Recent Advances in Topological Manifolds

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Lent 1971
Introduction

A topological $n$-manifold is a Hausdorff space which is locally $n$-Euclidean (like $\mathbb{R}^n$).

No progress was made in their study (unlike that in PL,differentiable) until 1968 when Kirby, Siebenmann and Wall solved most questions for high-dimensional manifolds (at least as much for PL and differentiable cases).

Question (1) Can compact $n$-manifolds be triangulated?

Yes, if $n \leq 3$ (Moise 1950's).

Unknown in general.

However, if manifolds (dim $\geq 5$) which don't have PL structures. (Might still have triangulations in which links of simplexes aren't PL spheres).

A machinery for deciding whether manifold of dim $\geq 5$ has a PL structure. (Obstruction gp. $H^4(M,\mathbb{Z})$)

$4$-manifolds quite unknown in topological, differentiable and PL cases.

(2) **Annulus conjecture.** Generalised Schoenflies theorem.

Let $B^n = \{ x \in \mathbb{R}^{n+1} | \|x\| \leq 1 \}$, $S^n = \{ x \in \mathbb{R}^{n+1} | \|x\| = 1 \}$

Embedding $f : B^n \rightarrow S^n$ (i.e. $1$-$1$ continuous map)

Is $\tilde{f}(B^n) = \mathbb{S}^n$? No - Alexander horned sphere.

Outline of course:

Basic facts about topological manifolds

Morton Brown's theorem - first recent result

Kirby's trick Homeo$(S^1)$ is a topological group (compact-open topology).

This is locally contractible: any homeo $h$ near $1$ can be joined by a path in Homeo$(S^1)$ to $1$

Product structure theorem: if $M$ is a topological manifold, $H$ topological group, $H$ acts on $M$ then $M$ has PL structure (n $\geq 5$)

Sketch of proof of annulus conjecture. (complete except for deep PL theorem)
§1. Basic properties of topological manifolds

Let \( \mathbb{R}^n_+ = \{ (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n > 0 \} \).
Identify \( \mathbb{R}^{n-1} \) with \( \{(x_1, -x_n) \in \mathbb{R}^n \mid x_n = 0\} = \partial \mathbb{R}^n \).

**Definition 1.1** A (topological) \( n \)-manifold (with boundary) is a Hausdorff space \( M \) such that each point of \( M \) has a neighbourhood homeomorphic to \( \mathbb{R}^n_+ \). The interior of \( M \), \( \text{int} M \), is the set of points in \( M \) which have neighbourhoods homeomorphic to \( \mathbb{R}^n_+ \). The boundary of \( M \), \( \partial M = M \setminus \text{int} M \).

\( \text{int} M \) is an open set in \( M \), \( \partial M \) is a closed set in \( M \).

\( M \) is an open manifold if it is non-compact and \( \partial M = \emptyset \).\( M \) is a closed manifold if it is compact and \( \partial M = \emptyset \).

**Examples**
1. Any open subset of an \( n \)-manifold is an \( n \)-manifold.
2. Let \( M \) be a connected manifold with \( \partial M = \emptyset \). If \( x, y \in M \) then there exists a homeomorphism \( h : M \to M \) with \( h(x) = y \).

**Theorem 1.2** (Hausdorff property of open manifolds)
Let \( U, V \subset \mathbb{R}^n \) be subsets such that \( U \neq V \). Then \( U \) is open in \( \mathbb{R}^n \), then \( \text{int} U \) is open in \( \mathbb{R}^n \).

**Proof:** later (2).

**Corollary 1.3** If \( M \) is an \( n \)-manifold, then \( \partial M \) is a non-manifold without boundary.

**Proof:** Suppose \( x \in \partial M \) and \( f : \mathbb{R}^n_+ \to M \) be a homeomorphism.

(A) \( x \notin f(\mathbb{R}^n_+) \to x \notin f(\mathbb{R}^n_+ \setminus \mathbb{R}^n_+) \), hence \( x \notin \text{int} M \).

(B) \( x \notin \partial M \to x \notin \text{int} M \), i.e., \( U \) is a nbhd of \( x \) in \( M \) homeo to \( \mathbb{R}^n_+ \), so \( \partial M \) is a non-manifold without boundary.

**Corollary 1.4** If \( M^n, N^n \) are manifolds, then \( M \times N \) is an \((m+n)\)-manifold with boundary, \( \partial (M \times N) = (\partial M \times N) \cup (M \times \partial N) \).

**Proof:** If \( x \in M \times N \), then \( x \) has a neighbourhood homeomorphic to \( \mathbb{R}^m_+ \times \mathbb{R}^n_+ \), so \( M \times N \) is an \((m+n)\)-manifold.

Clearly, \( \text{int} (M \times N) = \text{int} M \times \text{int} N \).

If \( x \in \partial (M \times N) \), then \( x \) has a neighbourhood homeomorphic to \( \mathbb{R}^m_+ \times \mathbb{R}^n_+ \), all homeomorphic to \( \mathbb{R}^{m+n-1}_+ \). By (B), \( x \in \partial (M \times N) \).

Hence, result.
Examples of manifolds:
1. \( \mathbb{R}^m \) is an \( m \)-manifold without boundary - open
2. \( S^m \) is an \( m \)-manifold (spherical projection gives a disc) - closed
3. \( B^n \) is a compact manifold, with boundary \( S^{n-1} \)
4. \( \mathbb{R}^n_+ \) is an \( n \)-manifold with boundary \( \mathbb{R}^n \)
5. Products of these
6. **\( \mathbb{R}^n \)**: orthogonal group \( O(n) \) are manifolds.

There are all differentiable manifolds. Topological manifolds which do not possess a differentiable structure.

**Lemma 1.5** If \( X \subset S^n \) is homeomorphic to \( B^k \), then \( \tilde{H}_k(S^n \setminus X) = 0 \) for all \( k \).

**Proof:** by induction on \( k \).
- True if \( k = 0 \): \( S^n \setminus \{pt\} \cong \mathbb{R}^n \)
- Assume true if \( k \leq l \): we prove it for \( k = l + 1 \).
- Choose homeomorphism \( f: B^l \to X \)
  \[
  \begin{array}{ccc}
  S^n & \to & X \\
  S^n \setminus f(B^l) & \mapsto & \tilde{H}_k(S^n \setminus X)
  \end{array}
  \]
  \( S^n \setminus f(B^l) \)

Take \( \alpha \in \tilde{H}_k(S^n \setminus X) \):
- \( \alpha \) is represented by the boundary of some singular chain lying in \( S^n \setminus f(B^l) \)
- Embed \( N_k \) of \( f(B^l) \) in \( S^n \) such that \( C \) lies in \( S^n \setminus N_k \)
- Open interval \( J \) containing \( t \) such that \( C \) lies in \( S^n \setminus f(B^l \setminus J) \)

Since unit interval is compact, we can cover by finitely many of the \( J_k \).

Let \( \phi_{p,q} : \tilde{H}_k(S^n \setminus X) 
\mapsto \tilde{H}_k(S^n \setminus f(B^l \setminus [t_p,t_q])) \) (where \( p < q \)).

Now \( \phi_{p,p}(n) = 0 \) \( \forall p \).

Suppose inductively that \( \phi_{p,q}(n) = 0 \) starts with \( i = 1 \).

By induction hypothesis, \( \tilde{H}_k(S^n \setminus f(B^l \setminus t_i)) = 0 \) \( s = i, i+1 \).

Sets \( S^n \setminus f(B^l \setminus [t_p,t_q]) \) are open.

We have lattice
\[
\begin{align*}
S^n \setminus f(B^l \setminus t_i) & \\
S^n \setminus f(B^l \setminus [t_p,t_q]) & \\
S^n \setminus f(B^l) & \\
S^n \setminus X & \\
S^n \setminus f(D_1) & \\
S^n \setminus f(D_2) & \\
S^n \setminus f(D_2 \cup D_1) & \\
S^n \setminus X & \\
\end{align*}
\]

**Mayer-Vietoris sequence**
\( 0 \to \tilde{H}_k(S^n \setminus f(D_1 \cup D_2)) \to \tilde{H}_k(S^n \setminus X) \to 0 \)

Since \( \tilde{H}_k(S^n \setminus f(D_1)) = \tilde{H}_k(S^n \setminus f(D_2)) = 0 \) by previous lemma.

Whence result by inductive hypothesis.

\( \square \)
Corollary 1.7 If \( f : S^{n-1} \to S^n \) is 1-1 and continuous, then 
\( S^n \setminus f(S^{n-1}) \) has just two components.

Proof: By Lemma 1.6, \( H_n(S^n \setminus f(S^{n-1})) \approx H_0(S^n) \approx \mathbb{Z} \)
\( S^n \setminus f(S^{n-1}) \) has two components.

\[ \square \]

Corollary 1.8 If \( f : B^n \to S^n \) is 1-1 and continuous, then \( f(\text{int } B^n) \) is open in \( S^n \).

Proof: By Lemma 1.5, \( H_n(S^n \setminus f(B^n)) = 0 \), so \( S^n \setminus f(B^n) \) is connected.

Now \( S^n \setminus f(B^n) = f(\text{int } B^n) \cup S^n \setminus f(B^n) \)
and \( f(\text{int } B^n) \) and \( S^n \setminus f(B^n) \) are connected, while \( S^n \setminus f(B^n) \) is not (by Corollary 1.7). Thus \( f(\text{int } B^n) \) and \( S^n \setminus f(B^n) \) are the components of \( S^n \setminus f(S^n) \), and are closed in \( S^n \setminus f(S^n) \).

Now \( S^n \setminus f(\text{int } B^n) \) open in \( S^n \setminus f(S^n) \), therefore open in \( S^n \).

\[ \square \]

Proof of Theorem 1.2:

We have \( U, V \subseteq \mathbb{R}^n \), homeo \( f : U \to V \), \( U \) open in \( \mathbb{R}^n \).
Choose \( x \in U \) for closed \( n \)-ball \( B^n \) containing \( x \).

Define \( g : \mathbb{R}^n \to S^n \) which is homeo onto \( g(B^n) \)
(e.g. inverse stereographic projection).

\( g : U \to S^n \) is 1-1 and continuous, so by 1.7 \( g(\text{int } B^n) \) is open in \( S^n \),
and \( f(B^n) \) is open in \( \mathbb{R}^n \).

Now \( f(x) \in f(\text{int } B^n) \cap f(U) = V \), so \( V \) is a neighbourhood of \( f(x) \).
Since \( V = f(U) \), \( V \) is open in \( \mathbb{R}^n \).

\[ \square \]

Section 2.1

The generalized Schönflies theorem

Definition 2.1 If \( M, N \) are manifolds, an embedding of \( M \) in \( N \) is 
a map \( f : M \to N \) which is a homeomorphism onto \( f(M) \).
([If \( M \) is compact then any 1-1 continuous map \( f : M \to N \) is 
an embedding, but this is not true in general].)

Theorem 2.2 (Morton Brown's Schönflies theorem)

If \( f : S^{n-1} \times [0,1] \to S^n \) is an embedding, then each component
of \( S^n \setminus f(S^{n-1} \times 0) \) has closure homeomorphic to \( B^n \).

Proof: below.

Definition 2.3 Let \( M \) be a manifold and \( X \subset \text{int } M \).

\( X \) is cellular if it is closed and, for any open set \( U \) containing \( X \),
\( \exists \) set \( Y \subset U \) such that \( Y \approx B^n \) and \( X \subset \text{int } Y \).

Examples: i) Any collapsible polyhedron in \( \mathbb{R}^n \) is cellular.
ii) If \( f : B^n \to S^n \) is any embedding, then \( S^n \setminus f(B^n) \) is cellular.

Lemma 2.4 If \( M \) is a manifold, and \( X \subset M \) is cellular then \( M \setminus X \)
by a homeomorphism fixed on \( \partial M \).

Proof: since \( X \) cellular, \( \forall Y_0 \subset \text{int } M \), \( Y_0 \approx B^n \) and \( X \subset \text{int } Y_0 \).
\( Y_0 \) has metric d. Let \( U_r = \{ y \in Y_0 : d(x, y) < \frac{r}{2} \} \). Define \( Y_r \)
inductively: assume \( Y_{r-1} \subset M \) constructed with \( X \subset \text{int } Y_{r-1} \).

\( X \) cellular \( \Rightarrow \exists Y_r \subset \text{int } Y_{r-1} \cap U_r \) such that \( Y_r \approx B^n \) and \( X \subset \text{int } Y_r \).

