The topological rigidity of the torus

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Preface

One of the main goals in topology is the classification of manifolds up to some equivalence relation (homotopy equivalence, homeomorphism, PL-homeomorphism, diffeomorphism...). A very natural question arising from this is to decide whether two manifolds identical up to some equivalence relation are still the same under a stronger equivalence relation. In this thesis, we will be interested in topogical rigidity. A manifold M is said to be *topologically rigid* if any topological manifold homotopy equivalent to M is actually homeomorphic to M. Maybe the most famous example of a topological rigidity phenomenon is the Poincaré conjecture, now proven in all dimensions, which asserts that $\mathbb{S}^n$ is topologically rigid. An example more related to the subject of this thesis is the following result of Mostow.

Mostow Rigidity Theorem. If $f : M \to N$ is a homotopy equivalence between two complete hyperbolic n-manifolds of finite volume ($n \geq 3$), then f is homotopic to an isometry.

Motivated by this result, Borel formulated the following conjecture, which can be thought as a topological analogue of the Mostow Rigidity Theorem.

Borel Conjecture. A compact aspherical manifold is topologically rigid.

The first example was developed during the sixties. In 1964, Bass, Heller and Swan [. H. 64] proved the vanishing of $Wh(\mathbb{Z}^n)$. At the same period, Farrell and Hsiang developped the theory of codimension one splitting obstructions [FH73], using the ideas introduced in the doctoral dissertation of Farrell (1967). This in turn was used by Hsiang and Shaneson [HS70] to classify the PL-structures on a PL-manifold homotopy equivalent to a high-dimensional torus. This result is a cornerstone in high-dimensional topology. It was of crucial importance to Kirby and Siebenmann [KS77] who used it to develop the theory of topological manifolds.

This in turn was used by Hsiang and Wall to prove the topogical rigidity of the torus in 1969 in [HW69].

Since then, this circle of idea has been extensively studied, especially by Farrell and Jones, who were able to prove the Borel Conjecture for a large class of manifolds.
The aim of this thesis is to present the surgical proof of the original result, namely the topologicl rigidity of the torus. It is organised as follows. The first chapter presents a proof of the Bass-Heller-Swan theorem for $\mathbb{Z}^n$, which will be used in Chapter 2 to split homotopy equivalences along codimension one submanifolds. This splitting theorem will be a key ingredient for the computation of the PL-structure set $\mathcal{S}_{PL}(\mathbb{T}^k \times \mathbb{D}^n)$ presented in Chapter 4. Chapter 5 deals with the various PL-structure a topological manifold might carry. Finally, we prove the topological rigidity of the torus in Chapter 6.

While trying to be as self-contained as possible, I was forced to outsource some results and some of the most technical lemmas to keep this thesis reasonably long. I shall give references anytime I do that.

I would like to thank here my advisor, Andrew Ranicki. During these few months I spent in Edinburgh, he guided me through this beautiful world of surgery theory, showing me its beautiful landscapes, while encouraging me to venture on some (algebraic) roads I would not have taken alone.

I would like also to thank my advisor at the Ecole Normale Supérieure, Frédéric Paulin. Since my very first year at the ENS he helped me and encouraged me to work on a field that is not so fashionable in France. I could not thank him enough for that!

Contents

Chapter 1

The Bass-Heller-Swan Theorem.

The Whitehead torsion $\tau(f) \in Wh(\pi_1(N))$ of a homotopy equivalence $f : M \to N$ vanishes if f is homotopic to a homeomorphism. Thus the vanishing of $Wh(\pi_1(M))$ is generally a first step in proving the topological rigidity of a high-dimensional manifold M. This chapter is devoted to prove the vanishing of $Wh(\mathbb{Z}^n)$.

In Section 1.1, we define lower algebraic K-groups for suitable categories. The interest of such a general approach will become apparent in Section 1.2 where we prove the Resolution Theorem for K_1, which will allow us to work with more tractable categories. We prove in Section 1.3 a theorem of Grothendieck which will be needed to prove the vanishing of $Wh(\mathbb{Z}^n)$ in Section 1.4. This chapter is greatly inspired by [Ros94] and [. H. 64].

