THE POINCARÉ DUALITY THEOREM
AND ITS CONVERSE I.
Andrew Ranicki (Edinburgh)
http://www.maths.ed.ac.uk/~aar

FLOER CENTER OF GEOMETRY

Festive Opening Colloquium
Bochum, 7th December, 2011
Local to global and, if possible, global to local

- There are many theorems in TOPOLOGY of the type

  \[ \text{local input } \implies \text{global output} \]

- Theorems of the type

  \[ \text{global input } \implies \text{local output} \]

  are even more interesting, and correspondingly harder to prove! This frequently requires ALGEBRA.

- *Algebra is a pact one makes with the devil!*  
  (Sir Michael Atiyah)

- *I rather think that algebra is the song that the angels sing!*  
  (Barry Mazur)

- *One thing I’ve learned about algebra … don’t take it too seriously* (Peanuts cartoon)
Poincaré duality and its converse

- The Poincaré duality of an $n$-dimensional topological manifold $M$
  \[ H^*(M) \cong H_{n-*}(M) \]
  is a local $\implies$ global theorem.

- **Theorem** Let $n \geq 5$. A space $X$ with $n$-dimensional Poincaré duality $H^*(X) \cong H_{n-*}(X)$ is homotopy equivalent to an $n$-dimensional topological manifold if and only if $X$ has sufficient local Poincaré duality.

- Modern take on central result of the Browder-Novikov-Sullivan-Wall high-dimensional surgery theory for differentiable and $PL$ manifolds, and its Kirby-Siebenmann extension to topological manifolds (1962-1970)

- Will explain "sufficient" over the course of the lectures!
The Seifert-van Kampen Theorem and its converse

Local $\implies$ global. The fundamental group of a union

$$X = X_1 \cup_Y X_2, \ Y = X_1 \cap X_2$$

is an amalgamated free product

$$\pi_1(X) = \pi_1(X_1) \ast_{\pi_1(Y)} \pi_1(X_2).$$

Global $\implies$ local. Let $n \geq 6$. If $X$ is an $n$-dimensional manifold such that $\pi_1(X) = G_1 \ast_H G_2$ then $X = X_1 \cup_Y X_2$ for codimension 0 submanifolds $X_1, X_2 \subset X$ with

$$\partial X_1 = \partial X_2 = Y = (n-1)\text{-dimensional manifold},$$

$$\pi_1(X_1) = G_1, \ \pi_1(X_2) = G_2, \ \pi_1(Y) = H.$$
The Vietoris Theorem and its converses

- **Theorem** If \( f : X \rightarrow Y \) is a surjection of compact metric spaces such that for each \( y \in Y \) the restriction
  \[
  f| : f^{-1}(y) \rightarrow \{y\}
  \]
  induces an isomorphism in homology
  
  \[
  H_*(f^{-1}(y)) \cong H_*(\{y\})
  \]
  then \( f \) induces isomorphisms in homology
  
  \[
  f_* : H_*(X) \cong H_*(Y)
  \]

- **Local input:** each \( f^{-1}(y) \) \( (y \in Y) \) is acyclic
  
  \[
  \tilde{H}_*(f^{-1}(y)) = 0
  \]

- **Global output:** \( f_* \) is an isomorphism.

- Would like to have converses of the Vietoris theorem! For example, under what conditions is a homotopy equivalence homotopic to a homeomorphism?
Manifolds and homology manifolds

- An \textit{n-dimensional topological manifold} is a topological space $M$ such that each $x \in M$ has an open neighbourhood homeomorphic to $\mathbb{R}^n$.

- An \textit{n-dimensional homology manifold} is a topological space $M$ such that the local homology groups of $M$ at each $x \in M$ are isomorphic to the local homology groups of $\mathbb{R}^n$ at 0.

\[
H_\ast(M, M \setminus \{x\}) \cong H_\ast(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) = \begin{cases} 
\mathbb{Z} & \text{if } \ast = n \\
0 & \text{if } \ast \neq n
\end{cases}
\]

- A topological manifold is a homology manifold.
- A homology manifold need not be a topological manifold.
- Will only consider compact $M$ which can be realized as a subspace $M \subset \mathbb{R}^{n+k}$ for some large $k \geq 0$, i.e. a compact ENR. This is automatically the case for topological manifolds.
The triangulation of manifolds

- A \textbf{triangulation} of a space \( X \) is a simplicial complex \( K \) together with a homeomorphism

\[
X \cong |K|
\]

with \( |K| \) the polyhedron of \( K \).

- \( X \) is compact if and only if \( K \) is finite.

- Triangulation of \( n \)-dimensional topological manifolds:
  - Exists and is unique for \( n \leq 3 \)
  - Known: may not exist for \( n = 4 \)
  - Unknown: if exists for \( n \geq 5 \)
    (Update: now known. \textbf{Manolescu 2013}: Nontriangulable topological manifolds in each dimension \( n \geq 5 \))
  - Differentiable and PL manifolds are triangulated for all \( n \geq 0 \)

- Triangulation of \( n \)-dimensional homology manifolds:
  - Exists and is unique for \( n \leq 3 \)
  - Known: may not exist for \( n \geq 4 \).
The naked homeomorphism

- Poincaré, for one, was emphatic about the importance of the naked homeomorphism - when writing philosophically - yet his memoirs treat DIFF or PL manifolds only. in L. Siebenmann’s 1970 ICM lecture on topological manifolds.