\( \text{int } Y_r = \text{interior of } Y_r \) in \( M \)
\( Y_0 \subset \text{int } Y_r \cap Y_{r-1} \subset \ldots \subset X = \bigcap_{r=0}^{\infty} Y_r \)
We construct homeomorphisms \( h_r : M \to M \) such that

1. \( h_0 = 1 \)
2. \( h_r \big|_{M \setminus Y_{r-1}} = h_{r-1} \big|_{M \setminus Y_{r-1}} \)
3. \( h_r(Y_r) \) has diameter \( < \frac{1}{r} \) (w.r.t. metric \( d \))

Suppose \( h_r \) is defined. Choose a homeomorphism
\[
 f : h_r^{-1}(Y_r) \to B^n
\]
Now \( Y_r \subset \text{int} Y_{r-1} \), so \( f(h_r(Y_r)) \subset \text{int} B^n \) and
\[
 \exists \lambda < 1 \text{ s.t. } f(h_r(Y_r)) \subset \lambda B^n, \text{ and also } \exists \epsilon > 0 \text{ s.t. } f^{-1}(\epsilon B^n) \text{ has diameter } < \frac{1}{r}.
\]

Define \( h_r : M \to M \) by
\[
 h_r(x) = \begin{cases} h_r^{-1}(x) & x \in M \setminus Y_{r-1} \\ f^{-1}g f h_r^{-1}(x) & x \in Y_r \end{cases}
\]

To verify (iii):
\[
h_r(Y_r) \subset f^{-1}g f h_r^{-1}(Y_r) \subset f^{-1}g(\lambda B^n) \subset f^{-1}(\epsilon B^n)
\]
has diameter \( < \frac{1}{r} \).

Define \( h(x) = \lim_{r \to \infty} h_r(x) \) for each \( x \in M \).
If \( x \in M \setminus Y_r \), then
\[
h_r(x) = h_{r+1}(x) = \ldots = h(x) \text{ by i), so } h(x) \text{ exists.}
\]
Since
\[
h_r(Y_r) \supset h_{r+1}(Y_r) \supset \ldots \text{ with diam } h_r(Y_r) \to 0
\]
Then \( h_r(Y_r) = \{y\} \) for some \( y \in M \).
If \( x \in X, h_r(x) \in h(Y_r) \) so \( d(h_r(x), y) < \frac{1}{r} \) by (ii), so \( h_r(x) \to y \text{ as } r \to \infty \) and \( h(x) = y \).

Thus \( h \) induces a continuous map
\[
 \hat{h} : M/\sim \to M \quad \hat{h} \big|_{\partial M} = 1.
\]

Since \( h \) coincides with some \( h_r \) outside \( X \)
\[
h \big|_{M \setminus X} \to M \setminus \{y\} \text{ is a homeomorphism}
\]
\[
h(x) = y \text{, so } \hat{h} \text{ is bijective.}
\]
\[
\hat{h} \big|_{M \setminus X} \text{ is open}
\]
If \( U \) is a nhbd. of \( X \) in \( M \) then \( U \supset Y_r \), some \( r \),
\[
\text{so } \forall \epsilon > 0 \text{ nhbd. of } Y_r \subset h(U)
\]
and \( h(U) \) is a nhbd. of \( y \), so \( h(U) \) is open.
\[.\]\[.\]\[.\]
\[
\therefore \hat{h} \text{ is a homeomorphism.}
\]

Lemma 2.5: If \( X \subset \text{int } B^n \) is closed and \( B^n/\sim \) is homeo.
to some subset of \( S^n \), then \( X \) is cellular.
Proof: Let \( f : B^n \to S^n \) induce an embedding \( \hat{f} : B^n/\sim \to S^n \).
suppose \( f(x) = y \).
Then \( f(B^n) = \hat{f}(B^n/\sim) \subset S^n \).
(Apply Thm 1.2 to nhbs. of pts. of \( B^n \).
Let \( U \) be any nhbd. of \( X \) in \( B^n \), \( f(U) \) is a nhbd. of \( y \) in \( S^n \).)
\((11)\) \(f(B^n)\) is a proper closed subset of \(S^n\).

\(\exists\) homeomorphism \(h : S^n \to S^n\) such that \(h\) is defined on some neighborhood of \(y\) and \(h(f(B^n)) \subseteq f(U)\):

\(\forall Y \subseteq S^n, Y \supseteq B^n, f(B^n) \subseteq \text{Int} Y\).

Let \(Z\) be a small convex ball with \(y \in \text{Int} Z\). Then radial map gives \(h\).

Define \(g : B^n \to B^n\) by

\[
g(x) = \begin{cases} f^{-1}f(y) \times x & (y \times x \in f(U) \text{ well-defined}) \\ x & x \in X \end{cases}
\]

Then \(g\) is continuous since \(f = 1\) in \(U\) in \(y\). Also, \(g\) is \(1\).

Now \(g(B^n) \subseteq B^n\) and \(g(B^n) \subset f^{-1}f(U) \subseteq f(U) = U\), and \(y = 1\) on a neighborhood of \(y\). Therefore, \(g(B^n) \cap X\) is cellular.

**Proof of Theorem 2.2.** \(f : S^n \times [-1, 1] \to B^n\) is embedding.

\(S^n \setminus (S^n \times 0)\) has two components, \(D_+\) and \(D_-\).

Say \(f(S^{n-1} \times 1) \subseteq D_+\).

Let \(X_+ = D_+ \setminus f(S^{n-1} \times (0, 1))\)

\[X_- = D_- \setminus f(S^{n-1} \times (-1, 0))\]

Then \(X_+\) and \(X_-\) are both closed, and \(X_+ \cup X_- = S^n \setminus (S^{n-1} \times 1)\).

Note that \(S^n \setminus (S^n \times y) \cong (S^{n-1} \times [-1, 1]) / S^{n-1} \times 1\)

\(\exists\) map \(g : S^n \to S^n\) so that \(g(x) = y\), \(g(y) = y\) and \(g|S^n \times (x \cup X_-)\) is a homeomorphism onto \(S^n \setminus \{y, y\}\).

\((12)\) \(y_+, y_-\) are the poles of \(S^n\).

\(X_+ \cup X_-\) is an embedded subset of \(S^n\).

\(\exists\) homeomorphism \(h : S^n \to S^n\), such that \(h = 1\) on a neighborhood of \(y_+\), and \(h(g(Y)) \subseteq S^n \setminus \{y_+, y_-\}\).

Define \(\phi : Y \to S^n\) by

\[
\phi(x) = \begin{cases} g^{-1}h(y) \times x & x \in X_- \\ x & x \in X_+ \end{cases}
\]

Continuous since \(h = 1\) on a neighborhood of \(y_+\).

\(\phi\) is injective on \(Y \setminus X_+\) and \(\phi(X_+) = g^{-1}h(y_+)\)

\(\phi\) induces an embedding \(\hat{\phi} : Y / X_+ \to S^n\), \(Y \supseteq B^n\).

By Lemma 2.5, \(X_+\) is cellular.

\(\overline{D}_+\) is a manifold with \(X_+ \subseteq \overline{D}_+ \subseteq \text{Int} \overline{D}_+\).

By Lemma 2.4, \(\overline{D}_+ \cong \overline{D}_+ / X_+ \cong S^{n-1} \times \{0, 1\} \cong B^n\)

Similarly for \(D_-\).

**Corollary 2.6** If \(f, g : S^{n-1} \times [1, 1] \to S^n\) are embeddings, then \(S^n \times [1, 1] \to S^n\) is an embedding.
\[ h_f|_{S^{n-1} \times 0} = g|_{S^{n-1} \times 0} \]

**Proof:** If \( \phi : \partial B^n \to \partial B^n \) is a homeomorphism, then \( \phi \) extends to a homeomorphism \( \phi : B^n \to B^n \).

(In obvious way along radii: \( \phi(r \cdot x) = r \cdot \phi(x) \quad 0 < r < 1 \) in \( B^n \).

If \( Y_1, Y_2 \) are homeomorphic to balls and \( \phi : \partial Y_1 \to \partial Y_2 \) is a homeomorphism, then \( \phi \) extends to a homeomorphism \( \phi : Y_1 \to Y_2 \).

Let \( D_+, D_- \) be components of \( S^n \setminus f(S^{n-1} \times 0) \)

\[ E_+, E_- \]

Define \( h|_{f(S^{n-1} \times 0)} \) to be \( g^{-1} \), so \( h : \partial D_+ \to \partial E_+ \).

Since \( \partial D_+ \cong \partial E_+ \cong B^n \), \( h \) can be extended to a homeomorphism \( h : \overline{D_+} \to \overline{E_+} \).

Extend \( h \mid_{\overline{D_-}} \to \partial E_- \) (already defined) to homeo \( h \mid_{\overline{D_-}} \to \overline{E_-} \).

Obtain homeo \( h : S^n \to S^n \) with \( h|_{S^{n-1} \times 0} = g|_{S^{n-1} \times 0} \).

**Definition 2.7** A collar of \( \partial M \) in \( M \) is an embedding \( f: \partial M \times I \to M \) such that \( f(x, 0) = x \quad (x \in \partial M) \).

Exercise \( f(\partial M \times 1) \) is a neighbourhood of \( \partial M \) in \( M \).

**Remark:** From now on, we only consider metrizable manifolds (i.e., ones which are 2nd countable).

Exercise Compact manifolds are metrizable.

Theorem 2.8 (Morton Brown) If \( M \) is metrizable, then \( \partial M \) has a collar in \( M \).

If \( U \) is an open set in \( \partial M \), say that \( U \) is collared if \( U \) has a collar in the manifold \( M \setminus U \).

Let \( V \subseteq U \) be a smaller open set.

Let \( \lambda : U \to [0, 1] \) be a continuous map such that \( \lambda(x) = 0 \iff x \notin V \).

Define a **spindle neighbourhood** of \( V \) in \( U \times I \) to be

\[ S(V, \lambda) = \{ (x, t) \in U \times I \mid t < \lambda(x) \} \]

(open: so a neighbourhood of \( V \times 0 \)).

**Lemma 2.9** Let \( f : S(V, \lambda) \to U \times I \) be an embedding with \( f|_{V \times 0} = 1 \). Then \( \exists \) homeo \( h : U \times I \to U \times I \) such that:

i) \( h_f = 1 \) on \( S(V, \mu) \) for some \( \mu \) s.t. \( \mu < \lambda \)

ii) \( h|_{V \times I \setminus f(S(V, \lambda))} \) is identity.

**Proof:** Spindle nbhds form a base of nbhds.
Lemma 2.10 If \( U, V \subset \mathfrak{A} M \) are collared, then \( U u V \) is collared.

Proof: Let \( f : U \times I \rightarrow M \), \( g : V \times I \rightarrow M \) be collared.

Choose \( \lambda : U u V \rightarrow I \) so that \( s(U \cap V, \lambda) = f^\lambda g^\lambda(x, t) \).

Apply Lemma 2.9 to the embedding

\[ g^\lambda f^\lambda : s(U \cap V, \lambda) \hookrightarrow V \times I. \]

\( \exists S(U \cap V, \mu) \subset S(U \cap V, \lambda) \) and a homeomorphism

\[ h : V \times I \rightarrow V \times I \]

s.t. \( \text{sg}^\lambda f^\lambda \big|_{S(U \cap V, \mu)} = 1 \). Then \( gh^{-1} \) and \( f \) agree on \( S(U \cap V, \mu) \).

Define open set \( U_1 \subset U \times I \) by

\[ U_1 = \{ (x, t) \in U \times I | d(x, (U \cap V, 0)) < d(x, (V \cap V, 0)) \}. \]

Define \( V_2 \subset V \times I \) similarly: then

\[ (x, t) \in \text{sg}^\mu f^\mu(x, t) \]

Continuity of \( h \) is simply verified.

In fact, \( h \) is a homeomorphism s.t. \( h f = 1 \) on \( S(V, \mu) \) and \( h = 1 \) on \( f(S(V, \mu)) \).

\[ \square \]
Then $U_3, V_3$ open, $U_3 \cap V_3 = \emptyset$, $f(U_3) \cap g^{-1}(V_3) = \emptyset$
$(U \setminus V) \times 0 \subset U_3$, $(V \setminus U) \times 0 \subset V_3$ so
$W = U_3 \cup S(U \cup V, \mu) \cup V_3$ is a neighbourhood of $(U \cup V) \times 0$ in $(U \cup V) \times I$.
Define $\phi : W \to M$ by $\phi(x) = \begin{cases} f(x) & x \in U_3 \cup S(U \cup V, \mu) \\ g(x) & x \in S(U \cup V, \mu) \cup V_3 \end{cases}$.
Then $\phi$ is well-defined, and continuous.
Continuous and 1-1.

$\exists \nu : U \cup V \to I$ such that $S(U \cup V, \nu) \subset W$.
Define $\psi : (U \cup V) \times I \to M$;
$(x, t) \mapsto \phi(x, t \nu(x))$. This is continuous and 1-1, and hence an embedding (invariance of domain).

Proof of Theorem 2.8:

1) collared sets cover $\partial M$ because
$x \in \partial M \Rightarrow \exists \text{homeo } f : \mathbb{R}^n \times [0, 1) \to M$ onto a nbhd of $x$ in $M$
We proved in Cor 3.3 $f(\mathbb{R}^n \times 0)$ contains a nbhd $U(x) \times 0$ of $x$ in $M$.

Then $U$ has a collar given by
$g : U \times I \to M$; $(y, t) \mapsto f(g^{-1}(y), t)$.