Throughout this chapter R will be a **commutative** ring **with unit**.

1.1 K_0, K_1.

Definition 1.1.1. (i) Let **Proj** R be the category of finetaly generated projective R-modules.
(ii) Let R-$\mathbf{mod_{fg}}$ be the category of finitely generated R-modules. It is an abelian category if R is noetherian.

In order two define the algebraic K-groups K_0, K_1 of a category, we need some additional properties.

Definition 1.1.2. A *category with exact sequence* is a full additive subcategory $\mathcal{P}$ of an abelian category $\mathcal{A}$ such that

- $\mathcal{P}$ is closed under extensions, *i.e*, if

$$0 \to P_1 \to P \to P_2 \to 0$$

is a short exact sequence in $\mathcal{A}$ and $P_1, P_2 \in \mathrm{Obj}\ \mathcal{P}$, then $P \in \mathrm{Obj}\ \mathcal{P}$.

- $\mathcal{P}$ has a small skeleton, *i.e*, $\mathcal{P}$ has a full subcategory $\mathcal{P}_0$ which is small and for which the inclusion $\mathcal{P}_0 \hookrightarrow \mathcal{P}$ is an equivalence.

Proposition 1.1.3. *Let R be a Noetherian ring. Then R-$\boldsymbol{mod_{fg}}$ and **Proj** R are categories with exact sequences.*

Proof. These categories are clearly closed under extensions. **Proj** R has for (small) skeleton the set of direct summands in $R^n, n \geq 0$. R-$\mathbf{mod_{fg}}$ has for (small) skeleton the set of quotient modules of $R^n, n \geq 0$. □

We now define the first algebraic K-groups for **Proj** R and R-$\mathbf{mod_{fg}}$.

Definition 1.1.4. Let $\mathcal{P}$ be a category with exact sequences, and $\mathcal{P}_0$ its small skeleton. Let $\Lambda_0(\mathcal{P})$ be the free abelian group with generators $[P], P \in \mathrm{Obj}\ \mathcal{P}_0$. We define $K_0(\mathcal{P})$ as the quotient of $\Lambda_0(\mathcal{P})$ by the subgroup generated by the following relations:

$[P] = [P_1] + [P_2]$ if there is a short exact sequence in $\mathcal{P}$ of the form

$$0 \to P_1 \to P \to P_2 \to 0.$$

Note that since every object of $\mathcal{P}$ is isomorphic to an object of $\mathcal{P}_0$, the notation $[P]$ makes sense for every $P \in \mathrm{Obj}\ \mathcal{P}$.

Definition 1.1.5. Let $\mathcal{P}$ be a category with exact sequences, and $\mathcal{P}_0$ its small skeleton. Let $\Lambda_1(\mathcal{P})$ be the free abelian group with generators $(P, \alpha), P \in \mathrm{Obj}\ \mathcal{P}_0, \alpha \in \mathrm{Aut}\ P$. We define $K_1(\mathcal{P})$ as the quotient of $\Lambda_1(\mathcal{P})$ by the subgroup generated by the following relations:

(1) $[P, \alpha\beta] = [P, \alpha] + [P, \beta]$.

(2) If there is a commutative diagram in $\mathcal{P}$ with exact row

$$\begin{tikzcd}
0 \arrow[r] & P_1 \arrow[r, "\iota"] \arrow[d, "\alpha_1"'] & P \arrow[r, "\pi"] \arrow[d, "\alpha"'] & P_2 \arrow[r] \arrow[d, "\alpha_2"'] & 0 \\
0 \arrow[r] & P_1 \arrow[r, "\iota"] & P \arrow[r, "\pi"] & P_2 \arrow[r] & 0
\end{tikzcd}$$

where $\alpha \in \text{Aut } P$, $\alpha_1 \in \text{Aut } P_1$ and $\alpha_2 \in \text{Aut } P_2$, then

$$[P, \alpha] = [P_1, \alpha_1] + [P_2, \alpha_2].$$

Once again, since every object of $\mathcal{P}$ is isomorphic to an object of $\mathcal{P}_0$, the notation $[P, \alpha]$ makes sense for every $P \in \text{Obj } \mathcal{P}$, $\alpha \in \text{Aut } P$.