- ... topological manifolds bear the simplest possible relation to their underlying homotopy types. This is a broad statement worth testing. (ibid.)

- Will describe how surgery theory manufactures the homotopy theory of topological manifolds of dimension $> 4$ from Poincaré duality spaces and chain complexes.

- Poincaré duality is the most important property of the algebraic topology of manifolds.
The original statement of Poincaré duality

- Analysis Situs and its Five Supplements (1892–1904)

- Originally proved for a differentiable manifold $M$, but long since established for topological and homology manifolds.

- $h = n$, the dimension of $M$.

- $P_p = \dim\mathbb{Z}H_p(M)$, the $p$th Betti number of $M$.

- Happy birthday! 2011 is the 100th anniversary of Brouwer’s proof that homeomorphic manifolds have the same dimension. Also true for homology manifolds.
Orientation

A local fundamental class of an $n$-dimensional homology manifold $M$ at $x \in M$ is a choice of generator

$$[M]_x \in \{1, -1\} \subset H_n(M, M\{x\}) = \mathbb{Z}.$$

The local Poincaré duality isomorphisms are defined by

$$[M]_x \cap - : H^*(\{x\}) \cong H_{n-*}(M, M\{x\}).$$

An $n$-dimensional homology manifold $M$ is orientable if there exists a fundamental homology class $[M] \in H_n(M)$ such that for each $x \in M$ the image

$$[M]_x \in H_n(M, M\{x\}) = \mathbb{Z}$$

is a local fundamental class.

We shall only consider manifolds which are orientable, together with a choice of fundamental class $[M] \in H_n(M)$. 
Poincaré duality in modern terminology

**Theorem** For an $n$-dimensional manifold $M$ the cap products with the orientation $[M] \in H_n(M)$ are Poincaré duality isomorphisms

$$[M] \cap - : H^*(M) \cong H_{n-*}(M).$$

**Idea of proof** Glue together the local Poincaré duality isomorphisms

$$[M]_x \cap - : H^*(\{x\}) \cong H_{n-*}(M, M \setminus \{x\}) \ (x \in M)$$

to obtain the global Poincaré duality isomorphisms

$$[M] \cap - = \lim_{x \in M} [M]_x \cap - :$$

$$H^*(M) = \lim_{x \in M} H^*(\{x\}) \cong H_{n-*}(M) = \lim_{x \in M} H_{n-*}(M, M \setminus \{x\})$$

Need to work on the chain level, rather than directly with homology.
Poincaré duality spaces

Definition An $n$-dimensional Poincaré duality space $X$ is a finite CW complex $X$ with a homology class $[X] \in H_n(X)$ such that cap product with $[X]$ defines Poincaré duality isomorphism

$$[X] \cap - : H^*(X; \mathbb{Z}[\pi_1(X)]) \cong H_{n-*}(X; \mathbb{Z}[\pi_1(X)]) .$$

In the simply-connected case $\pi_1(X) = \{1\}$ just

$$[X] \cap - : H^*(X) \cong H_{n-*}(X) .$$

Homotopy invariant: any finite CW complex homotopy equivalent to an $n$-dimensional Poincaré duality space is an $n$-dimensional Poincaré duality space.

A triangulable $n$-dimensional homology manifold is an $n$-dimensional Poincaré duality space.

A nontriangulable $n$-dimensional homology manifold is homotopy equivalent to an $n$-dimensional Poincaré duality space.
Floer’s Diplom thesis

- Floer’s 1982 Bochum Diplom thesis (under the supervision of Ralph Stöcker) was on the homotopy-theoretic classification of \((n - 1)\)-connected \((2n + 1)\)-dimensional Poincaré duality spaces for \(n > 1\).


Klassifikation hochzusammenhängender Poincaré-Räume

Andreas Floer

Diplomarbeit
Ruhr-Universität Bochum
Abteilung für Mathematik
1982
Manifold structures in the homotopy type of a Poincaré duality space

- (Existence) When is an $n$-dimensional Poincaré duality space homotopy equivalent to an $n$-dimensional topological manifold?

- (Uniqueness) When is a homotopy equivalence of $n$-dimensional topological manifolds homotopic to a homeomorphism?

- There are also versions of these questions for differentiable and $PL$ manifolds, and also for homology manifolds.

- But it is the topological manifold version which is the most interesting! Both intrinsically, and because most susceptible to algebra, at least for $n > 4$. 

Surfaces

- Surface = 2-dimensional topological manifold.
- Every orientable surface is homeomorphic to the standard surface $\Sigma_g$ of genus $g \geq 0$.
- Every 2-dimensional Poincaré duality space is homotopy equivalent to a surface.
- A homotopy equivalence of surfaces is homotopic to a homeomorphism.
- In general, the analogous statements for false for $n$-dimensional manifolds with $n > 2$. 
# Bundle theories

<table>
<thead>
<tr>
<th>spaces</th>
<th>bundles</th>
<th>classifying spaces</th>
</tr>
</thead>
<tbody>
<tr>
<td>differentiable</td>
<td>manifolds</td>
<td>vector bundles</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>topological</td>
<td>manifolds</td>
<td>topological bundles</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>homotopy theory</td>
<td>Poincaré duality spaces</td>
<td>spherical fibrations</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- Forgetful maps downwards. Difference between the first two rows = finite (but non-zero) = exotic spheres (Milnor).

