1,2,\ldots$ is a sequence of real k-plane bundles and $h_k : \zeta_k + 0^1 \longrightarrow \zeta_{k+1}$ are bundle maps, where 0^1 is the trival line bundle. For example ζ_k might be the universal bundle for O_k, U_k $Spin_k$, etc. Let $T(\zeta_k)$ denote the Thom space of ζ_k and U_k its Thom class. If M is an m-manifold, a ζ structure on M is a map $h : \nu \longrightarrow \zeta_k$ where ν is the normal bundle of M embedded in R^{m+k}. In the usual way, one forms cobordism groups $\Omega_m(\zeta)$ consisting of cobordism classes of pairs (M,h) and one has $\Omega_m(\zeta) \approx \pi_{m+k}(T(\zeta_k))$.

Suppose M is a closed 2n-manifold and $h : \nu \longrightarrow \zeta_k$ is a ζ structure. Recall $T(\nu)$ is the $2n+k$, S-dual of M^+.

Consider

$$\{M^+, K_n\} \overset{d}{\approx} \{S^{2n+k}, T(\nu) \wedge K_n\} \xrightarrow{T(h)_*} \{S^{2n+k}, T(\zeta_k) \wedge K_n\}$$

where d is the S-duality isomorphism.

Let w be image of α under the map

$$\{S^{2n}, K_n\} = \{S^{2n+k}, S^k \wedge K_n\} \longrightarrow \{S^{2n+k}\ T(\zeta_k) \wedge K_n\}$$

induced by the inclusion of S^k into $T(\zeta_k)$ as a "fibre".

Lemma 10. (i) $w = T(h)_* \, d \, \lambda^* \alpha$ (Recall $\lambda : M^+ \longrightarrow K_n$ is the map of degree 1.)

(ii) $w \neq 0$, if and only if $\chi(Sq^{n+1})\, U_k = 0$, where χ is the canonical anti-automorphism of the Steenrod algebra. Furthermore, w is divisable at most by 2.

Suppose $\chi(Sq^{n+1})\, U_k = 0$. Then we may choose a homomorphism

$$\gamma : \{S^{2n+k}, T(\zeta_k) \wedge K_n\} \longrightarrow Z_4$$

such that $\gamma(w) = 2$. By lemma 5,

$$\varphi_h : H^n(M) \longrightarrow Z_4$$

defined by $\varphi_h(u) = \gamma \, T(h)_* \, d\theta(u,0)$ is a quadradic function. Let $A_\gamma(M,h) = A(\varphi_h)$.

Theorem 11. A_γ defines a homomorphism

$$A_\gamma : \Omega_{2n}(\zeta) \longrightarrow Z_8$$

$A(\{M,h\}) =$ Euler characteristic of M mod 2. If $g : \Omega_{2n}$ (Framed) $\longrightarrow \Omega_{2n}(\zeta)$ is the obvious map, $Ag = 4K$ where K is the Kervaire invariant.

Examples

(1) If ζ_k is the trival bundle over a point, $\Omega_{2n}(\zeta) = \Omega_{2n}$ (Framed), $\{S^{2n+k}, T(\zeta_k) \wedge K_n\} = \{S^{2n}, K_n\} = Z_2$ γ is unique and

$$A_\gamma = 4K$$

(2) If ζ_k is the universal Spin_k bundle and $n \equiv 1 \mod 4$,

$$\chi(Sq^{n+1})U_k = \chi(Sq^2 Sq^{n-1} + Sq^1 Sq^2 Sq^{n-2})U_k$$

$$= (\chi(Sq^{n-1})\ Sq^2 + \chi(Sq^{n-2})\ Sq^2 Sq^1)U_k$$

$$= 0$$

since $Sq^2 U_k = w_2 U_k$ and $Sq^1 U_k = w_1 U_k$. A_γ is then 4 times the invariant constructed in [2]. In [2], the construction depended on the choice of a secondary cohomology operation. This choice corresponds to the choice of γ .

(3) Let ζ_k' be the universal O_k bundle over BO_k and let U_k' be its Thom class. $\chi(Sq^{n+1})U_k = v_{n+1}U_k$ where v_{n+1} is the W^u class corresponding to Sq^{n+1}. (If M is a closed m-manifold,

$$Sq^{n+1} : H^{m-n-1}(M) \longrightarrow H^m(M)$$

is given by $Sq^{n+1}\chi = v_{n+1}\ \chi$. If $m = 2n$, v_{n+1} is zero on M.)

Let $p : B_{k,n} \longrightarrow BO_k$ be the fibration with fibre K_n and k-invariant v_{n+1}. Let $\zeta_k = p^*\zeta_k'$. $\Omega_{2n}(\zeta_k)$ is the cobordism theory considered by Browder in [3] and A_γ, in this case, is 4 times Browders invariant, when the latter is defined.

Another way of constructing this cobordism theory is as follows: Let M be a closed 2n-manifold and let $\varphi : H^n(M) \longrightarrow Z_4$ be a quadradic function. Define (M,φ) to be cobordant to zero if $M = \partial N$ and $\varphi j^* = 0$ where $j^* : H^n(N) \longrightarrow H^n(M)$. Let $\tilde{\Omega}_{2n}$ be the resulting cobordism group. Then the map $\Omega_{2n}(\zeta) \longrightarrow \tilde{\Omega}_{2n}$ given by $\{M,h\} \longrightarrow \{M,\varphi_h\}$ is an isomorphism.

(4) The following was suggested to me by Dennis Sullivan: Let ζ_1 be the universal real line bundle and let $\zeta_k = \zeta^1 + 0^{k-1}$. $\Omega_2(\zeta)$ is then the cobordism group of surfaces immersed in R^3.
$\{S^{2+k}, T(\zeta_k) \wedge K_1\} \approx Z_4$. One choose γ so that A of Boy's surface is 1. One then obtains Tony Phillips' result that $\Omega_2(\zeta) \approx Z_8$.

<u>Theorem 12</u>. $A_\gamma : \Omega_2(\zeta) \approx Z_8$. One may describe $\varphi : H^1(S\ ) \longrightarrow Z_4$ for a surface S immersed in R^3 as follows: Let $u \in H'(S)$ and let $S^1 \subset S$ represent the dual of u. Let T be a tubular neighborhood of S^1 in S. Then T is a twisted strip in R^3. Then $\varphi(u)$ = number of twists of T in R^3 modulo 4. (The Moelius band has one twist. The number of twists only

makes sense mod (4).) $A(\varphi)$ is the obstruction to making $H_1(S)$ zero, by surgery, in R^4.

Bibliography

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