This category-theoretic approach to K_1 might surprise (or alarm) the topologist reader. Note however that it also yields the common definition of K_1 that we briefly recall here.

Definition 1.1.6. A $n \times n$ matrix is said to be *elementary* if it is of the form $I_n + rE_{i,j}$, where $r \in R$ and $E_{i,j} = (\delta_{ik,jl})_{1 \leq k \leq n, 1 \leq l \leq n}$ $(1 \leq i \leq n, 1 \leq j \leq n)$. The subgroup of $GL_n(R)$ generated by elementary matrices is denoted $E_n(R)$. Using the natural embedding $GL_n(R) \hookrightarrow GL_{n+1}(R)$ given by

$$M \mapsto \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix},$$

$E_n(R)$ embeds in $E_{n+1}(R)$. We denote $GL(R)$ (resp. $E(R)$) the direct limit of the $GL_n(R)$ (resp. $E_n(R)$).

Lemma 1.1.7 (Whitehead's lemma)**.** $E(R) = [GL(R), GL(R)]$.

Definition 1.1.8. (i) Let $K_1(R) = GL(R)/E(R) = GL(R)_{ab}$.
(ii) If π is a group, let $Wh(\pi)$ be the quotient of $K_1(\mathbb{Z}\pi)$ by the image of $GL_1(\mathbb{Z}\pi)$ in $GL(\mathbb{Z}\pi)$, called the *Whitehead group* of π.

Proposition 1.1.9. *$K_1(R)$ is an abelian group and $K_1(R) \simeq K_1(\mathbf{Proj}\ R)$.*

For a proof of these results, we refer to [Ros94].

Definition 1.1.10. We set $K_i(R) = K_i(\mathbf{Proj}\ R)$, $i = 0, 1$.

Given a functor between two categories with exact sequences, it is natural to ask if it yields a map between algebraic K-groups. The following is an immediate consequence of the definitions.

Proposition 1.1.11. *Given two categories with exact sequences $\mathcal{P}$ and $\mathcal{M}$, an exact functor $F : \mathcal{P} \to \mathcal{M}$ induces homomorphisms $F_* : K_i(\mathcal{P}) \to K_i(\mathcal{M}), i = 0, 1$.*

The main advantage of defining algebraic K-groups for a general category is the Resolution theorem proved in the next section, which will allow us to work in R-$\mathbf{mod_{fg}}$, a more tractable category than $\mathbf{Proj}\ R$.

In Section 1.2, we will construct an explicit inverse of $i_* : K_1(\mathbf{Proj}\ R) \to K_1(R\text{-}\mathbf{mod_{fg}})$ under some assumptions on R. Recall that every finitely generated R-module admits a projective resolution. To extract some information from such a resolution it would be preferable to have a projective resolution with finitely many nonzero modules. This motivates the following definition.

Definition 1.1.12. A noetherian ring is called *regular* if every finitely generated R-module M admits a projective resolution of finite type (or simply a *finite resolution*), *i.e*, if there exists an exact sequence

$$0 \to P_n \to \ldots \to P_1 \to M$$

with each P_i a finitely generated projective R-module.

We recall a famous theorem of Hilbert.

Theorem 1.1.13 (Hilbert's Syzygy Theorem)**.** *If R is a regular ring, then so is $R[t]$.*