- An $n$-dimensional differentiable manifold $M$ has a tangent bundle $\tau_M : M \to BO(n)$ and a stable normal bundle $\nu_M : M \to BO$.

- Similarly for a topological manifold $M$, with $BTOP(n)$.

- An $n$-dimensional Poincaré duality space $X$ has a Spivak normal fibration $\nu_X : X \to BG$. 
The Hirzebruch signature theorem

- The **signature** of a $4k$-dimensional Poincaré duality space $X$ is

  $$\sigma(X) = \text{signature}(H^{2k}(X), \text{intersection form}) \in \mathbb{Z}$$

- The **Hirzebruch $L$-genus** of a vector bundle $\eta$ over a space $X$ is a certain polynomial $L(\eta) \in H^{4*}(X; \mathbb{Q})$ in the Pontrjagin classes $p_*(\eta) \in H^{4*}(M)$.

- **Signature Theorem** (1953) If $M$ is a $4k$-dimensional differentiable manifold then

  $$\sigma(M) = \langle L(\tau_M), [M] \rangle \in \mathbb{Z}$$

- There have been many extensions of the theorem since 1953!
The Browder converse of the Hirzebruch signature theorem

- **Theorem** (Browder, 1962) For $k > 1$ a simply-connected $4k$-dimensional Poincaré duality space $X$ is homotopy equivalent to a $4k$-dimensional differentiable manifold $M$ if and only if $\nu_X : X \to BG$ lifts to a vector bundle $\eta : X \to BO$ such that

  $$\sigma(X) = \langle \mathcal{L}(-\eta), [X] \rangle \in \mathbb{Z}.$$ 

- Novikov (1962) initiated the complementary theory of necessary and sufficient conditions for a homotopy equivalence of simply-connected differentiable manifolds to be homotopic to a diffeomorphism.

- Many developments in the last 50 years, including versions for topological manifolds and homeomorphisms.
The Browder-Novikov-Sullivan-Wall surgery theory I.

- Is an \( n \)-dimensional Poincaré duality space \( X \) homotopy equivalent to an \( n \)-dimensional topological manifold?

- The surgery theory provides a 2-stage obstruction for \( n > 4 \), working outside of \( X \), involving normal maps \( (f, b) : M \to X \) from manifolds \( M \), with \( b \) a bundle map.

- Primary obstruction in the topological \( K \)-theory of vector bundles to the existence of a normal map \( (f, b) : M \to X \).

- Secondary obstruction \( \sigma(f, b) \in L_n(\mathbb{Z}[\pi_1(X)]) \) in the Wall surgery obstruction group, depending on the choice of \( (f, b) \) in resolving the primary obstruction. The algebraic \( L \)-groups defined algebraically using quadratic forms over \( \mathbb{Z}[\pi_1(X)] \).

- The mixture of topological \( K \)-theory and algebraic \( L \)-theory not very satisfactory!
The Browder-Novikov-Sullivan-Wall surgery theory II.

- Is a homotopy equivalence $f : M \to N$ of $n$-dimensional topological manifolds homotopic to a homeomorphism?
- For $n > 4$ similar 2-stage obstruction theory for deciding if $f$ is homotopic to a homeomorphism.
- The mapping cylinder of $f$

$$L = M \times [0, 1] \cup_{(x, 1) \sim f(x)} N$$

defines an $(n + 1)$-dimensional Poincaré pair $(L, M \sqcup N)$ with manifold boundary. The 2-stage obstruction for uniqueness is the 2-stage obstruction for relative existence.

- Again, the mixture of topological $K$-theory and algebraic $L$-theory not very satisfactory!
THE POINCARÉ DUALITY THEOREM
AND ITS CONVERSE II.
Andrew Ranicki (Edinburgh)
http://www.maths.ed.ac.uk/~aar

FLOER CENTER OF GEOMETRY

Festive Opening Colloquium
Bochum, 8th December, 2011
The total surgery obstruction

I. Existence of manifold structures

- The $S$-groups $S_\ast(X)$ are $\mathbb{Z}$-graded abelian groups defined for any space $X$. A map $f : X \to Y$ induces $f_\ast : S_\ast(X) \to S_\ast(Y)$. If $f$ is a homotopy equivalence, then $f_\ast$ is an isomorphism.

- The **total surgery obstruction** $s(X) \in S_n(X)$ of an $n$-dimensional Poincaré duality space $X$ with the following properties.

  - If $f : X \to Y$ is a homotopy equivalence of $n$-dimensional Poincaré duality spaces then $f_\ast s(X) = s(Y) \in S_n(Y)$.
  
  - If $X$ is an $n$-dimensional homology manifold then $s(X) = 0 \in S_n(X)$.

- **Main Theorem** If $n \geq 5$ and $s(X) = 0 \in S_n(X)$ then $X$ is homotopy equivalent to an $n$-dimensional topological manifold.

- Global input $\implies$ local output.

- Proof by Browder-Novikov-Sullivan-Wall theory.
The total surgery obstruction

II. Uniqueness of manifold structures

- The **total surgery obstruction** of a homotopy equivalence $h : N \rightarrow M$ of $n$-dimensional topological manifolds is an element $s(h) \in S_{n+1}(M)$ with the following properties.