If $\partial M$ is compact, then $\partial M$ is collared by
Lemma 2.10. Before proceeding to general case we prove:

Lemma 2.10. Let $U_\alpha (\alpha \in A)$ be a disjoint family
of open collared sets. Then $\cup U_\alpha$ is collared.

Proof: Let $V_\alpha = \{ y \in M | d(y, U_\alpha) < d(y, U_{\beta} \cup U_{\beta}) \}$.
This is an open neighbourhood of $U_\alpha$ in $M$ and
$\alpha \neq \beta \Rightarrow V_\alpha \cap V_\beta = \emptyset$.
Let $W_\alpha = f_\alpha^{-1}(V_\alpha)$, a neighbourhood of $U_\alpha \times 0$ in $U_\alpha \times I$.

$\exists \nu : U_\alpha \to I$ such that $S(U_\alpha, \nu) \subset W_\alpha$.
Define $g_\alpha : U_\alpha \times I \to M$ by
$g_\alpha(x, t) = f_\alpha(x, t \nu(x)/2) \in V_\alpha$.

Define $g = \bigcup_{\alpha \in A} (U_\alpha \times I) \to M$.
This is a collar of $\cup U_\alpha$ in $M$.
We have proved that if $X=\mathbb{M}$ then

i) $X$ is collared by collared sets
ii) finite union of collared sets collared
iii) disjoint union of collared sets is collared
iv) open sets of collared sets are collared.

Then

i) - iv) $\Rightarrow X$ metric $\Rightarrow X$ collared.

**Lemma 2.10**

Any countable union of collared sets is collared.

**Proof.**

To consider countable nested unions

$$U = \bigcup\limits_{n=1}^{\infty} U_n$$

with $U_1 \subset U_2 \subset \ldots$.

Put $V_n = \{x \in U_n \mid d(x, X \setminus U_n) > 2^{-n}\}$

Then $U = \bigcup\limits_{n=1}^{\infty} V_n$ for $x \in U_k \Rightarrow \exists n : B(x, 2^{-n}) < U_k$

$\Rightarrow d(x, X \setminus U_k) > 2^{-n} \Rightarrow d(x, X \setminus U_n) > 2^{-n} \Rightarrow x \in V_n$.

Now $V_n \subset V_{n+1}$ and

Let $A_k = V_{2k+1} \setminus V_{2k-1}$ $B_k = V_{2k+1} \setminus V_{2k}$.

Then $A = \bigcup\limits_{k=1}^{\infty} A_k$ is disjoint union of collared sets - hence collared. Similarly for $B = \bigcup\limits_{k=1}^{\infty} B_k$.

Now $U = A \cup B \cup V_2$ is collared.

**Lemma 2.10**

Every open cover of a metric space $X$ has a $\sigma$-disjoint refinement.

**Proof.** (cf. Kelley, p. 129)

Let $\mathcal{U}$ be an open cover of metric space $X$.

If $U \in \mathcal{U}$, let $U_n = \{x \in U \mid d(x, X \setminus U) > 2^{-n}\}$

Then $d(U_n, X \setminus U_n) > 2^{-(n+1)}$

Well-order $\mathcal{U}$ by relation $\prec$.

Let $U^*_n = U_n \setminus \bigcup_{V \in U \setminus \mathcal{U}} V_n$.

If $U \nless V$ then $U \prec V$ or $U \not\succ V$.

$$U^*_n \subset X \setminus V_{n+1}, U^*_n \subset X \setminus V_{n+1}$$

and in either case $d(U^*_n, U^*_n) > 2^{-(n+1)}$.

Let $U_n^* = \text{open } 2^{-(n+1)}$ neighbourhood of $U^*_n < (0, 1)$.

$U \prec V \Rightarrow U_n \cap V_n$ disjoint.

Enough to prove $\bigcup_{n=1}^{\infty} U_n = X$.

If $x \in X$ let $U$ be first (w.r.t. $\prec$) member of $\mathcal{U}$ containing $x$. Then $x \in U_n$ for some $n$ and
(21) \[ x \in U_n \subset U_n' \]

Now \( E \cup U_n' \) is a \( \sigma \)-disjoint refinement of \( H \).

\[ □ \]

\[ □ \text{Thm.} 2.8 \]

References:
2) Morton Brown: ”Locally flat embeddings of topological manifolds” Annaals of Maths. 75 (1962) 331-341

A shortened version of 2) is included in the book “Topology of 3-manifolds”.

Definition 2.11 Let \( M^n, N^n \) be manifolds without boundary. An embedding \( f: M^n \rightarrow N^n \) is locally flat if, for all \( x \in M \), a neighbourhood \( U \) of \( x \) and embedding \( F: U \times \mathbb{R}^{n-m} \rightarrow N^n \) s.t. \( F(y,0) = f(y) \) (\( y \in \partial U \)).

N.B. needn’t be an embedding \( g: M \times \mathbb{R}^{n-m} \rightarrow N^n \) s.t. \( g(y,0) = f(y) \) (\( y \)).

I.e. \( S^n \rightarrow \text{Möbius strip (along centre line). This is locally flat but \# embeddings } S^n \rightarrow M \text{ agreeing with previous one on } S^n \).

(22) Examples 1) If \( f: S^{n-1} \rightarrow S^n \) is locally flat then each component of \( S^n \setminus f(S^{n-1}) \) has closure homeo to \( B^n \).

2) If \( E \) is compact, and \( f, g: \Theta M \times I \rightarrow M \) are two collars, then \( \exists \) homeomorphism \( h: M \rightarrow M \) such that \( hf \) agrees with \( g \) on \( \Theta M \times [0, \frac{1}{3}] \), and \( h = 1 \) outside \( f(\Theta M \times I) \cup g(\Theta M \times I) \).

"Collaring of \( \Theta M \) in \( M \) is unique"

Not true if \( \Theta M \) non-compact; (Milnor’s rising sun).

Exercise: suggest a generalization that does work.

Given two manifolds \( M^n, N^n \) let \( E(M,N) \) be the set of embeddings of \( M \) in \( N \) with the compact-open topology.

A map \( f: X \rightarrow Y \) is proper if \( C \subseteq Y \) compact \( \Rightarrow f^{-1}(C) \subseteq X \) compact.

Let \( E_p(M,N) \) be the set of embeddings \( \mathfrak{a} \) which are proper maps.

We shall be interested in \( E_p(\mathbb{R}^n \setminus \text{Int } B^n, \mathbb{R}^n) \), which consists of embeddings \( f: \mathbb{R}^n \setminus \text{Int } B^n \rightarrow \mathbb{R}^n \) onto \( \mathbb{R}^n \setminus \text{Int } B^n \).

\[ □ \]
Let $\hat{\mathbb{R}}^n$ be the 1-pt. compactification of $\mathbb{R}^n$. 
$f: \hat{\mathbb{R}}^n \setminus \text{int } B^n \to \mathbb{R}^n$ extends to a continous map 
\[ \hat{f}: \hat{\mathbb{R}}^n \setminus \text{int } B^n \to \mathbb{R}^n \quad (\hat{f}(\infty) = 0) \iff f \text{ is proper} \] 
(in general $f: X \to Y$ extends to a continous map 
\[ f: \hat{X} \to \hat{Y} \quad (\hat{f}(\infty) = 0) \iff f \text{ is proper} \). 

**Theorem 2.12** There is a neighbourhood $U$ of 1 in 
\[ E(6B^n \setminus \text{int } B^n, \mathbb{R}^n) \] 
and a continuous map 
\[ \Theta: U \to E_p(\mathbb{R}^n \setminus \text{int } B^n, \mathbb{R}^n) \] 
s.t. $\Theta(f)|_{S^{n-1}} = f|_{S^{n-1}}$.

**Proof:** Take $U = \{f \in E(6B^n \setminus \text{int } B^n, \mathbb{R}^n) \mid d(x, f(x)) < 1, \quad x \in 6B^n \setminus \text{int } B^n \}$ If $f \in U$, then $f(2B^n \setminus \text{int } B^n) \subseteq \text{int } 3B^n \setminus \partial B^n$ and $f(6B^n \setminus \text{int } 5B^n) \subseteq f(7B^n \setminus \text{int } 4B^n)$.

Define inductively 
\[ f_k: (4k+6)B^n \setminus \text{int } B^n \to \mathbb{R}^n \] 
\[ f_k|_{(4k+6)B^n \setminus \text{int } B^n} = f_k|_{(4k+5)B^n \setminus \text{int } B^n} \]

i) $f_0 = f$

ii) $f_k|_{(4k+6)B^n \setminus \text{int } B^n} = f_k|_{(4k+5)B^n \setminus \text{int } B^n}$

Suppose $f_k$ constructed.

If $a, b, c, d \in (a, b)$ are fixed outside $6B^n \setminus a B^n$, taking $C B^n$ onto $d B^n$.

Let $g_k(c, d) = p(4k+6, 4k+7, 4k+8, 4k+9, 4k+10)$.

Define $g_k: (4k+6)B^n \setminus \text{int } B^n \to \mathbb{R}^n$ by 
\[ g_k = p_k(3, 11, 4, 8) f_k p_k(1, 5^{3/2}, 5^{3/2}, 2) \]

Define homeo $h_k: \mathbb{R}^n \to \mathbb{R}^n$ by 
\[ h_k(x) = \begin{cases} f_k p_k(1, 5^{3/2}, 2, 5^{3/2}) f_k^{-1}(x) & (x \in \text{image } f_k) \\ x & (\text{otherwise}) \end{cases} \]

Define $h_k: (4k+10)B^n \to (4k+6)B^n$ be a radial homeo fixed on (4k+5)B^n, sending 
\[ (4k+6)B^n \to (4k+7)B^n \]

Define 
\[ f_{k+1} = h_k g_k \sigma_k: (4k+10)B^n \setminus \text{int } B^n \to \mathbb{R}^n \]

Check i) let $x \in (4k+5)B^n$ so $g_k(x) = x$, $p_k(1, 5^{3/2}, 5^{3/2}, 2) p_k^{-1}(x) = y = f_k p_k(1, 5^{3/2}, 5^{3/2}, 2)(x) \in (4k+3)B^n$ (induction hypothesis), $g_k(y) = p_k(3, 11, 4, 8)(y) = z$.
Similarly, we can verify iii)  

To prove iv), that $f_k$ depends continuously on $f$, it is enough to show that $h_k$ depends continuously on $f_k$. Let $f_k'$ be near $f_k$, and let

$$h_k' = f_k' p_k f_k^{-1}$$

on $w_{f_k}$ where $p_k = p_k(1/2^k)$.  

If $C$ is a compact set in $\mathbb{R}^n$, must prove that $\sup_{y \in C} d(h_k x, h'_k x)$ can be made $< \varepsilon$ by requiring $d(f_k(y), f_k'(y)) = 0$ ($\forall y \in A_k$)

$$f_k \rightarrow h_k \in (68k! \mid \int B, \mathbb{R}) \rightarrow \text{Homeo} \mathbb{R}^n$$

Let $A_k = (2k+1) B^n \setminus \text{int} B^n = \text{domain of } f_k$

Given $\varepsilon > 0$  $\exists r > 0$ s.t. $y, y' \in A \Rightarrow d(y, y') < \eta$ \Rightarrow $d(f_k(y), f_k(y')) < \varepsilon$. Since $f_k$ injective, $\forall s > 0$

s.t. $y, y' \in A \Rightarrow d(y, y') > \eta \Rightarrow d(f_k(y), f_k(y')) > \delta$.

We suppose $\delta < \varepsilon/2$.

Suppose $d(f_k(y), f_k'(y)) < \delta/2$ for all $y \in A$

Let $x \in C$. We split into cases.

i) $x \in \text{inf } f_k \setminus \text{inf } f_k'$, say $x = f_k(y), f_k'(y) = f_k(y')$

Then $d(f_k(y), f_k'(y')) = d(f_k'(y'), f_k(y')) < \delta/2 < \delta$

$\Rightarrow d(x, y') < \eta$, so

$$d(h_k x, h'_k x) = d(f_k p_k y_k, f'_k p_k y'_k)$$

$$< d(f_k p_k y_k, f_k p_k y_k) + d(f_k p_k y_k, f'_k p_k y_k)$$

$$< \varepsilon/2 + \delta/2 < \varepsilon$$

ii) If $x \in \text{inf } f_k \setminus \text{inf } f_k'$, say $x = f_k(z)$

Then $d(x, f_k(y)) < \delta/2$, so $\exists z \in \text{int} A$ s.t. $d(f_k(z), f_k(y)) < \delta/2$.

But $d(f_k(z), f_k'(z)) < \delta/2$, so $d(x, f_k'(z)) < \delta$, so $d(y, z) < \eta$ 

$\Rightarrow d(h_k x, h'_k x) = d(f_k p_k y_k, x) 

< d(f_k p_k y_k, f_k p_k z) + d(f_k p_k z, f'_k p_k y_k) 

< \varepsilon/2 + \delta/2 < \varepsilon$.

iii) $f_k(x) \leq \text{inf } f_k$ or $f_k'(x) \leq \text{inf } f_k'$ similar

iv) If $x \notin \text{inf } f_k \cup \text{inf } f_k'$, nothing to prove.
We have proved \( f_k \to h_k \) continuous.