This implies the following

Proposition 1.1.14. *If R is a regular ring, then so is $R[t, t^{-1}]$.*

Proof. $R[t, t^{-1}]$ is noetherian as a localization of the noetherian ring R. Let M be a finitely generated $R[t, t^{-1}]$-module. Choose a finite set of generators for M, and let M_1 be the finitely generated $R[t]$-module they generate. By the Syzygy theorem, let

$$0 \to P_n \to \ldots \to P_1 \to M_1$$

be a finite resolution of the $R[t]$-module. $R[t, t^{-1}]$ is flat over $R[t]$, since $R[t, t^{-1}] = \lim t^{-n} R[t]$ and $t^{-n}R[t]$ is free over $R[t]$, so

$$0 \to R[t, t^{-1}] \otimes_{R[t]} P_n \to \ldots \to R[t, t^{-1}] \otimes_{R[t]} P_1 \to R[t, t^{-1}] \otimes_{R[t]} M_1 \simeq M$$

is a finite resolution of the $R[t, t^{-1}]$-module M. □

Corollary 1.1.15. *$\mathbb{Z}[\mathbb{Z}^n]$ is a regular ring for all $n \geq 0$.*

Proof. By induction, since $\mathbb{Z}$ is clearly regular and $\mathbb{Z}[\mathbb{Z}^{n+1}] \simeq \mathbb{Z}[\mathbb{Z}^n][t, t^{-1}]$. □

1.2 The Resolution theorem for K_1.

In this section we will prove that for a regular ring R, $K_1(R)$, or equivalently $K_1(\mathbf{Proj}\ R)$, is naturally isomorphic to $K_1(R\text{-}\mathbf{mod_{fg}})$. The main advantage of this theorem is that it allows us some useful constructions as quotients, since we now work in an abelian category. This considerations will be of crucial importance in the proof of the vanishing of $Wh(\mathbb{Z}^n)$.

The notation $[\ ]_{fg}$ (resp. $[\ ]_{proj}$) will denote an element of $K_*(R\text{-}\mathbf{mod_{fg}})$ (resp. $K_*(\mathbf{Proj}\ R)$.

Our starting point, while trying to build a map $K_1(R\text{-}\mathbf{mod_{fg}}) \to K_1(\mathbf{Proj}\ R)$, is the fact that every finitely generated R-module (with R-regular) admits a finite resolution. We need the following result, which allows to lift an automorphism of a finitely-generated R-module to an automorphism of some finite resolution.

Proposition 1.2.1. *Let R be a regular ring, M a finitely generated R-module, and α an automorphism of M. Then there exists a finite resolution*

$$0 \to P_r \to \ldots \to P_1 \to M$$

and elements $\alpha_i \in Aut\ P_i, 1 \leq i \leq r$, such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & P_r \arrow[r] \arrow[d, "\alpha_r"'] & \cdots \arrow[r] & P_1 \arrow[r] \arrow[d, "\alpha_1"'] & M \arrow[r] \arrow[d, "\alpha"'] & 0 \\
0 \arrow[r] & P_r \arrow[r] & \cdots \arrow[r] & P_1 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

Proof. Choose an epimorphism $P \to M \to 0$, with P projective. Since P is projective, every endomorphism of M lifts to an endomorphism of P. However, an automorphism of M does not necessarily lift to an automorphism of P. To avoid this difficulty, let us consider the automorphism $\alpha \oplus \alpha^{-1} \in \text{Aut}\ M \oplus M$. Note that we have (it is convenient to adopt a matricial notation here):

$$\alpha \oplus \alpha = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\alpha^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Thus, lifting α (resp. α^{-1}) to any endomorphism β (resp. β') of P, the formula

$$\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\beta' & 1 \end{pmatrix} \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

yields a lifting of $\alpha \oplus \alpha^{-1}$ to an automorphism of α_1 of $P \oplus P$. We thus have the following commutative diagram

$$\begin{tikzcd}
P \oplus P \arrow[r, "\pi\oplus 0"] \arrow[d, "\alpha_1"'] & M \arrow[r] \arrow[d, "\alpha"'] & 0 \\
P \oplus P \arrow[r, "\pi\oplus 0"] & M \arrow[r] & 0.
\end{tikzcd}$$