- If the point inverses $h^{-1}(x) \subset N$ ($x \in M$) are acyclic

  $$h|_{h^{-1}(x)} : H_*(h^{-1}(x)) \cong H_*(\{x\})$$

  then $s(h) = 0 \in S_{n+1}(M)$.

- If $n \geq 5$ and $s(h) = 0 \in S_{n+1}(M)$ then $h$ is homotopic to a homeomorphism. (Need also Whitehead torsion $\tau(h) = 0$). Every $s \in S_{n+1}(M)$ is $s = s(h)$ for some $h$.

- Global input $\implies$ local output.

- (A.R.) **The total surgery obstruction**

(Proc. 1978 Aarhus Topology Conference, Springer Lecture Notes)
The Wall surgery obstruction

In 1969 C.T.C. Wall constructed the surgery obstruction groups $L_n(A)$ of a ring with involution $A$, using quadratic structures on f.g. free $A$-modules.

4-periodic: $L_n(A) = L_{n+4}(A)$

$L_0(A) =$ Witt group of quadratic forms over $A$.

$L_1(A) =$ stable automorphism group of quadratic forms over $A$.

$L_2(A) =$ Witt group of symplectic-quadratic forms over $A$.

$L_3(A) =$ stable automorphism group of symplectic-quadratic forms over $A$.

A normal map $(f, b) : M \to X$ from an $n$-dimensional manifold $M$ to an $n$-dimensional Poincaré duality space $X$ has a surgery obstruction $\sigma(f, b) \in L_n(\mathbb{Z}[\pi_1(X)])$ such that $\sigma(f, b) = 0$ if (and for $n \geq 5$ only if) $(f, b)$ is normal bordant to a homotopy equivalence.
Local $\implies$ global in surgery theory

- The algebraic $L$-groups $L_\ast(\mathbb{Z}[\pi_1(X)])$ depend only on the fundamental group $\pi_1(X)$ of a space $X$, so are global.

- The Witt groups of sheaves of quadratic forms over $X$ define the generalized homology groups $H_\ast(X;\mathbf{L}(\mathbb{Z}))$, which are local. Here $\mathbf{L}(\mathbb{Z})$ is a spectrum with

  $$\pi_\ast(\mathbf{L}(\mathbb{Z})) = L_\ast(\mathbb{Z}) = \mathbb{Z}, 0, \mathbb{Z}_2, 0, \ldots \text{ (4-periodic)}$$

  the simply-connected surgery obstruction groups.

- For any space $X$ there is an exact sequence

  $$\cdots \rightarrow H_n(X;\mathbf{L}(\mathbb{Z})) \xrightarrow{A} L_n(\mathbb{Z}[\pi_1(X)]) \rightarrow S_n(X) \rightarrow H_{n-1}(X;\mathbf{L}(\mathbb{Z})) \rightarrow \cdots$$

  $A$ is the local $\implies$ global assembly map in $L$-theory. Originally defined geometrically by Quinn.

- The $S$-groups $S_\ast(X)$ measure the failure of $A$ to be an isomorphism.
The failure of local Poincaré duality

- Let $X$ be an $n$-dimensional Poincaré duality space. The failure of local Poincaré duality at $x \in X$ is measured by the groups $K_*(X, x)$ in the exact sequences

$$
\cdots \to H^{n-r-1}(\{x\}) \xrightarrow{[X]_x \cap -} H_{r+1}(X, X\setminus\{x\}) \to K_r(X, x) \to H^{n-r}(\{x\}) \to \cdots
$$

- $X$ is a homology manifold if and only if $K_*(X, x) = 0$ ($x \in X$).
- Roughly speaking, the total surgery obstruction $s(X) \in S_n(X)$ is the cobordism class of a sheaf over $X$ of chain complexes with quadratic Poincaré duality over $\mathbb{Z}$ with $K_*(X, x)$ the stalk at $x \in X$.
- Chain complex with quadratic Poincaré duality
  $\equiv$ chain complex with quadratic structure
  $\equiv$ generalization of quadratic form.
Bringing in the sheaves
(From *The Night of the Hunter*)

- The book
  A.R. *Algebraic L-theory and manifolds* (CUP, 1992) developed the theory for simplicial complexes $K$, with an assembly map

  \[ A : \{ (\mathbb{Z}, K) \text{-modules} \} \rightarrow \{ \mathbb{Z}[\pi_1(K)] \text{-modules} \} \]

  to provide the passage from local to global in algebra. This is sufficient for applications, since every Poincaré duality space is homotopy equivalent to one which is triangulated.

- Unfortunately, have not yet been able to develop the necessary sheaf theory. However, the paper
  A.R. + Michael Weiss *On the construction and topological invariance of the Pontryagin classes* (Geometriae Dedicata 2010) points in the right direction!
Rings with involution

- **An involution** on a ring $A$ is a function
  
  $$A \to A \ ; \ a \mapsto \bar{a}$$

  such that
  
  $$a + b = \bar{a} + \bar{b} , \ \bar{a} \bar{b} = \bar{b} \bar{a} , \ \bar{a} = a \ (a, b \in A) .$$

