Oliver's Formula and Minkowski's Theorem

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In this note we give an elementary verification of an unpublished formula of Bob Oliver. This leads to a three line proof of Minkowski's theorem, that a finite group acting effectively on a surface of genus at least two is represented faithfully by its action on homology.

Theorem. (Oliver)

If $\mathbb{Z}_n$ acts cellularly on a finite complex X then

$$\chi(X^{\mathbb{Z}_n}) = L(g)$$

where L(g) is the Lefshetz number of a generator.

Proof.

Examine the equivariant chain complex of X

$$C_*(X) \simeq C_*(X^{\mathbb{Z}_n}) \oplus \overline{C}_*(X)$$

where $\overline{C}_*(X)$ is freely generated by cells not in $X^{\mathbb{Z}_n}$. Thus

$$L(g) = \Sigma\ (-1)^i\ \mathrm{Tr}\ g_{i_*}$$

$$= \Sigma\ (-1)^i\ \mathrm{Tr}\ g|C_i(X)$$

$$= \Sigma\ (-1)^i\ \mathrm{Tr}\ g|C_i(X^{\mathbb{Z}_n}) + \Sigma\ (-1)^i\ \mathrm{Tr}\ g\ \overline{C}_i\ (X)$$

$$= \chi(X^{\mathbb{Z}_n})$$

since $g|C_i(X^{\mathbb{Z}_n})$ is the identity and $g|\overline{C}_i(X)$ has only zeroes along the diagonal. □

Corollary. (Minkowski)

If π acts effectively on a surface M of genus at least two then $\pi \to \mathrm{Aut}\, H_1(M;\mathbb{Q})$ is injective.

Proof.

Let $\mathbb{Z}_n \subset$ Kernel. Since the action is effective $M^{\mathbb{Z}_n}$ is a union of circles and points so that $\chi(M^{\mathbb{Z}_n}) \geq 0$. On the other hand, by assumption $g_* = 1$ so $L(g) = \chi(M) < 0$. □

An easy consequence is that only finitely many groups act effectively on any fixed surface of genus larger than one.

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