By commutativity of the diagram, $\ker(\pi \oplus 0)$ is stable under α_1, and α_1 induces an isomorphism of $\ker(\pi \oplus 0)$. Thus we are back to the same situation, with finitely generated R-module $\ker(\pi \oplus 0)$ and $\alpha_1 \in \mathrm{Aut}\ \ker(\pi \oplus 0)$. Thus repeating the same argument, possibly infinitely many times, we have a lifting of projective resolution

$$\begin{tikzcd}
\cdots \arrow[r, "d_{r+1}"] & P_r \arrow[r, "d_r"] \arrow[d, "\alpha_r"'] & \cdots \arrow[r] & P_1 \arrow[r] \arrow[d, "\alpha_1"'] & M \arrow[r] \arrow[d, "\alpha"'] & 0 \\
\cdots \arrow[r, "d_{r+1}"] & P_r \arrow[r, "d_r"] & \cdots \arrow[r] & P_1 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

with $\alpha_i \in \mathrm{Aut}\ P_i$, for all $i \geq 0$. But R is a regular ring, so admits a projective resolution of finite lenght, say r. Now, by a basic lemma of homological algebra, this implies that any projective resolution can be shortened at its r-th stage. More precisely, considering the projective resolution

$$\cdots \xrightarrow{d_{r+1}} P_r \xrightarrow{d_r} \cdots \longrightarrow P_1 \longrightarrow M \longrightarrow 0,$$

this implies that $\ker d_r$ is projective. Since α_{r+1} induces an isomorphism on $\ker d_r$, we have the following finite resolution:

$$\begin{tikzcd}
0 \arrow[r] & \ker d_r \arrow[r, "d_{r+1}"] \arrow[d, "\alpha_{r+1}"'] & P_r \arrow[r, "d_r"] \arrow[d, "\alpha_r"'] & \cdots \arrow[r] & P_1 \arrow[r] \arrow[d, "\alpha_1"'] & M \arrow[r] \arrow[d, "\alpha"'] & 0 \\
0 \arrow[r] & \ker d_r \arrow[r, "d_{r+1}"] & P_r \arrow[r, "d_r"] & \cdots \arrow[r] & P_1 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

□

Proposition 1.2.2. *Define $\Phi : K_1(R\text{-}\boldsymbol{mod_{fg}}) \to K_1(\boldsymbol{Proj}\ R)$ as follows. Given an element $[M, \alpha] \in K_1(R\text{-}\boldsymbol{mod_{fg}})$, we can lift to an isomorphism of some finite projective resolution, as in 1.2.1,*

$$\begin{tikzcd}
0 \arrow[r] & P_r \arrow[r] \arrow[d, "\alpha_r"'] & \cdots \arrow[r] & P_1 \arrow[r] \arrow[d, "\alpha_1"'] & M \arrow[r] \arrow[d, "\alpha"'] & 0 \\
0 \arrow[r] & P_r \arrow[r] & \cdots \arrow[r] & P_1 \arrow[r] & M \arrow[r] & 0.
\end{tikzcd}$$

Set

$$\Phi\left([M,\alpha]_{fg}\right) = \sum_{i\geq 1}(-1)^i[P_i,\alpha_i]_{proj} \in K_1(\boldsymbol{Proj}\ R).$$

Then Φ is a well defined homomorphism.

We first prove that this is independent of the chosen resolution. To achieve this, we need the following lemmas

Lemma 1.2.3. *Given two finitely generated R-module M, M', a morphism $\alpha : M \to M'$, and a finite projective resolution*

$$0 \to P_r \to \ldots \to P_0 \to M \to 0,$$

there exists a lift

$$\begin{tikzcd}
0 \arrow[r] & \cdots \arrow[r] & P'_{r+1} \arrow[r] \arrow[d, "\alpha_{r+1}"'] & P'_r \arrow[r] \arrow[d, "\alpha_r"'] & \cdots \arrow[r] & P'_1 \arrow[r] \arrow[d, "\alpha_1"'] & M' \arrow[r] \arrow[d, "\alpha"'] & 0 \\
0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & P_r \arrow[r] & \cdots \arrow[r] & P_1 \arrow[r] & M \arrow[r] & 0.
\end{tikzcd}$$

whose rows are finite projective resolutions.