- **Example 1** A commutative ring $A$, with $\bar{a} = a$.
- **Example 2** A group ring $A = \mathbb{Z}[\pi]$ with $\bar{g} = g^{-1} \ (g \in \pi)$.
- Regard a left $A$-module $P$ as a right $A$-module with
  
  $$P \times A \to P \ ; \ (x, a) \mapsto \bar{a}x .$$

- The tensor product of left $A$-modules $P, Q$ is the abelian group defined by
  
  $$P \otimes_A Q = P \otimes_{\mathbb{Z}} Q / \{ax \otimes y - x \otimes ay \mid a \in A, x \in P, y \in Q\}$$

  with transposition isomorphism

  $$P \otimes_A Q \to Q \otimes_A P \ ; \ x \otimes y \mapsto y \otimes x .$$
Duality over a ring with involution

The dual of a left $A$-module $P$ is the left $A$-module

$$P^* = \text{Hom}_A(P, A), \quad A \times P^* \to P^* ; (a, f) \mapsto (x \mapsto f(x)a).$$

The natural $A$-module morphism

$$P \to P^{**} ; x \mapsto (f \mapsto f(x))$$

is an isomorphism for f.g. free $P$.

For $A$-modules $P, Q$ the abelian group morphisms

$$P^* \otimes_A Q \to \text{Hom}_A(P, Q) ; f \otimes y \mapsto (x \mapsto f(x)y),$$

$$* : \text{Hom}_A(P, Q) \to \text{Hom}_A(Q^*, P^*) ; f \mapsto (f^* : g \mapsto (x \mapsto g(f(x))))$$

are isomorphisms for f.g. free $P, Q$. 
Quadratic forms on chain complexes I.

- The \textit{n-dual} of a f.g. free $A$-module chain complex

$$C : \ldots \to C_r \xrightarrow{d} C_{r-1} \to \ldots \to C_1 \xrightarrow{d} C_0 \to \ldots$$

is the f.g. free $A$-module chain complex

$$C^{n-*} : \ldots \to C^0 \xrightarrow{d^*} C^1 \to \ldots \to C^{r-1} \xrightarrow{d^*} C^r \to \ldots$$

with $C^r = C_r^*$.

- An ‘algebraic Poincaré complex’ is a f.g. free $A$-module chain complex $C$ with a chain equivalence $C^{n-*} \simeq C$ satisfying extra conditions. There are two flavours: symmetric and quadratic. Will ignore the difference today, using algebraic for both!
Quadratic forms on chain complexes II.

For any f.g. free $A$-module chain complex $C$ there is defined an isomorphism of $A$-module chain complexes

$$C \otimes_A C \rightarrow \text{Hom}_A(C^{-*}, C) ; \; x \otimes y \mapsto (f \mapsto f(x).y) .$$

The homology group

$$H_n(C \otimes_A C) = H_0(\text{Hom}_A(C^{n-*}, C))$$

is the group of chain homotopy classes of chain maps $\phi : C^{n-*} \rightarrow C$.

The action of $T \in \mathbb{Z}_2$ by the transposition involution

$$T : C \otimes_A C \rightarrow C \otimes_A C ; \; x \otimes y \mapsto (-)^{pq} y \otimes x \; (x \in C_p, y \in C_q)$$

corresponds to the duality involution

$$T : \text{Hom}_A(C^{-*}, C) \rightarrow \text{Hom}_A(C^{-*}, C) ; \; f \mapsto (-)^{pq} f^* ,$$

$$(f : C^p \rightarrow C_q) \mapsto ((-)^{pq} f^* : C^q \rightarrow C_p) , \; y(f^*(x)) = x(f(y)) .$$
Algebraic Poincaré cobordism

- An \( n \)-dimensional algebraic Poincaré complex over \( A \) \((C, \phi)\) is an \( n \)-dimensional f.g. free \( A \)-module chain complex \( C \) together with a chain equivalence \( \phi : C^{n-*} \rightarrow C \) such that there exists a chain homotopy \( T\phi \simeq \phi : C^{n-*} \rightarrow C \).
- If \( 1/2 \notin A \) need additional structure: either symmetric or quadratic.
- A cobordism \((L; M, M')\) of \( n \)-dimensional manifolds has Poincaré-Lefschetz duality

\[
[L] \cap - : H^{n+1-*}(L, M) \cong H_*(L, M')
\]

- **Proposition** (Mishchenko, R., 1970’s) The Wall group \( L_n(A) \) is the group of cobordism classes of \( n \)-dimensional algebraic Poincaré complexes \((C, \phi)\) over \( A \), with \((C, \phi) \sim (C', \phi')\) if \( C \oplus C' \subset D \) for an \((n + 1)\)-dimensional f.g. free \( A \)-module chain complex \( D \) such that \( H^{n+1-*}(D, C) \cong H_*(D, C') \).
The polyhedron of a simplicial complex

- A simplicial complex $K$ is a collection of finite subsets $\sigma \subseteq K^{(0)}$ of an ordered vertex set $K^{(0)}$ such that:
  
  a) $v \in K$ for each $v \in K^{(0)}$,
  
  b) if $\sigma \in K$ and $\tau \subseteq \sigma$ then $\tau \in K$.

- The dimension of $\sigma \in K$ is

  $|\sigma| = \text{(no. of vertices in } \sigma \text{)} - 1$

Let $K^{(n)}$ denote the set of $n$-simplexes in $K$.