\( f \to f_k \) is continuous if \( f \to f_k \) is 

\( \forall \) by ii) on (28)

\[
\Theta(\hat{f}) : \mathbb{R}^n \setminus \text{int } B^n \to \mathbb{R}^n \text{ by } \\
\Theta(\hat{f})(x) = \hat{f}(x) \quad (x \text{ large } x \in (k+1)B^n)
\]

Then \( \Theta(\hat{f}) \) is proper (interleaving prop iii) on (24)

and also \( \Theta(\hat{f}) \) is an embedding, \( \Theta(\hat{f}) \in \mathcal{E}_p(\mathbb{R}^n \setminus \text{int } B^n, \mathbb{R}^n) \)

\( \Theta(\hat{f}) \) depends continuously on \( f \) because \( f_k \) agrees with \( f \) on \( (k+1)B^n \) and \( f_k \) depends continuously on \( f \).

\[ \square \]

**Corollary 2.13** If \( 0 < \lambda < 1 \), then \( \exists \) neighbourhood \( V \)

of \( 1 \in \mathcal{E}(\mathbb{R}^n \setminus \text{int } B^n, \mathbb{R}^n) \) and a continuous map

\[ f : V \to \mathcal{E}(\mathbb{R}^n \setminus \text{int } B^n, \mathbb{R}^n) \text{ s.t. } \forall f, \quad \Theta(f) = f \big|_{\partial V} \]

\[ \text{Proof: } \hat{X} \text{ is 1-pt. compactification of } X \]

if \( g : X \to Y \) proper \( \iff \) \( g \) extends to \( \hat{g} : \hat{X} \to Y \) with \( \hat{g}(\infty) = -\infty \).
Have to show $\hat{g}(C) \subseteq U$ for all $g \in N$
$\hat{g}(C) = \hat{g}(C \cap (B^n)_0) \cup \hat{g}(R^n \setminus B^n)$

$C \subseteq U \cup$ one of complimentary domain

In fact $U \cup$ outside domain $C \cap (B^n)_0$

Hence the map $g \rightarrow \hat{g}$ is continuous.

$E(B^n \cup B^n, R^n) \subseteq U \rightarrow E_p(R^n \cup R^n, R^n)$

$\exists$ homeomorphisms $h : \hat{R^n} \rightarrow \hat{R^n}$

\[ h(x) = \begin{cases} \frac{x}{\|x\|_2} x + o, & x \neq 0, \\ x, & x = 0 \end{cases} \]

causes $B^n \cup B^n$ onto $B^n \cup B^n$

Let $\hat{2}$ taking $\hat{R^n} \cup B^n \rightarrow B^n$

Hence result

\[ \square \]

\section{Properties of Tori}

\textbf{Definition 3.1} Let $\mathbb{Z}^n$ be integer lattice in $\mathbb{R}^n$. Then $T^n = \mathbb{R}^n / \mathbb{Z}^n$ is the \textit{n-dimensional torus} (Clearly $\mathbb{R}^n = S^1 \times \ldots \times S^1$ $n$-times.)

Let $e : \mathbb{R}^n \rightarrow T^n$ be projection map.

If $a \in \mathbb{Z}^n$, let $\tau_a : \mathbb{R}^n \rightarrow \mathbb{R}^n, x \rightarrow a + x$.

\textbf{Proposition 3.2} $e : \mathbb{R}^n \rightarrow T^n$ is a universal covering of $T^n$. If $X$ is a simply-connected space and $f : X \rightarrow T^n$ is any map then $\hat{f} : X \rightarrow \mathbb{R}^n$ such that $f = e \hat{f}$. ($\hat{f}$ is a lift of $f$).

If $\hat{f}_1, \hat{f}_2$ are lifts of $f$ then $\hat{f}_1 = \tau_a \hat{f}_2$

for some $a \in \mathbb{Z}^n$. \[ \square \]
If \( X \) is simply connected \( f: X \times \mathbb{T}^n \to X \times \mathbb{T}^n \) is a map, then \( \tilde{f} \) is a map \( X \times \mathbb{R}^n \to X \times \mathbb{R}^n \) such that \( e\tilde{f} = f\).

**Lemma** 3.3 If \( f \) is a homeomorphism so is \( \tilde{f} \); if \( f \) is identity, then \( \tilde{f} \) commutes with the covering translations.

**Proof:** \( f \) homeo, inverse \( g \). Form \( \tilde{f}, \tilde{g} \).

\[
e \tilde{f} \tilde{g} = f \tilde{g} = g f e = e
\]

\[
\therefore \tilde{f} \tilde{g} = \tilde{a} \text{ for some } a . \text{ Similarly } \tilde{g} \tilde{f} = \tilde{b} .
\]

\[
\therefore \tilde{f} \text{ commutes with } T_a . \]

**Definition** 3.4 Let \( M, N \) be manifolds. An immersion \( f: M \to N \) is a map such that each point \( x \in M \) has a neighbourhood \( U_x \) with \( f|_{U_x} \) an embedding. If \( U_x \) can be chosen so that \( f|_{U_x} \) is locally flat then \( f \) is locally flat immersion.
Theorem 3.5 There is an immersion of 
\((T^n \setminus \text{pt})\) in \(R^n\).

Proof: (1) \(T^n \setminus \text{pt}\) is open parallelizable manifold

By Hirsch's theory of immersions \(F^m\) immersion \(T^n \setminus \text{pt} \to R^n\).

Alternatively:

(2) Regard \(T^n\) as the product of \(n\) circles. \(T = T^1\) circle.
Let \(J\) be a closed interval in \(T\).
\(T^n \setminus J^n \cong T^n \setminus \text{pt}\) (all open manifolds).
\(T^n \setminus (2J)^n \implies T^n \setminus J^n \to T^n \setminus \text{pt}\).

Assume inductively that \(J^n\) immersion \(T^n \setminus J^n \to R^n\)

such that \(f_n \times 1 : (T^n \setminus J^n) \to [1,1] \to R^n \times R = R^{n+1}\)
extends to an immersion \(g_n : T^n \times [1,1] \to R^{n+1}\).

Induction starts with \(n=1\).

(\text{in fact we can find embeddings)  \(\nabla\) \(\nabla\) \(\nabla\) \(\nabla\))

Let \(\xi_0 : T \setminus J \to [-1,1]\) be a homeo.

Choose embedding \(\xi : T \to R \times R\) s.t. \(\xi(0,0) = 0\).

Then \(\xi(x,0) = \xi(x,1) = (x, f_0(x))\).

Extend \(\xi_0 : [-1,1] \to T \setminus J\) to an embedding \(\psi : R \to T\).

Suppose \(f_n, g_n\) constructed

\(T^{n+1} \setminus J^{n+1} = (T^n \setminus J^n) \times T \cup T^n \times (T \setminus J)\)

Define \(f_{n+1} : T^{n+1} \setminus J^{n+1} \to R^{n+1}\) by

\[f_{n+1} = (1 \times f_n \times \psi) \cup (1 \times \psi) g_n (1 \times f_n \times \psi^{-1})\]

On \((T^n \setminus J^n) \times (T \setminus J)\), \(g_n = f_n \times 1\)

so \((1 \times \psi) g_n (1 \times \psi^{-1}) = (1 \times \psi) (f_n \times 1) (1 \times \psi^{-1}) = f_n \times 1\)

Let \(J' = T \setminus \Phi([-\frac{1}{4}, \frac{1}{4}])\), \(\Phi \subset \text{int } J\).

We shall construct immersion \(g_{n+1} : T^{n+1} \times [1,1] \to R^{n+2}\)

which agrees with \(f_n \times 1\) on \(T^{n+1} \setminus J^{n+1} \times [-\frac{1}{4}, \frac{1}{4}]\).

This will be enough, since \(T^{n+1} \setminus J^{n+1} \cong T^{n+1} \setminus J^{n+1}\)

Define \(\Theta_c : T \to R^3 (\subset R^3)\) by

\[\Theta_c(z) = \begin{cases} \frac{z}{2} e^{2(121z - \frac{1}{2}) \pi} & 121 \leq \frac{1}{2} \\ \frac{z}{2} e^{2(121z - \frac{1}{2}) \pi} & \frac{1}{2} \leq 121 \leq \frac{3}{4} \\ 121 \geq \frac{3}{4} \end{cases}\]

Let \(J^0 = T \setminus \psi([-\frac{3}{4}, \frac{3}{4}])\)

Let \(\lambda : T^n \to [0,1]\) be continuous s.t. \(\lambda^{-1} \frac{1}{4} \geq 1\).

\[\int_{T^n \setminus J^0} = 0.\]
S 4. Local contractibility

Definition 4.1 A space $X$ is locally contractible if for each point $x \in X$ and each nbhd $U$ of $x$, if $\text{nbhd } V$ of $x$ and homotopy $H: V \times I \rightarrow U$ such that $H_0 = 1$, $H_1(V) = x$. 

Let $X^I$ be set of paths in $X$ ending at $x$. Enough to find nbhd $V$ and map $\phi: V \rightarrow X^I$ s.t. $\phi(y)$ is a path from $y$ to $x$ and $\phi(x) = \text{constant path at } x$.

Given open nbhd $U$ of $x$, $U^I$ is open set in $X^I$, so if nbhd $V$ of $x$ in $X$ s.t. $\phi(V') \subset U^I$.

If $M$ is a manifold, let $\mathcal{H}(M)$ be the space of homeomorphisms of $M$, together with the compact-open topology.
Definition 4.2. An isotopy of $M$ is a path in $H(M)$. Equivalently, an isotopy is a homeomorphism $H: M \times I \to M \times I$ such that $p_2 H = p_2$. We say that $H$ is an isotopy from $H_0$ to $H_1$, and $H_0, H_1$ are isotopic.

\[\square\]

Theorem 4.3. $H(M)$ is locally contractible.

(Cermsky, Kirby).

Proof: $H(\mathbb{R}^n)$ is a group, so it is enough to show that it is locally contractible at 1.

Choose embedding $i: 4B^n \to T^n$ and choose immersion $f: T^n \setminus i(0) \to \mathbb{R}^n$.

$T^n \setminus i(\text{Int } B^n)$ is compact, if $\delta > 0$ s.t. for all $x \in T^n \setminus i(\text{Int } B^n)$, $\delta \cdot N_\delta(x)$ is injective.

We may suppose $\delta < d(i(3B^n \setminus \text{Int } 2B^n), 4S^{n-1} \cup S^{n-1})$.

Since $f$ is open, $\varepsilon_x = d(f(x), \mathbb{R}^n \setminus N_\delta(f(x))) > 0$.

$E = \inf \{\varepsilon_x : x \in T^n \setminus i(\text{Int } B^n)\} > 0$.

If $x \in T^n \setminus i(\text{Int } B^n)$ and $u \in \mathbb{R}^n$ are such that $d(f(x), u) < \varepsilon$, then there is unique $w \in N_\delta(x)$ such that $fu = w$.

Let $h \in H(\mathbb{R}^n)$ such that $d(h(f(x)), f(u)) < \varepsilon$ for all $x \in T^n \setminus i(\text{Int } B^n)$.

For $x \in T^n \setminus i(\text{Int } B^n)$, let $h(x)$ be the unique point in $N_\delta(x)$ such that $fh(x) = h(f(x), x) \in T^n \setminus i(\text{Int } B^n)$.

Since $f$ is an open immersion, $h$ is an open immersion. If $h(x) = h(y)$, then

\[
x, y \in N_\delta(h(x)) \Rightarrow f(x) = f(y) = h(x) \Rightarrow h(x) + h(y) \neq h(x) + h(y).
\]

$k$ is an embedding depending continuously on $h \in H(\mathbb{R}^n)$.

Consider $i, k: 3B^n \setminus \text{Int } 2B^n \to \text{Int } 4B^n$.

By Cor 2.13, for $V : W \to S(3B^n \setminus \text{Int } 2B^n, \text{Int } 4B^n)$ and continuous map $\phi: W \to S(3B^n, \text{Int } 4B^n)$ s.t.

$\phi(g) |_{3B^n} = g |_{3B^n}$.

Define $k': T^n \to T^n$ by $x \mapsto \{i(k'(x)), x \in i(3B^n)\}$.

Then $k': T^n \to T^n$ is a homeo, depending continuously on $h \in V$, where

\[V = \{h \in H(\mathbb{R}^n) : h \text{ is defined and } h \cdot i \in W\}\]

If $V$ is sufficiently small, then $h \in V \Rightarrow h \equiv 1$ homotopic only.

Let $e: \mathbb{R}^n \to T^n$ be the (universal) covering map.

By 3.8, $h: \mathbb{R}^n \to \mathbb{R}^n$ such that $e = e^{h_{1}}$.