Proof. We start by lifting the following diagram

$$\begin{tikzcd}
 & M' \arrow[d, "\alpha"'] & \\
P_0 \arrow[r, "d_0"] & M \arrow[r] & 0
\end{tikzcd}$$

to a diagram

$$\begin{tikzcd}
P'_0 \arrow[r] \arrow[d, "\alpha_0"'] & M' \arrow[d, "\alpha"'] & \\
P_0 \arrow[r, "d_0"] & M \arrow[r] & 0.
\end{tikzcd}$$

Let $B = \ker P_0 \oplus M' \xrightarrow{-d_0 \oplus \alpha}$. Since d_0 is surjective, the projection $B \to M'$ is surjective. Now consider an epimorphism $P'_0 \to B$, with P'_0 projective. Composing with the projections $B \to P_0$ and $B \to M'$ yields a diagram

$$\begin{tikzcd}
P'_0 \arrow[r, "d'_0"] \arrow[d, "\alpha_0"'] & M' \arrow[r] \arrow[d, "\alpha"'] & 0 \\
P_0 \arrow[r, "d_0"] & M \arrow[r] & 0.
\end{tikzcd}$$

The lemma now results by induction. Namely, suppose we have constructed

$$\begin{tikzcd}
& P'_i \arrow[r, "d'_i"] \arrow[d, "\alpha_i"'] & \cdots \arrow[r] & P'_0 \arrow[r] \arrow[d, "\alpha_0"'] & M' \arrow[d, "\alpha"'] \\
P_{i+1} \arrow[r] & P_i \arrow[r, "d_i"] & \cdots \arrow[r] & M \arrow[r] & 0,
\end{tikzcd}$$

then we apply the same reasoning to the diagram

$$\begin{tikzcd}
& \ker d_i \arrow[d, "\alpha"'] & \\
P_{i+1} \arrow[r] & \ker d_i \arrow[r] & 0
\end{tikzcd}$$

. Since $P_i = 0$ for $i \geq r+1$, we conclude by adding a finite (projective) resolution of $\ker d_{r+1}$. $\square$

Corollary 1.2.4. $\Phi([M, \alpha]_{fg})$ *is independent of the finite projective resolution used.*

Proof. Suppose we have to lifts of α to some automorphism of some finite resolution

$$\begin{tikzcd}
0 \arrow[r] & P_r \arrow[r] \arrow[d, "\alpha_r"'] & \cdots \arrow[r] & P_1 \arrow[r] \arrow[d, "\alpha_1"'] & M \arrow[r] \arrow[d, "\alpha"'] & 0 \\
0 \arrow[r] & P_r \arrow[r] & \cdots \arrow[r] & P_1 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & P'_r \arrow[r] \arrow[d, "\alpha'_r"'] & \cdots \arrow[r] & P'_1 \arrow[r] \arrow[d, "\alpha'_1"'] & M \arrow[r] \arrow[d, "\alpha"'] & 0 \\
0 \arrow[r] & P'_r \arrow[r] & \cdots \arrow[r] & P'_1 \arrow[r] & M \arrow[r] & 0.
\end{tikzcd}$$

By applying 1.2.2 to the diagram

$$\begin{tikzcd}
& & & & M \arrow[d, "\Delta"'] & \\
0 \arrow[r] & P_r \oplus P'_r \arrow[r, "d_r \oplus d'_r"] & \cdots \arrow[r] & P_1 \oplus P'_1 \arrow[r, "d_1 \oplus d'_1"] & M \oplus M \arrow[r] & 0,
\end{tikzcd}$$

where Δ is the diagonal map, we obtain a finite resolution

$$\ldots \to P''_1 \to M$$

and chain maps $f_\bullet : P''_\bullet \to P_\bullet$, $f'_\bullet : P''_\bullet \to P'_\bullet$ covering the identity of M, where $P_\bullet$ is the chain complex

$$\ldots \to P_2 \xrightarrow{d_2} P_1 \to 0$$

($P'_\bullet$ and $P''_\bullet$ being defined in a similar way). Note that since we have ommited the $M's$ at the end, the various chain complexes defines have zero homology except in degree one where it is isomorphic to M. Moreover, by commutativity of the diagrams