- The polyhedron of $K$ is the usual identification space

  $|K| = \left( \coprod_{n=0}^{\infty} \Delta^n \times K^{(n)} \right)/\sim$

with $\Delta^n$ the convex hull of $(0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^{n+1}$. 
The simplicial chain complex

The **simplicial chain complex** $C(K)$ has

$$d : C(K)_n = \mathbb{Z}[K^{(n)}] \to C(K)_{n-1} = \mathbb{Z}[K^{(n-1)}];$$

$$(v_0 v_1 \ldots v_n) \mapsto \sum_{i=0}^{n} (-)^i (v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n)$$

$$(v_0 < v_1 < \cdots < v_n)$$

The homology and cohomology groups of the polyhedron are the same as those of the simplicial complex

$$H_*(|K|) = H_*(K) = H_*(C(K)),$$

$$H^*(|K|) = H^*(K) = H^*(C(K)).$$

For any simplicial complexes $K, L$ $H_n(|K| \times |L|)$ is the group of chain homotopy classes of chain maps $C(K)^{n-*} \to C(L)$. 
Polyhedral Poincaré complexes

- A **triangulated $n$-dimensional Poincaré space** is a finite simplicial complex $K$ with universal cover $\tilde{K}$ and a homology class $[K] \in H_n(K)$ satisfying the equivalent conditions:
  - (a) the cap products
    \[ [K] \cap - : H^{n-*}(\tilde{K}) = H_*(C(\tilde{K})^{n-*}) \to H_*(\tilde{K}) \]
    are $\mathbb{Z}[\pi_1(K)]$-module isomorphisms.
  - (b) The image $\Delta[K] \in H_n(X)$ under the diagonal map
    \[ \Delta : |K| \to X = |\tilde{K}| \times_{\pi_1(K)} |\tilde{K}| ; x \mapsto (\tilde{x}, \tilde{x}) \]
    is a chain homotopy class of $\mathbb{Z}[\pi_1(K)]$–module chain equivalences $\phi = \Delta[K] : C(\tilde{K})^{n-*} \to C(\tilde{K})$.
  - (c) The cap product $[X] \cap - : H^n(X) \to H_n(X)$ is an isomorphism, with $\Delta[K]^* \in H^n(X)$ a $\mathbb{Z}[\pi_1(K)]$-module chain homotopy inverse $\phi^{-1} : C(\tilde{K}) \to C(\tilde{K})^{n-*}$.
  - $(C(\tilde{K}), \phi)$ is an $n$-dimensional algebraic Poincaré complex over $\mathbb{Z}[\pi_1(K)]$. 
Dual cells

▶ The **barycentric subdivision** of \( K \) is the simplicial complex \( K' \) with \( K'(0) = K \) and

\[
K'(n) = \{(\sigma_0, \sigma_1, \ldots, \sigma_n) \mid \sigma_0 \subset \sigma_1 \subset \cdots \subset \sigma_n\}.
\]

Homeomorphic polyhedron \( |K'| \cong |K| \).

▶ The **dual cells** of \( K \) are the contractible subcomplexes

\[
D(\sigma) = \{(\sigma_0, \sigma_1, \ldots, \sigma_n) \in K' \mid \sigma_0 \subseteq \sigma\} \subseteq K'.
\]

▶ The **boundary** of the dual cell \( D(\sigma) \) is

\[
\partial D(\sigma) = \{(\sigma_0, \sigma_1, \ldots, \sigma_n) \in D(\sigma) \mid \sigma_0 \neq \sigma\}.
\]

▶ **Proposition** The local homology groups of \( |K| \) at \( x \in |K| \) are the homology groups of the dual cells relative to boundaries

\[
H_*(|K|, |K| \setminus \{x\}) = H_{*-|\sigma|}(D(\sigma), \partial D(\sigma)) \quad (x \in \text{interior}(\sigma), \sigma \in K).
\]

For each \( \sigma \in K \) and \( x \in \text{interior}(\sigma) \) there are natural maps

\[
\partial_\sigma : H_*(|K|) = H_*(K) \to H_*(|K|, |K| \setminus \{x\}) = H_{*-|\sigma|}(D(\sigma), \partial D(\sigma)).
\]
The \((\mathbb{Z}, K)\)-category I. Modules

- A \((\mathbb{Z}, K)\)-module is a f.g. free \(\mathbb{Z}\)-module \(M\) with splitting
  \[
  M = \sum_{\sigma \in K} M(\sigma) .
  \]
- A morphism of \((\mathbb{Z}, K)\)-modules \(f : M \to N\) is a \(\mathbb{Z}\)-module morphism such that
  \[
  f(M(\sigma)) \subseteq \sum_{\tau \geq \sigma} N(\tau) \ (\sigma \in K) .
  \]
- Proposition A \((\mathbb{Z}, K)\)-module morphism \(f : M \to N\) is an isomorphism if and only if each
  \[
  f(\sigma, \sigma) : M(\sigma) \to N(\sigma) \ (\sigma \in K)
  \]
is a \(\mathbb{Z}\)-module isomorphism.
Assembly

Let \( p : \tilde{K} \to K \) be the universal cover of a connected simplicial complex \( K \). The **assembly** functor

\[
A : \{(\mathbb{Z}, K)\text{-modules}\} \to \{\text{f.g. free } \mathbb{Z}[\pi_1(K)]\text{-modules}\}
\]

is defined by

\[
A(M) = \sum_{\tilde{\sigma} \in \tilde{K}} M(p(\tilde{\sigma})).
\]

**Local \(\mathbf{\Rightarrow}\) global.**

**Example** For finite \( K \) the simplicial chain complex \( C(K') \) is a \((\mathbb{Z}, K)\)-module chain complex with

\[
C(K')(\sigma) = C(D(\sigma), \partial D(\sigma)) \ (\sigma \in K)
\]

The assembly is the simplicial \( \mathbb{Z}[\pi_1(K)] \)-module chain complex of \( \tilde{K}' \)

\[
A(C(K')) = C(\tilde{K}').
\]
The algebraic Vietoris theorem

Let $f : L \rightarrow K'$ be a simplicial map with $K, L$ finite.