If $V$ is sufficiently small, there is a unique choice of $h$ such that $d(h(0), 0) < \frac{1}{2}$. Then $h$ depends continuously on $h$.  

\[\square\]
Choose \( s \), \( 0 < s < r \). If \( V \) is a small enough,
\[
\text{he} V \rightarrow \tilde{h}(sB^n) \subset \text{int tB}^n.
\]

Define isotopy \( S_t \) of \( \text{IR}^n \) by
\[
S_t(x) = \sigma R_t \sigma^{-1}(x).
\]
This depends continuously on \( h \). \( S_0 = 1 \)
\[
S_t \mid f_\text{e}(sB^n) = h \mid f_\text{e}(sB^n)
\]
\[
x \rightarrow \sigma^{-1} x \overset{e}{\rightarrow} \sigma^{-1} x \rightarrow h \sigma^{-1} x \rightarrow \sigma h \sigma^{-1} x
\]
W.l.o.g. \( O \in \text{int fe}(sB^n) \). \( S_t, h \) is 1 on a nbhd of \( O \)
Define \( F_t : \text{IR}^n \rightarrow \text{IR}^n \) by
\[
F_t(x) = \left\{ \begin{array}{ll}
t' S^{-1} h(tx) & \quad (t > 0) \\
0 & \quad (t = 0)
\end{array} \right.
\]
Define \( H_t = S_t F_t \) \( \left( \text{i.e. } H_t(x) = S_t(F_t(x)) \right) \).
This is isotopy from 1 to \( h \).
\( H_t \) depends continuously on \( h \).
and
\[
h = 1 \Rightarrow H_t = 1,
\]
So \( H(\text{IR}^n) \) is locally contractible.
What about $H(M)$ for (say) $M$ compact?

(Use handle decomposition)

Let $\mathcal{E}(k\text{-handle})$ be space of embeddings of $B^k \times B^n$ leaving $(\partial B^k) \times B^n$ fixed.

**Theorem 4.4** There is a neighbourhood $V$ of 1 in $\mathcal{E}(k\text{-handle})$ and a homotopy $H: V \times I \to \mathcal{E}(k\text{-handle})$ s.t.

i) $H_t(1) = 1$ $\quad (\forall t)$

ii) $H_0(h) = h$ $\quad (\forall h \in V)$

iii) $H_t(h)|_{\partial B^k \times B^n} = 1$

iv) $H_t(h)|_{\partial B^k \times B^n} = h|_{\partial B^k \times B^n}$ $\quad \forall t, h$.

**Proof:** Let $i: T^n \to T^n$ be fixed embedding, $f: T^n \setminus \text{int} \to T^n$ fixed immersion.

Choose $r > 0$ s.t. $r < 1$, $f(e(rB^n))$ injective $\in (rB^n) \cap (4B^n)$.

Modify $f$ so that $f(e(\text{int} \cap B^n)) > \frac{1}{2} B^n$.

**End of proof.**

Let $h \in \mathcal{E}(k\text{-handle})$ be close to 1.

First preliminary homotopy $G_t$ from $h$ to $g \in \mathcal{E}(k\text{-handle})$ such that $\partial |_{B^k \setminus \frac{1}{2} B^k \times B^n} = 1$.

$G_t(x, y) = \begin{cases} 
(x, y) & (\text{if } 1 \leq t \leq \frac{1}{2}) \\
((1-\frac{t}{2})h_t((1-\frac{t}{2})x, y), h_t(x, y)) & \text{otherwise}
\end{cases}$

where $h_t: (x, y) \mapsto ((1-\frac{t}{2})x, y)$.

$1 \times 1 \leq 1 - \frac{t}{2}$

This definition has to be modified to have $G_0 = h$, $G_1 = g$

is an embedding fixed on $B^k \setminus \frac{1}{2} B^k \times B^n$. $G$ depends continuously on $h$ and $G_t|_{B^k \times \partial B^n} = h|_{B^k \times \partial B^n}$.

As in 4.3 construct embedding $g': B^k \times (T^n \setminus i(\text{int}B^n)) \to B^k \times (T^n \setminus i(\text{int}B^n))$ s.t. $h(x, y)$.

and $g'|_{B^k \setminus \frac{1}{2} B^k \times (T^n \setminus \text{int}B^n)} = 1$.

Put $g'|_{B^k \setminus \frac{1}{2} B^k \times \text{int}B^n} = 1$. This extends the $g'$ already defined.

Use 2.13 to extend $g'|_{B^k \times (\text{int}B^n \setminus \text{int}(\frac{1}{2} B^k \times i(2B^n)))}$

to embedding $g'': \frac{B^k}{4} \times i(3B^n) \to B^k \times i(4B^n)$.
s.t. \( g'' = g' \) on \( \partial (\frac{3}{4}B_k \times i(3B_n)) \).

Let \( \tilde{g} : B^k \times \mathbb{R}^n \rightarrow B^k \times \mathbb{R}^n \) be such that

\[
(1x) \tilde{g} = g''(1x) \quad \text{and} \quad \tilde{g} |_{\partial B^k \times B^n} = 1.
\]

\( \tilde{g} \) is bounded, i.e.

\[
d(x, \tilde{g}(x)) \leq A \quad (x \in B^k \times \mathbb{R}^n)
\]

Extend \( \tilde{g} \) to a homeo of \( \mathbb{R}^k \times \mathbb{R}^n \) by

\[
\tilde{g} |_{(\mathbb{R}^k \setminus B^k) \times \mathbb{R}^n} = 1.
\]

Define \( p : \text{ht}(2B^k \times 2B^n) \rightarrow \mathbb{R}^k \times \mathbb{R}^n \), homeomorphism fixing \( B^k \times B^n \),

\[
p(x, y) = \begin{cases} (x, y) & \text{if } (x, y) \in B^k \times B^n \\ (2 - \max(1, y_0))^{-1}(x, y) & \text{if } (x, y) \not\in B^k \times B^n \end{cases}
\]

Then \( \rho \circ \tilde{g} \circ p : \text{ht}(2B^k \times 2B^n) \rightarrow \mathbb{R}^k \times \mathbb{R}^n \) extends to a

homeomorphism of \( 2B^k \times 2B^n \), fixed on \( \partial (2B^k \times 2B^n) \).

In fact, \( \rho \circ \tilde{g} \circ p \) fixes \( (2B^k \setminus \text{ht } B^k) \times 2B^n \).

Thus \( \rho \circ \tilde{g} \circ p \) defines homeo of \( B^k \times 2B^n \) fixed on \( \partial (B^k \times 2B^n) \). Define isotopy \( R_t \) of \( B^k \times 2B^n \) by

\[
R_t(x, y) = \rho \circ (t \circ \tilde{g} \circ p)(x, y) \quad \max(1, y_0) \leq t
\]

Let \( \sigma : B^k \times 2B^n \rightarrow B^k \times \text{ht } B^n \) be an embedding with \( \sigma |_{B^k \times B^n} = \text{fe}. \)

Now define isotopy \( S_t \) of \( B^k \times B^n \) by

\[
S_t(x) = \begin{cases} R_t(x, 0) \quad (x \in \partial B^k) \\ x \quad (x \not\in \partial B^k) \end{cases}
\]

Then

\[
\sigma |_{B^k \times B^n} = \sigma \quad \text{and} \quad \sigma |_{B^k \times \text{ht } B^n} = \text{fe}.
\]

Suppose \( V \) is so small that

\[
h \in V \Rightarrow g \text{ is defined and also } g(4B^n) \subset \text{fe}(\text{ht } B^k).
\]

Then

\[
S_1 |_{B^k \times \frac{1}{2}B^n} = g.
\]

Define

\[
H_t : B^k \times B^n \rightarrow B^k \times \mathbb{R}^n, \quad x \rightarrow \begin{cases} G_{2t}(x) \quad (0 \leq t \leq \frac{1}{2}) \\ gS_{2t}(x) \quad (\frac{1}{2} < t) \end{cases}
\]

This does what is required.

Lemma 4.5 If \( C \subset \mathbb{R}^n \) is compact and \( \varepsilon > 0 \), then

\( C \) lies in the interior of a handlebody with

handles of diameter \( \leq \varepsilon \). Explicitly, infinitely

many embeddings \( h_i : B^k \times B^{n-k} \rightarrow \mathbb{R}^n \) \((i = 1, 2, \ldots) \)

such that, if \( W_j = \bigcup_{i \leq j} h_i(B^k \times B^{n-k}) \) then

i) \( h_i(B^k \times B^{n-k}) \cap W_{i-1} = h_i(\partial B^k \times B^{n-k}) \)

ii) \( W_i \) is a neighbourhood of \( C \)

iii) \( h_i(B^k \times B^{n-k}) \) has diameter \( \leq \varepsilon \) and \( h_i(B^k \times B^{n-k}) \subset N_{\varepsilon}(C) \).
Proof: Cover $C$ by a lattice of cubes of side $\varepsilon$. Since $C$ is compact, $C$ only needs a finite no. of these. Let $y_1, \ldots, y_l$ be all the faces of all the cubes of $C$ meeting $C$.

Let $k_i = \dim y_i$ and order $y_i$ so that $k_0 \leq k_1 \leq \ldots \leq k_l$.

Define $H_i = \frac{N_{\varepsilon/2}^{k_0}(y_i)}{\bigcup_{j<k_i} N_{\varepsilon/2}^{k_0}(y_j)}$.

Let $H_i = N_{\varepsilon/2}^{k_0}(y_i) \cup \bigcup_{j<k_i} N_{\varepsilon/2}^{k_0}(y_j)$.

and $\frac{1}{2}H_i = H_i \cap N_{\varepsilon/2}^{k_0}(y_i)$.

Then $H_i \cap y_i = y_i$ (radial projection) $\subseteq B^{k_i}$.

and clearly $H_i \cong (H_i \cap y_i) \times B^{n-k_i}$.

Define $h_i : B^{k_i} \times B^{n-k_i} \rightarrow H_i$ carrying $B^{k_i} \times \{0\} \cong B^{n-k_i}$ onto $\frac{1}{2}H_i$, and $(\partial B^{k_i}) \times B^{n-k_i}$ onto $H_i \cap \bigcup_{j<k_i} \{y_j\}$.

Then $h_i, h_2, \ldots, h_l$ do what is required.

\[ \square \]

Addendum 4.6 If $D \subseteq C$ is compact, then we can select $h_i$ so that $i)$ is still satisfied, $\frac{1}{2}$ and $iii)$ satisfied by $h_i$ w.r.t. $D$ instead of $C$, i.e. $U h_i : (B^{k_i} \times \{0\} \cong B^{n-k_i})$ is a nbhd of $D$, $h_i : (B^{k_i} \times B^{n-k_i})$ has diam $< \varepsilon$ and $\mathfrak{c} \in N_\varepsilon(D)$.

Proof: Select $h_i$ if $y_i$ is a face of a cube which meets $D$.

\[ \square \]

\section{Theorem 4.7 (Kirby - Edwards)} Let $C,D$ be compact in $\mathbb{R}^n$ and $U, V$ be nbhds of $C,D$. Let $\mathcal{E}$ be space of embeddings of $U$ in $\mathbb{R}^n$ which restrict to $1$ on $V$. There is a neighbourhood $N_{\varepsilon} \subseteq \mathcal{E}$ and a homotopy $H : N \times I \rightarrow H(\mathbb{R}^n)$ s.t.

i) $H_t(z) = z \quad \forall t$

ii) $H_0(g) = 1 \quad \forall g \in N$

iii) $H_t(g)|_{\partial C} = \partial C$

iv) $H_t(g)|_{\partial U(U \setminus V)} = 1 \quad \forall t, g$

Proof: Let $\varepsilon = \min(d(C, \mathbb{R}^n \setminus U), d(D, \mathbb{R}^n \setminus V))$.

Cover $C \cup D$ by a handlebody in $U \cup V$, handles of diameter $< \varepsilon$. Let $h_1 \ldots h_2$ be the handles $W_i$ as in 4.3. Select $h_1 \ldots h_2$ to form sub-handlebody covering $D$, contained in $V$.

Let $X = \bigcup h_i : (B^{k_i} \times B^{n-k_i}) \rightarrow W_i = U \cup D \subseteq R^n$.

Suppose inductively that we have constructed nbhd.

$N_{\varepsilon, i}$ of $1$ in $\mathcal{E}$, and homotopy $H^{(i)} : N_{\varepsilon, i} \times I \rightarrow H(\mathbb{R}^n)$ such that $i)$ and $ii)$ are satisfied.

\[ H^{(i)}(g)|_{W_{i-1}} = 1, \quad H^{(i)}(z) = 1, \quad H^{(i)}(g)|_{\partial U(U \setminus V)} = 1 \]
If \( h_i(B^k \times B^{n-k}) \subset X \), put \( N_i = N_{i-1} \) and \( H_i = H_{i-1} \). (This is consistent because if \( h_i(B^k \times B^{n-k}) \cap h_j(B^k \times B^{n-k}) = \emptyset \) for \( j < i \), then \( h_j(B^k \times B^{n-k}) \subset X \).)