Regard $C(L)$ as a $(\mathbb{Z}, K)$-module chain complex by

$$C(L)(\sigma) = C(f^{-1}D(\sigma), f^{-1}\partial D(\sigma)) \ (\sigma \in K).$$

Proposition $f$ has acyclic point inverses if and only if

$$f : C(L) \rightarrow C(K')$$

is a $(\mathbb{Z}, K)$-module chain equivalence.

Corollary If $f$ has acyclic point inverses then

$$\tilde{f} : C(\tilde{L}) \rightarrow C(\tilde{K}')$$

is a $\mathbb{Z}[\pi_1(K)]$-module chain equivalence.
The \((\mathbb{Z}, K)\)-category II. Products

- **The product** of \((\mathbb{Z}, K)\)-modules \(A, B\) is the \((\mathbb{Z}, K)\)-module

\[
A \otimes_{(\mathbb{Z}, K)} B = \sum_{\lambda, \mu \in K, \lambda \cap \mu \neq \emptyset} A(\lambda) \otimes_{\mathbb{Z}} B(\mu) \subseteq A \otimes_{\mathbb{Z}} B
\]

with

\[
(A \otimes_{(\mathbb{Z}, K)} B)(\sigma) = \sum_{\lambda, \mu \in K, \lambda \cap \mu = \sigma} A(\lambda) \otimes_{\mathbb{Z}} B(\mu).
\]

- **Example** For simplicial maps \(f : L \to K'\), \(g : M \to K'\) the pullback polyhedron

\[
L \times_K M = \{ (x, y) \in |L| \times |M| \mid f(x) = g(y) \in |K| \}
\]

has homology

\[
H_*(L \times_K M) = H_*(C(L) \otimes_{(\mathbb{Z}, K)} C(M))
\]

with

\[
C(L)(\sigma) = C(f^{-1}D(\sigma), f^{-1}\partial D(\sigma)),
\]

\[
C(M)(\sigma) = C(g^{-1}D(\sigma), g^{-1}\partial D(\sigma)).
\]
The \((\mathbb{Z}, K)\)-category III. Duality

- The **dual** of a \((\mathbb{Z}, K)\)-module \(M\) is the \((\mathbb{Z}, K)\)-module chain complex \(TM\) with

\[
TM(\sigma)_r = \begin{cases} 
\sum_{\tau \geq \sigma} M(\tau)^* & \text{if } r = -|\sigma| \\
0 & \text{otherwise.}
\end{cases}
\]

- The dual of a \((\mathbb{Z}, K)\)-module chain complex \(C\) is a \((\mathbb{Z}, K)\)-module chain complex \(TC\). Analogue of Verdier duality for sheaves.

- **Example** The dual of \(C(K')\) is \((\mathbb{Z}, K)\)-equivalent to the cochain complex of \(K\)

\[
TC(K') \simeq C(K)^-* \ , \ C(K)^r(\sigma) = \begin{cases} 
\mathbb{Z} & \text{if } r = -|\sigma| \\
0 & \text{otherwise.}
\end{cases}
\]

- For any \((\mathbb{Z}, K)\)-module chain complexes \(C, D\)

\[
H_*(C \otimes_{(\mathbb{Z}, K)} D) = H_*(\text{Hom}_{(\mathbb{Z}, K)}(TC, D)) .
\]
The assembly map

▶ **Proposition** (i) The generalized homology group $H_n(K; L(\mathbb{Z}))$ is the cobordism group of $n$-dimensional algebraic Poincaré complexes $(C, \phi : TC_{*-n} \to C)$ in the $(\mathbb{Z}, K)$-module category.

▶ (ii) The assembly functor

$$A : \{(\mathbb{Z}, K)\text{-modules}\} \to \{\mathbb{Z}[\pi_1(K)]\text{-modules}\}$$

induces assembly maps in algebraic $L$-theory

$$A : H_n(K; L(\mathbb{Z})) \to L_n(\mathbb{Z}[\pi_1(K)])$$

▶ (iii) $\mathbb{S}_n(K)$ is the cobordism group of $(n - 1)$-dimensional algebraic Poincaré complexes $(C, \phi)$ in the $(\mathbb{Z}, K)$-module category such that the assembly $A(C)$ is a contractible f.g. free $\mathbb{Z}[\pi_1(K)]$-module chain complex, $H_*(A(C)) = 0$. 
From local to global Poincaré duality, and back again!