Now suppose \( h_i(B^k \times B^{n-k}) \not\subset X \).

Choose \( N_i \) so that \( g^{-1} H^i_{(g)} h_i(B^k \times \frac{3}{4} B^{n-k}) \subset h_i(B^k \times h_i(B^{n-k})) \).

Let \( f = h_i^{-1} g_{(g)}^{-1} h_i^{(-)}(g) \), \( h_i : B^k \times \frac{3}{4} B^{n-k} \rightarrow B^k \times h_i(B^{n-k}) \).

Then \( f \) fixes \( \partial(B^k \times \frac{3}{4} B^{n-k}) \). Theorem 4.4 gives a continuously varying isotopy \( H^i_{(g)}(y) \) s.t.

1. \( H^i_{(g)}(1) = 1 \)
2. \( H^i_{(g)}(g) = f \)
3. \( H^i_{(g)}(g) | B^k \times \frac{1}{2} B^{n-k} = f | B^k \times \frac{1}{2} B^{n-k} \)
4. \( H^i_{(g)}(g) | \partial(B^k \times \frac{3}{4} B^{n-k}) = 1 \).

Let \( H^i_{(g)} : N_i \times I \rightarrow \mathcal{H}(\mathbb{R}^n) \) s.t.

1. \( H^i_{(g)}(1) = 1 \)
2. \( H^i_{(g)}(g) = 1 \)
3. \( H^i_{(g)}(x, 0) = 0 \)
4. \( H^i_{(g)}(x, 0) = 0 \)

Define \( H^i_{(g)}(x) = H^i_{(g)}(g) h_i^{-1} f^{-1}(H^i_{(g)} h_i^{-1}(x)) \).

and \( H^i_{(g)}(x) = H^i_{(g)}(g)(x) \) \( (x \notin \partial B^k \times \frac{3}{4} B^{n-k}) \).

\( W_i = W_{i-1} \cup h_i(B^k \times \frac{1}{2} B^{n-k}) \).

\( h_i(B^k \times \frac{3}{4} B^{n-k}) \cap X \subset h_i(\partial(B^k \times \frac{3}{4} B^{n-k})) \).

Completes induction.

\( H = H^e, N = N^e \) does what is required.

---

**Theorem 4.8** If \( M \) is a compact manifold then \( \mathcal{H}(M) \) is locally contractible.

**Proof:** First suppose \( M \) is closed. Cover \( M \) by finitely many embeddings \( f_i : \mathbb{R}^n \rightarrow M \) \( (i = 1, 2, \ldots, n) \).

In fact, assume \( M = \bigcup_i f_i(\mathbb{R}^n) \).

Let \( h : M \rightarrow M \) be homeo near 1.

Define inductively isotopy \( H^i(h) \) of \( M \) such that

1. \( H^i(h)(1) = 1 \)
2. \( H^i(h)(h) = 1 \)
3. \( H^i(h)(x, 0) = 0 \)
4. \( H^i(h)(x, 0) = 0 \)

Define

\( H^i(h) \) depends continuously on \( h \).

\( H^i(h)(1) = 1 \).

\( H^i(h)(h) = 1 \).

\( H^i(h) \) agrees with \( h \) on \( \bigcup f_i(\mathbb{R}^n) \).
Suppose $H_t^{(u)}$ defined. Let $C = (1 + 2^{-u})B^n$
$U = (1 + 2^{-u})B^n$, and let $D = f_i^{-1} \left( \bigcup_{j \in i} (H^{(u)} B^n) \right)$
$\cap 4B^n$

$V = f_i^{-1} \left( \bigcup_{j \in i} (1 + 2^{-u})B^n \right)$.

Suppose $h$ is so near $1$ that $h^{-1} H_t^{(u)} h f_i (U) \subset f_i (R^n)$

Apply Thm 4.7 to $g = f_i^{-1} h^{-1} H_t^{(u)} h f_i : U \rightarrow R^n$

If $h$ sufficiently near $1$, we get continuously

varying isotopy $H_t^{(u)}$ of $R^n$ s.t.

a) $H'_t (1) = 1 \forall t$

b) $H_0 (h) = 1 \forall h$

c) $H_t^{(u)} | C = \frac{h}{r}$

d) $H'_t (h) |_{S(U \setminus D)} = 1$ at $t, h$.

Define $H^{(u)} = H_t^{(u)} (h)$ by $H^{(u)} = H^{(u)}$.

$H^{(u)}_t (x) = H_t^{(u)} f_i (H_t^{(u)} h f_i (x))$ if $x \in f_i (R^n)$

$= H_t^{(u)} (x)$ (if $x \notin \cdot$)

Then $H^{(u)}$ satisfies i) - iii), and completes induction.

Now suppose $\partial M \neq \Phi$. Let

$\gamma : \partial M \times I \rightarrow M$

be a collar of $\partial M$ in $M$. $H(\partial M)$ is locally contractible. If $h \in H(M)$ near $1$ then we have isotopy $H_t^{(h)}$ of $\partial M$ with $H_0 (h) = 1$,

$H_t (h) = h | \partial M$.

Define isotopy $\overline{H}$ of $M$ by

$\overline{H}_t (y(x,u)) = y(H_{tw-u} (x), u)$

$x \in \partial M, u \in I$.

$\overline{H}_t (y) = y$ if $y \notin \gamma (\partial M \times I)$.

Then $\overline{H}_t$ is an isotopy of $M$ from $1$ to $H_1$

where $H_t$ agrees with $h$ on $\partial M$.

$\exists$ isotopy $G_t : M \rightarrow M$ from $H_t$ to $G_t$,

where $G_t$ agrees with $h$ on $\gamma (\partial M \times [0, \frac{1}{2}])$

Now argument goes as for closed manifolds.

Exercise: If $M$ is compact then $H_t (h | M)$ is locally contractible.

□
Theorem 4.9 (Isotopy extension) Let $M, N$ be $n$-manifolds with $M$ compact, $\partial N = \emptyset$ and $M \subset N$. Suppose given a path $H : I \to \mathcal{E}(M,N)$, if $U$ is a neighbourhood of $\partial M$ in $M$, then $\exists$ isotopy $\overline{H} : I \to \mathcal{H}(N)$ such that $\overline{H}_0 = 1$ and $\overline{H}_t|_{M \setminus U} = H|_{M \setminus U}$.

Proof: First use method of 4.8 to generalize 4.7 to closed deal with compact $C, D \in N$ (i.e. replace $\overline{H}^{(n)}$ by $N$).

Let $f \in \mathcal{E}(M,N)$. Then $f(M) \subset N$ is a neighbourhood of $f(M \setminus U)$ (assume $U$ open), and $\mathcal{F}_{\text{inbd}}(V_f)$ of $1$ in $\mathcal{E}(f(M), N)$ and homotopy $F^{(1)} : V_f \times I \to \mathcal{H}(N)$ s.t. $F^{(1)}(0)|_{M \setminus U} = g|_{M \setminus U}$ for $g \in V_f$.

Let $W_f = \{ gf | g \in V_f \}$. Then $W_f$ is a neighbourhood of $f$ in $\mathcal{E}(N,N)$. Now $\{ W_f \}_{f \in \mathcal{E}(M,N)}$ is an open cover of $\mathcal{E}(M,N)$.

Define $H_t$ for $\alpha_t, t \leq t, \frac{1}{2}, \ldots, t = 1$ of $I$ s.t.

$H_t \subset W_{f_i}$ for $f_i \in \mathcal{E}(M,N)$.

Define $\overline{H}_t$ for $\alpha_t, t \leq t, \frac{1}{2}, \ldots, t = 1$ by

$\overline{H}_t = F^{(1)}(H_t \circ f_i^{-1})(f_i \circ (H_t \circ f_i^{-1})^{-1})^{-1}_t \overline{H}_t_{\alpha_t}^{-1}$.

Then $\overline{H}_t = H_t$ on $M \setminus U$.

Addendum 4.10 $\overline{H}_t$ can be chosen to be the identity outside some compact set.

(C) (4.7 also produces isotopies of compact support).

Corollary 4.11 Let $f : B^n \to \text{int} 2B^n$ be isotopic to 1.

Then $2B^n \setminus f(\text{int} \frac{1}{2}B^n) \cong 2B^n \setminus \text{int} \frac{1}{2}B^n$.

Proof: Let $H_t$ be an isotopy from 1 to $f$.

By 4.10 $\exists$ isotopy $\overline{H}_t$ of int $2B^n$, fixed outside $\lambda B^n$ for some $\lambda < 2$, such that $\overline{H}_t \cdot f$ on $\frac{1}{2}B^n$.

$\overline{H}_t$, defines homeomorphism $2B^n$.

$2B^n \setminus \text{int} \frac{1}{2}B^n \to 2B^n \setminus f(\text{int} \frac{1}{2}B^n)$.

□
55. Triangulation Theorems

Definition 5.1: An $r$-simplex in $\mathbb{R}^n$ is the convex hull of $(r+1)$ linearly independent points.

Let $K \subset \mathbb{R}^n$ be compact. An embedding $f : K \to \mathbb{R}^n$ is PL if $K$ is a finite union of $r$-simplices, each mapped linearly by $f$.

If $M$ is an $n$-manifold and a PL structure on $M$ is a family $F$ of embeddings $f : \Delta^n \to M$ s.t.

i) Every point of $M$ has a neighbourhood of from $f(\Delta^n)$ ($f \in F$)

ii) If $f, g \in F$ then $g \circ f | f^{-1}(\Delta^n) : f^{-1}(\Delta^n) \to \mathbb{R}^n$ is PL.

iii) $F$ is maximal w.r.t. i) and ii).

If $M, N$ have PL structures $F, G$ an embedding $h : M \to N$ is PL if $f \in F \Rightarrow hf \in G$.

Example i) The composite of two PL embeddings is PL, and $\Delta^n \text{PL} \to \Delta^n \text{PL}$ is PL, i.e., PL is an equivalence relation.

ii) A PL structure on $M$ defines a PL structure $\mathcal{F}$ on $\partial M$.

54. A compact manifold has a PL structure if and only if it has a triangulation with

\[
\text{link (vertex)} \xrightarrow{\text{PL}} \partial \Delta^n.
\]

We need 3 deep theorems from PL topology.

Proposition 5.2

A) Suppose $M$ is a closed PL manifold which is homotopy equivalent to $S^n$. If $n \geq 5$, then $M$ is PL homeomorphic to $S^n \times \Delta^n$. 

B) Call a non-compact manifold $W$ simply-connected at $\infty$ if for every compact set $C_{\infty}$, $W \setminus C_{\infty}$ is also simply-connected. (Example: $\mathbb{R}^n$ is simply connected at $\infty$ iff $n \geq 3$)

Suppose $W^n$ is an open PL manifold which is simply connected at infinity. If $n \geq 6$

then $W$ is PL homeomorphic to $\mathbb{R}^n$ where $\partial W$ is some compact PL manifold. (Browder, hat)
(55) Let $M$ be a closed PL manifold which is homotopy equivalent to $T^n$. Then some finite covering of $M$ is PL homeomorphic to $T^n = (\partial B^n)$.

(Proof in Wall's book.)

Theorem 5.3 (Annulus conjecture) If $h: B^n \to \text{Int} B^n$ is an embedding and $n \geq 6$, then $B^n \setminus (\text{Int} B^n) \cong B^n \setminus \text{Int} B^n$.

Proof: Let $a \in T^n$ and let $f: T^n \to B^n$ be a PL immersion st. $f(T^n-a) \cong B^n-a$.

Let $h: B^n \to \text{Int} B^n$ be top homomorphism: we shall find PL structure $F'$ on $T^n-a$ st. $hf$ is PL w.r.t. $F'$.

Let $F_0 = \{ \phi: T^n \to T^n | (hf) \phi$ is PL embedding $\}$

Since $hf$ is an open immersion, $\bigcup \{ \phi(\text{Int} A^n) | \phi \in F \}$ covers $T^n-a$.

Let $\phi, \psi \in F_0$. Then $\phi(\text{Int} A^n) \cap \psi(\text{Int} A^n) = \emptyset$.

Extend $f$ to PL structure $F'$ on $T^n-a$. Let $(T^n-a, F')$.

For $n \geq 3$, $(T^n-a) \cong (T^n-a)$, so $(T^n-a)$ is simply-connected at $a$.

Since $n \geq 6$, by 5.2(B) compact PL manifold $W$ and PL homeo $g: (T^n-a) \to \text{Int} W$.

PL collar $\gamma: \partial W \times I \to W$ $\epsilon > 0$ small. Let $\Delta$ be a nbhd of $a$ in $T^n$ which is homeo to $B^n$, and so small that $g(\gamma(\partial W \times I)) \supset A-a \supset g(\gamma(\partial W \times \epsilon))$.

It follows that $\partial W \cong S^{n-1}$, so by 5.2(A) since $n \geq 6$, $\partial W$ is PL homeo to $S^{n-1}$.