For any simplicial complex $K$

$$H_n(K) = H_n(\text{Hom}(\mathbb{Z}, K)(TC(K'), C(K'))).$$

The cap product with any homology class $[K] \in H_n(K)$ is a $(\mathbb{Z}, K)$-module chain map

$$\phi = [K] \cap - : TC(K')_{*-n} \to C(K')$$

with diagonal components

$$\phi(\sigma, \sigma) = \partial_\sigma [K] \cap - : TC(K')_{*-n}(\sigma) = C(D(\sigma))^{n-\ast - |\sigma|} \to C(K')(\sigma) = C(D(\sigma), \partial D(\sigma)) \ (\sigma \in K),$$

with assembly

$$[K] \cap - : TC(\tilde{K}')_{*-n} \simeq C(\tilde{K})^{n-\ast} \to C(\tilde{K}') \simeq C(\tilde{K}).$$

$K$ is a homology manifold if and only if $[K] \cap -$ is a $(\mathbb{Z}, K)$-module chain equivalence. This is essentially Poincaré’s original proof of duality!
The total surgery obstruction

- The **total surgery obstruction** of a polyhedral $n$-dimensional Poincaré duality space $K$ is the cobordism class

$$s(K) = (C(\phi)_{*+1}, \psi) \in \mathbb{S}_n(K),$$

with $C(\phi)$ the $\mathbb{Z}[\pi_1(K)]$-contractible algebraic mapping cone of the $(\mathbb{Z}, K)$-module chain map

$$\phi = [K] \cap - : TC(K')_{n-*} \to C(K').$$

- The image

$$t(K) = [s(K)] \in H_{n-1}(K; \mathbb{L}(\mathbb{Z}))$$

is such that $t(K) = 0$ if and only if there exists a normal map $(f, b) : M \to |K|$, $M$ an $n$-dimensional topological manifold.

- $s(K) = 0$ if and only if there exists a normal map $(f, b)$ with surgery obstruction $\sigma(f, b) = 0 \in L_n(\mathbb{Z}[\pi_1(K)])$.

- For $n \geq 5$ $s(K) = 0$ if and only if $|K|$ is homotopy equivalent to an $n$-dimensional topological manifold, by B-N-S-W theory.
The symmetric signature

▶ The **symmetric signature** of a triangulated $n$-dimensional Poincaré space $K$ is the algebraic Poincaré cobordism class

$$
\sigma(K) = (C(\tilde{K}), \phi) \in L_n(\mathbb{Z}[\pi_1(K)]) .
$$

▶ The symmetric signature is a homotopy invariant, generalizing the signature.

▶ Modulo 2-torsion, the total surgery obstruction is the image

$$
s(K) = [\sigma(K)] \in \text{im}(L_n(\mathbb{Z}[\pi_1(K)]) \to \mathbb{S}_n(K)) .
$$

▶ **Theorem** (A.R., 1992) Modulo 2-torsion, if $n \geq 5$ $|K|$ is homotopy equivalent to an $n$-dimensional topological manifold if and only if $s(K) = 0 \in \mathbb{S}_n(K)$, if and only if

$$
\sigma(K) \in \text{im}(A : H_n(K; \mathbb{L}(\mathbb{Z})) \to L_n(\mathbb{Z}[\pi_1(K)])) .
$$

▶ For $n = 4k$, $\pi_1(K) = \{1\}$ this is just Browder’s converse of the Hirzebruch signature theorem.
The homotopy types of topological manifolds

- For $n \geq 5$ the homotopy types of $n$-dimensional topological manifolds $M$ fit into a fibre square

\[
\begin{array}{ccc}
\text{topological manifolds} & \xrightarrow{} & \text{cobordism of local APC’s} \\
\downarrow & & \downarrow A \\
\text{PD spaces} & \xrightarrow{} & \text{cobordism of global APC’s}
\end{array}
\]

with PD = Poincaré duality, APC = algebraic Poincaré complexes, A = assembly.

- Local = in the $(\mathbb{Z}, K)$-module category, for a finite simplicial complex $K$ with a surjection $|K| \to M$ with acyclic point inverses, and $\pi_1(|K|) \cong \pi_1(M)$,

- Global = in the $\mathbb{Z}[\pi]$-module category, $\pi = \pi_1(|K|) = \pi_1(M)$. 
Three conjectures

- The **Novikov conjecture** (1969) on the homotopy invariance of the higher signatures of manifolds with fundamental group $\pi$ is equivalent to the injectivity of the local $\Rightarrow$ global assembly map $1 \otimes A : H_*(B\pi; L(\mathbb{Z})) \otimes \mathbb{Q} \to L_*(\mathbb{Z}[\pi]) \otimes \mathbb{Q}$.

  History and survey of the Novikov conjecture.

- The **Borel conjecture** (1953) on the existence and rigidity of topological manifold structures on aspherical Poincaré complexes $B\pi$ is essentially equivalent to the assembly map $A : H_*(B\pi; L(\mathbb{Z})) \to L_*(\mathbb{Z}[\pi])$ being an isomorphism, so that local $\iff$ global.

  1953 letter from Borel to Serre.

- The **Farrell-Jones conjecture** (1982) that a generalized assembly map from equivariant homology to the $L$-theory of $\mathbb{Z}[\pi]$ is an isomorphism for all groups $\pi$. 

Conclusion

▶ Starting with Novikov himself, many authors in the last 40 years have proved many special cases of the Novikov, Borel and Farrell-Jones conjectures, using a wide variety of algebraic, geometric and analytic methods.
▶ Some (though not all) have used the algebraic $L$-theory assembly map defined here.
▶ There is still much work to be done to understand the relationship between all these methods of proof, and maybe even prove new results!