By Schönflies theorem, $a \cup g(\gamma(\partial W \times (0,\epsilon))) \cong B^n$.

(56) Extend $F'$ to $T^n - (\text{aug } \gamma(\partial W \times (0,\epsilon)))$ to PL structure $F''$ on $T^n$.

[For $F'$ induces PL structure on $\partial (a \cup g(\gamma(\partial W \times (0,\epsilon))))$; extend 'conewise' to PL structure on $a \cup g(\gamma(\partial W \times (0,\epsilon)))$]

By 5.2(C), finite covering of $(T^n)$ which is PL homeo to $T^n$. Let $\varepsilon: T^n \to (T^n)$ be a finite cover.

Let $\varepsilon: T^n \to T^n$ be corresponding covers of $T^n$.

By theory of covering spaces $\exists$ homeo $h: T^n \to T^n$ (not $\text{Int} B^n$) such that $\varepsilon = \varepsilon \cdot h$ (h homotopic to $I$). Now let $\tilde{h}: \mathbb{R}^n \to \mathbb{R}^n$ be a homeo st. $\tilde{h} = h \cdot e$. Then $d(x, \tilde{h}x)$ is bounded uniformly for $x \in \mathbb{R}^n$.

Let $\rho: \text{Int} B^n \to \mathbb{R}^n$ be a PL radial homeomorphism.

Now $\eta = \rho \cdot \tilde{h}: \text{Int} B^n \to \text{Int} B^n$ extends to a homeo of $B^n$ fixing $\partial B^n$.

Let $U$ be a non-empty open set in $\text{Int} B^n$ such that $\eta(e \rho(U)) \cap B = \emptyset$ and $\gamma = f \cdot e \rho \mid U$ maps $U$ injectively into $\frac{1}{2}B^n$.

Let $\gamma = hf \cdot e \rho \mid \gamma(U) \to \text{Int} B^n$.

Then $\gamma$ are PL embeddings and $\gamma \cdot \gamma = h \cdot f$. The PL annulus conjecture is true (proof by regular-neighbourhood theory).

$\exists$ $n$-simplex $\Delta \subset U$ s.t. $\gamma(\Delta) \subset$ some $n$-simplex $\Delta' \subset \gamma(U)$. Theorem 5.3 is true.
By PL annulus theorem,
\[ \frac{1}{2}B^n \setminus \sigma(\Delta) \cong \text{std. annulus} \cong \frac{1}{2}B^n \]

\[ B^n - h(\text{int} \frac{1}{2}B^n) \cong B^n - h(\text{int} \Delta) \]

(Obtained by gluing standard annulus
\[ h(\frac{1}{2}B^n) - h(\text{int} \Delta) \text{ onto } B^n - h(\text{int} \frac{1}{2}B^n) \]

\[ \cong B^n - \sigma''(\Delta') - \sigma''(\text{int} \Delta) \]

\[ \cong \Delta'' \cdot \sigma''(\text{int} \Delta) \cong B^n - \sigma''(\text{int} \Delta) \]

\[ \cong B^n - \text{int} \Delta \cong B^n - \text{int} \frac{1}{2}B^n \]

The proof depends only on knowing that given embeddings \( f, g : B^n \to T^n \) \( \exists ! h : T^n \to T^n \) carrying \( f(\frac{1}{2}B^n) \) onto \( g(\frac{1}{2}B^n) \). If we could do this purely geometrically (i.e., without PL theory) for all dimensions, we would have then proved the annulus conjecture in all dimensions.

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New notation:
\[ W_i \text{ any manifold} \]
\[ J = \text{subset } (\partial W \times I) \cup (W \times 1) \text{ of } W \times I \]

**Theorem 5.4** Let \( M \) be a PL manifold and let \( \partial : I \times B^k \times \mathbb{R}^n \to M \) be a homeomorphism which is PL on \( J \). If \( k + n \geq 6 \) then \( \exists \text{ isotopy} \)

\[ H_t : I \times B^k \times \mathbb{R}^n \to M \text{ s.t.} \]

1. \( H_0 = \partial \)
2. \( H_t \) is PL on \( I \times B^k \times B^n \)
3. \( H_t - \partial \) on \( J \) and outside \( I \times B^k \times 2B^n \).

**Proof**: Let \( a \in T^n \) and let \( f : T^n - a \to \mathbb{R}^n \) be a PL immersion. As in 5.3, let \( F' \) be a PL structure on \( I \times B^k \times (T^n - a) \) s.t.

\[ h(1 \times f) : (I \times B^k \times (T^n - a))' \to M \text{ is PL} \]

Then \( F' \) agrees with \( F \) near \( J \).

Let \( A \) be a ball nbhd of \( a \) in \( T^n \).
Let extend \( F' \) over a nbhd of \( \partial A \) in \( T^n \).
Fix \( I \) in \( I \times B^k \times T^n \) (using std. structure).
As in 5.3 extend \( F \) over \( \mathbb{R}^k \times T^n \), obtaining structure \( \tilde{F} \).

Grafting argument \( \Rightarrow \) can extend \( \tilde{F} \) over \( \partial(D \times B^n) \cap \partial(D \times T^n) \) in \( \partial(D \times B^n \times T^n) \).

As in 5.3 extend to PL structure over \( \partial(D \times B^n \times T^n) \), agreeing with standard structure near \( \partial \) and with \( F' \) on \( D \times B^n \times (T^n - A) \).

We can take \( F' \) to be the standard structure near \( D \times B^n \times T^n \). Now \( F' \) is defined near \( \partial(D \times B^n \times A) \); we extend over \( D \times B^n \times A \) as in 5.3, obtaining a PL manifold \( (D \times B^n \times T^n)' \).

The inclusion \( (D \times B^n \times (T^n - A))' \hookrightarrow (D \times B^n \times T^n)' \) is PL except on \( D \times B^n \times A \), and the identity map

\[ D \times B^n \times T^n \rightarrow (D \times B^n \times T^n)' \]

is PL near \( D \).

Now we need another result from PL topology:

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**Proposition 5.5** Let \( W, V_1, V_2 \) be compact PL manifolds with \( \partial W = V_1 \cup V_2 \), and \( V_i \cap V_j = \partial V_i = \partial V_j \).

Suppose the inclusions \( V_i \rightarrow W \) are hty. equivalent if \( i \neq j \).

If \( \Pi_i(W) \) is free abelian, and \( \dim W \geq 6 \), then \( W \) is PL homeomorphic to \( V_i \times I \).

Apply this result with \( W = (D \times B^n \times T^n)' \), \( V_1 = D \) and \( V_2 = (D \times B^n \times T^n)' \). We obtain a PL homeomorphism \( (D \times B^n \times T^n)' \rightarrow D \times I \).

Since \( D \times I \cong D \times B^n \times T^n \) by a PL homeomorphism taking \((x, 0)\) to \( x \), we can find a PL homeomorphism \( g : D \times B^n \times T^n \rightarrow (D \times B^n \times T^n)' \) which is the identity near \( D \).

Let \( \tilde{h} : D \times B^n \times \mathbb{R}^n \rightarrow D \times B^n \times \mathbb{R}^n \) be such that \( \tilde{h} \cdot \tilde{g} = 1 \) on \( D \) and \( \tilde{h} = 1 \) on \( D \). Then \( \tilde{h} \) is a bounded homeomorphism.

Extend \( \tilde{h} \) over \( [0, \infty) \times D \times \mathbb{R}^n \) by putting \( \tilde{h} = 1 \) inside \( D \times B^n \times \mathbb{R}^n \).

Extend further over \( D \times B^n \times \mathbb{R} \times \mathbb{R}^n \) by putting

\( \tilde{h}(t, x, y) = (t, p_1 \tilde{h}(0, x, y), p_3 \tilde{h}(0, x, y)) \) for \( t \leq 0 \).

Note that \( d(x, \tilde{h}(x)) \) remains bounded for \( x \in D \times B^n \times \mathbb{R}^n \).
Suppose $0 < r < 1$, $e(rB^n) \cap A = \emptyset$ and $f \mid_{rB^n}$ is injective. We may also suppose $e(rB^n) \supseteq sB^n$ for some $s > 0$. There is a PL radial homeomorphism $p : (-1, 1) \times \text{int} (2B^k \times rB^n) \to \mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^n$, fixed near $I \times B^k \times rB^n$. Then $p \circ p'$ extends to a homeomorphism of $[1, 2] \times 2B^k \times rB^n$ fixing the boundary.

Let $\eta = \text{res}^{-1} : I \times B^k \times rB^n \to I \times B^k \times B^n$. Note that $I \times B^k \times B^n \times I$ is the join of $(\frac{1}{2}, 0, 0, \frac{1}{2})$ to $(I \times I) \cup (I \times B^k \times B^n \times I)$. Define PL homeomorphism $R$ of $I \times B^k \times B^n \times I$ by

$$R(\frac{1}{2}, 0, 0, \frac{1}{2}) = (\frac{1}{2}, 0, 0, \frac{1}{2})$$

$$R(I \times I) \cup (I \times B^k \times B^n \times I) = 1,$$ $R \mid_{I \times B^k \times B^n \times 0} = \gamma$, and extending conewise. Then $R$ defines a PL isotopy $R_t$ of $I \times B^k \times B^n$, fixed near $I$, with $R_0 = \gamma$ and $R_1 = 1$.

Let $\sigma : I \times B^k \times B^n \to I \times B^k \times rB^n$ be a PL embedding which agrees with $1 \times f$ near $I \times B^k \times rB^n$. Then $h \circ \eta^{-1}$ agrees with $h(1 \times f)$ gap near $\gamma (I \times B^k \times rB^n)$, so it is PL there.

Now $s$-cobordism theorem $\Rightarrow I \times B^k \times B^n - N \cong \overline{I \times B^k \times B^n - I \times B^k \times rB^n}$ (Prop 5.5).

Now follows that $\exists \sigma : I \times B^k \times B^n \to M$, a PL embedding such that $\sigma \mid_{\gamma} = h \circ \eta$ near $I \times B^k \times rB^n$ (regard $I \times B^k \times B^n - N$ as a collar of $F_1 N$).

$$W = I \times B^k \times B^n - \gamma (I \times B^k \times rB^n)$$ is a PL manifold (since an open subset of PL manifold).

If $n \geq 3$, $W$ is simply-connected at infinity, so if $n \geq 3$, Brouwer-Levine-Livescy thm (5.2(b)) implies that

$W \cong$ open subset of compact manifold.

If $n \leq 2$, some result, using instead Siebenmann's thesis. Follow that $\gamma (I \times B^k \times rB^n)$ has a neighborhood which is a compact PL manifold such that

$\partial N \subseteq I \times B^k \times B^n - N$ is a hty equivalence.
Let \( R_t \) be isotopy from \( \eta \) to \( 1 \) rel \( J \).

Define \( S_t : I \times B^k \times B^n \to M \) by

\[
S_t(x) = \begin{cases} 
\sigma^* R_t \gamma^{-1}(\sigma^*)^{-1} h(x) & h(x) \in \text{im} \sigma^* \\
h(x) & h(x) \notin \text{im} \sigma^*
\end{cases}
\]

Then \( S_0 = h, S_t \mid_{I \times B^k \times sB^n} = \sigma^* R_t \gamma^{-1} \sigma^{-1} \mid_{(r \leq r(I \times B^k \times sB^n))} \)

\[
= \sigma^{-1} \mid_{I \times B^k \times sB^n} \to M
\]

which is PL.

Then \( S_t = h \) on \( J \) and also outside \( h^{-1}(\text{image } \sigma^*) \) which is compact.

\( \therefore S_t = h \) on \( J \) and outside \( I \times B^k \times sR^n \) for some \( R > 0 \).

Trivial to replace \( S_t \) by isotopy \( H_t \) satisfying 1)-3).

\[ \square \]

**Theorem 5.6** Let \( C, D \) be closed subsets of \( \mathbb{R}^n \) and let \( U \) be an open neighbourhood of \( C \). Let \( F \) be a PL structure on \( U \times I \subseteq \mathbb{R}^n \times I \) which agrees with the standard PL structure near \((U \cap D) \times I\), and near \( U \times 0 \). If \( n \geq 6 \), then there is an isotopy \( H_t \) of \( \mathbb{R}^n \times I \) such that:

1) \( H_0 = 1_{\mathbb{R}^n \times I} \)

2) \( H_t : (U \times I, \text{standard}) \to (U \times I, F) \) is PL near \( C \times I \)

3) \( H_t = 1 \) near \((D \cap (\mathbb{R}^n - U)) \times I\) and near \( \mathbb{R}^n \times 0 \).

\[ \square \]
Theorem 5.7 (Product structure theorem). Let $M^n$ be a topological manifold, let $C \subseteq M$ be a closed subset, and let $U$ be an open neighborhood of $C$. Let $F_0$ be a PL structure on $U$, and let $G$ be a PL structure on $M \times \mathbb{R}^k$, such that $G$ agrees with $F_0 \times \mathbb{R}^k$ on $U \times \mathbb{R}^k$. If $n \geq 6$, then there exists a PL structure $F$ on $M$, agreeing with $F_0$ on $C$, and a PL homeomorphism $(M \times \mathbb{R}^k, F \times \mathbb{R}^k) \to (M \times \mathbb{R}^k, G)$ which is isotopic to 1 by an isotopy fixing a neighborhood of $C \times \mathbb{R}^k$. □

Definition 5.8 PL structures $F_1, F_2$ on $M$ are isotopic if there is a PL homeomorphism $h : (M, F_1) \to (M, F_2)$ which is isotopic to 1.

Let $\text{PL}(M)$ be the set of isotopy classes of PL structures on $M$.

Corollary 5.9 If $\dim M \geq 6$, the natural map $\text{PL}(M) \to \text{PL}(M \times \mathbb{R}^k)$ is a bijection. In particular, if $M \times \mathbb{R}^k$ has a PL structure, and $\dim M \geq 6$, then $M$ has a PL structure. □

Lemma 5.10 Any two PL structures on $\mathbb{R}^n$ are isotopic.

Proof: Let $F$ be a PL structure on $\mathbb{R}^n$. By Prop 5.2(B) $(\mathbb{R}^n, F)$ is PL homeomorphic to $\text{ht} W$, where $W$ is a compact PL manifold $\partial W \cong S^{n-1}$, so by 5.2(A) $\partial W \cong \mathbb{R}^n$. W is contractible, so by 5.2(A), $W \cong \mathbb{R}^n$. So there exists a PL homeomorphism $h : \mathbb{R}^n \to \text{ht} B^n \to \text{ht} W \to (\mathbb{R}^n, F)$

Can assume $h$ is orientation-preserving.

Must prove $h$ is isotopic to 1.

Let $R > r > 0$ be chosen so that $h(r B^n) \subseteq \text{ht} h(R B^n)$.

By annulus theorem 5.3, there exists a homeomorphism $f : R B^n - \text{int} r B^n \to R B^n$ such that $f|_{R B^n} = 1$. Since $h$ is orientation-preserving, and using the proof of 5.3 we can choose $f$ so that $f = h$ on $r B^n$.

Extend $f$ over $\mathbb{R}^n$ by $f(x) = \begin{cases} 0 & \text{if } x \in R \\ x & \text{if } |x| \leq R \end{cases}$. Extend $h$ over $\mathbb{R}^n$ by $h(x) = \begin{cases} x & \text{if } |x| \leq R \\ 0 & \text{if } x \in R B^n \end{cases}$.
Since $f = 1$ outside $R^m$, $f$ isotopic to 1, so $h$ isotopic to $f^*h$.
Since $f^*h = 1$ in $R^n$, $f^*h$ is isotopic to 1.

$h$ isotopic to 1 as required.

\[ \square \]

**Proof of Thm 5.7**

Clearly sufficient to prove for case $k = 1$. Assume first $M = R^n$.

$G = \text{PL structure on } R^n 	imes R \to R^{n+1}$. By 6.10, there exists isotopy $H_t$ s.t. $H_t : R^{n+1} \to R^{n+1}$ such that $H_t = 1$ for $t < 1/2$. $H$ defines homeomorphism $H : R^n \times R \times I \to R^n \times R \times I$ where

$H(x, t) = (H_t(x), t)$

Let $H = H(\text{std. PL structure})$. Then $H$ agrees with standard structure near $R^n \times R \times 0$.

Apply Theorem 5.6 to $R^n \times R \times I$ with $C, U, D, F$ replaced by $R^n \times (-\infty, 0)$, $R^n \times (-\infty, \frac{1}{2})$, $\emptyset$, $\emptyset$. $H$ agrees on $R^n \times R \times 1$

\[ \text{Let } g : R^n \to R \text{ be defined by } (g(x, t)) = g(x) \]

\[ \text{Let } G'' = (g \times 1) G'' \]
Near $C \times I$, $G''$ agrees with $(g \times 1)$ (standard structure) $= F_g \times I$ (in nbhd of $C \times I$).

Define $F = G'' |_{\mathbb{R}^n \times 0}$.

$F$ agrees with $F_0$ near $C$.

Remains constant isotopy (rel $C \times \mathbb{R}$) from $F \times \mathbb{R}$ to $G$.

Choose a PL isotopy (of embeddings)

$j_t : \mathbb{R} \to \mathbb{R}$ s.t. $j_t = 1$ ($t \leq \frac{1}{2}$) and $j_t(\mathbb{R}) < (1, \infty)$.

$J : \mathbb{R}^n \times \mathbb{R} \times I \to \mathbb{R}^n \times \mathbb{R} \times I$ defined by

$J(x, y, t) = (x, j_t(y), t)$.

Then PL structure $J^{-1}(G'' \times I)$ agrees with $G'' \times 0$ on $\mathbb{R}^n \times \mathbb{R} \times 0$ and agrees with $F_0 \times \mathbb{R} \times I$ near $C \times \mathbb{R} \times I$.

Apply theorem 5.6 (using fact that $G''$ isotopic to standard structure by lemma 5.10) to obtain isotopy from $G''$ to $J^{-1}(G'' \times I)$, fixed near $C \times \mathbb{R}$. We have $J^{-1}(G'' \times I) = J^{-1}(G \times I)$.

Similarly, $G'' \times \mathbb{R}$ isotopic (relative neighbourhood of $C \times \mathbb{R}$).

For general $M : 25$ with $\partial M = \emptyset$.

(W.l.o.g. connected)

We know that $M$ is metrizable $\Rightarrow$ $M$ is 2nd countable.

So $M = \bigcup f_i(B^n)$ where $f_i : \mathbb{R}^n \to M$ is an embedding.

Let $C_i = C \cup f_i(B^n) \cup \ldots \cup f_i(B^n)$.

Suppose inductively we have PL structure $F_i$, on nbhd of $C_i$, in $M$, extending $F_0$ and PL structure $G_{i-1}$ on $M \times \mathbb{R}$ extending $F_{i-1} \times \mathbb{R}$ and isotopic to $G$ by isotopy fixed near $C \times \mathbb{R}$.

Apply this result for $M - \mathbb{R}^n$ to $F' = F_i'(F_{i-1})$

(near $C - f_i'(C_{i-1})$) and $(f_i \times 1)^{-1}(G_{i-1}) = G'$.

We obtain a PL structure $F''$ on $\mathbb{R}^n(=F$ near $C)$.
and isotopy $H_t$ of $R^n \times R$ with $H_t = 1 (t < \frac{1}{2})$
and $H_t^{-1}(g') = F^t \times R$, and $H_t$ fixes a nbhd of $C'$. 

$H$ defines homeo $H : R^n \times R \times I \to I$:

let $H = H^1 (G' \times I)$. $H$ agrees with $G'$ near $R^n \times R \times 0$, with $F^t \times R$ on 
$R^n \times R \times 1$, and near $C' \times R \times I$. Apply
Theorem 5.6 to this: replace $C, U, D, F, B, B^n \times R$, $(\text{Int } 2B^n \times R)$, $C' \times R$, $\mathcal{F}$ to obtain
PL structure $G'$ on $R^n \times R$ which agrees
with $F^t \times R$ near $(C' \cup B^n) \times R$ and
which is isotopic to $G'$ rel $(C' \cup (R^n - \text{Int } 2B^n)) \times R$.

Define $F_i = F_{i-1} \cup \Phi_i (F^t)$ and extend 
$(F_i \times 1)(G')$ to structure $G_i$ on $M \times R$
agrees with $G_{i-1}$ off $F_i (R^n \times I)$.

Then $G_i$ agrees with $F_i \times R$ near $G_i \times R$
and $G_i$ isotopic to $G_{i-1}$ fixing nbhd of $G_{i-1} \times R$.

So $F_i = F_{i-1}$ near $G_{i-1}$. 

Since $F_i = F_{i-1}$ near $G_{i-1}$, $\mathcal{F}$ PL structure
For $M$ agreeing with $F_i$ near $G_i$, $\mathcal{G}$ agreeing with $F_i \times R$ near $C \times R$, $\mathcal{F}$ agrees with $F_i$ near $G_i$.
Since $G_i$ isotopic to $G_{i-1}$ (fixing nbhd of $C_i \times R$), hence all isotopies together
obtain isotopy $F_i \times R$ to $G_i$, fixing nbhd of $C \times R$. This proves product theorem
when $M$ has no boundary.

If $M$ has non-empty boundary $\partial M$, then
apply theorem for $M$ unbounded to $\partial M$, and then to $\text{Int } M$ using a different argument.

Seem to need $\text{dim } M \geq 7$ (to ensure $\text{dim } M \geq 6$)

(In fact theorem can be proved for all unbounded 5-manifolds and all 6-manifolds).
Applications: If $M$ is a topological manifold, we can embed $M$ in $\mathbb{R}^N$ with a neighborhood $E$ which fibers over $M$, i.e., $\mathcal{F}$ map $\pi: E \to M$ which is locally the projection of product, with fiber $\mathbb{R}^n$ (structural group $\mathcal{H}(\mathbb{R}^n) = \text{Top}_n$).

Let $\psi = (\pi, E \to M)$.

A necessary condition for $M$ to have a PL structure is that $\psi$ come from a PL bundle over $M$.

Also sufficient (if $\dim M \geq 6$)

$E(\psi)$ is open subset of $\mathbb{R}^N$ so that it inherits a PL structure. Suppose $\mathcal{F}$ PL bundle $\gamma$ over $E(\psi)$ which is equivalent to a topological bundle to $\psi$. $\mathcal{F}$ PL bundle $\gamma$ over $E(\psi)$ s.t. $\xi \oplus \gamma$ is trivial. Then total space $E(\gamma)$ is homeomorphic to $M \times \mathbb{R}^k$ and has a PL structure.

By product structure theorem, $M$ has a PL structure.

Classifying space $\text{BTop}$ classifying such topological structure bundles: $\mathcal{F}$ immaterial.

So take $\text{BTop} = \lim_{\to} \text{BTop}_k$.

Similarly for $\text{BPL}$, $\text{BPL}$ is also the limit of $\text{BPL}_k$.

$M$ has a PL structure if the map

$\nu: M \to \text{BTop}$ factors (up to homotopy) as

$$
\begin{array}{ccc}
M & \xrightarrow{\nu} & \text{BPL} \xrightarrow{\text{id}} \text{BTop} \\
\downarrow & & \downarrow \\
& & \\
\end{array}
$$

$M$ has a PL structure iff the classifying map of the stable normal bundle $\nu$ of $M$ lies in the image of $[M, \text{BPL}] \to [M, \text{BTop}]$.

To show that $\text{PL} \neq \text{Top}$:

Let $k$ be an integer, and $p_k: T^n \to T^n$ be induced by $\mathbb{R}^n \to \mathbb{R}^n, x \mapsto kx$.

Then $p_k$ is a $k^n$-fold covering (fiber bundle with discrete fiber $\mathbb{Z}/k\mathbb{Z}$).
Theorem 5.11 If \( h : T^n \to T^n \) is a homeomorphism homotopic to 1, then \( h_k \) is topologically isotopic to 1 for sufficiently large \( k \).

Proof: First isotop \( h \) until \( h(0) = 0 \) (where \( 0 = e(0) \in T^n \)). Choose \( h_k \) so that \( h_k(0) = 0 \). Let \( \tilde{h} : \mathbb{R}^n \to \mathbb{R}^n \) be homeo such that \( e \tilde{h}_k = h_k e \) and \( h_k(0) = 0 \).

Since \( h \equiv 1 \), \( \tilde{h}_k(0) = 0 \) and \( \tilde{h} \) is bounded.

\[ \tilde{h}_k(x) = \frac{1}{k} \tilde{h}(x) \] because \( p_k e \tilde{h}_k = p_k h_k e = h p_k e \),

\[ \tilde{h}_k(0) = 0 \] & these characterise \( \tilde{h}_k \).

\[ \sup_{x \in T^n} d(x, \tilde{h}_k(x)) = \frac{1}{k} (\sup_{x \in T^n} d(x, \tilde{h}(x)) \to 0 \text{ as } k \to \infty} \]

Proposition 5.12 (Wall) Let \( n \geq 5 \) \( \exists \) PL homeomorphism \( h : T^n \to T^n \) s.t. \( h \approx 1 \) and \( h_k \) is not PL isotopic to 1 for any odd \( k \).

Proof (Sketch)

\[ \frac{1}{k} \text{ PL isotopy classes of PL homeos of } T^n \] \[ \cong = H^2(T^n; \mathbb{Z}_2) \]

Exercise Show that, for \( k \) odd, \( H^2(T^n; \mathbb{Z}_2) \)
Exercise: Show that, if \( h : T^n \to T^n \) is PL and (topologically) isotopic to \( 1 \), but not PL isotopic to \( 1 \), then \( T^n \times I / (x,0) \sim (h^t,1) \)

is topologically homeomorphic to \( T^{n+1} \), but not PL homeomorphic to \( T^{n+1} \).