$$\begin{tikzcd} P''_1 \arrow[r] \arrow[d, "f_1"'] & M \arrow[r] \arrow[d, "Id"'] & 0 \\ P_1 \arrow[r] & M \arrow[r] & 0 \end{tikzcd} \qquad \begin{tikzcd} P''_1 \arrow[r] \arrow[d, "f'_1"'] & M \arrow[r] \arrow[d, "Id"'] & 0 \\ P'_1 \arrow[r] & M \arrow[r] & 0, \end{tikzcd}$$

f_1 and f'_1 induce isomorphism on the first homology groups. It follows that f and f' are homology equivalences, hence their mapping cones are acyclic. But an easy induction shows that for any exact sequence in **Proj** R

$$0 \to Q_n \to \ldots \to Q_1 \to Q_0 \to 0,$$

the Euler characteristic $\chi(Q_\bullet) = \sum_i (-1)^i [Q_i]_{proj}$ vanishes. Using the fact that $\chi(C_f) = \chi(P_\bullet) - \chi(P''_\bullet)$ and $\chi(C_{f'}) = \chi(P'_\bullet) - \chi(P''_\bullet)$, the result follows. □

Theorem 1.2.5 (Resolution theorem for K_1)**.** *There is an isomorphism $K_1($**Proj** $R) \xrightarrow{\iota_*} K_1(R\text{-}\boldsymbol{mod_{fg}})$ induced by the inclusion.*

Proof. For any finitely generated projective R-module P, $0 \to P \to P \to 0$ is a projective resolution, so $\Phi \circ \iota_* ([P]_{proj}) = \Phi ([P]_{fg}) = [P]_{proj}$. Moreover, an easy induction shows that for any exact sequence in R-$\mathbf{mod_{fg}}$

$$0 \to M_n \to \ldots \to M_1 \to M_0 \to 0,$$

the Euler characteristic $\chi(M_\bullet) = \sum_i (-1)^i [M_i]_{fg}$ vanishes. Hence for any finitely generated R-module and any projective resolution

$$0 \to P_n \to \ldots \to P_1 \to M \to 0,$$

we have

$$\iota_* \circ \Phi ([M]_{fg}) = \iota_* \left(\sum_{i \geq 1} (-1)^{i+1} [P_i]_{proj} \right) = \sum_{i \geq 1} (-1)^{i+1} [P_i]_{fg} = [M]_{fg}.$$

The result follows. □

1.3 The Grothendieck's Theorem

In this section we prove that every finitely generated projective R-module is stably trivial, where $R = \mathbb{Z}[\mathbb{Z}^n]$. This amounts to proving that the inclusion $\mathbb{Z} \hookrightarrow R$ induces an isomorphism $K_0(\mathbb{Z}) \xrightarrow{\approx} K_0(R)$. By induction, it suffices to prove that the inclusion induces an isomorphism $K_0(R) \xrightarrow{\approx} K_0(R[t, t^{-1}])$ for any regular ring R.

From now on, R will be a regular ring. Give R the trivial grading, and give $R[t]$ the canonical grading given by the degree.

Definition 1.3.1. For any graded $R[t]$-module M, define $\varphi_i(M) \subset M_i$ by $M_i = \sum_{j \geq 1} Rt^j \ M_{i-j}$, and define $\varphi_i(M) = M_i/D_i(M)$. Denote by $\varphi(M)$ the graded R-module $\sum_i \varphi_i(M)$.

Lemma 1.3.2. *If M is a graded module bounded below such that $\varphi(M) = 0$, then $M = 0$.*

Proof. Let M be a non zero graded module bounded below, and let n denote the lowest integer such that $M_n \neq 0$, then by definition $\varphi_n(M) = 0$, and $\varphi_n(M) = M_n$. $\square$

It is not hard to see that φ is additive and sends free modules into free modules. Hence it sends projective modules into projective modules. Actually, we have the following

Proposition 1.3.3. *The functor $Q \rightsquigarrow R[t] \otimes_R Q$ establishes a bijection between isomorphism classes of graded projective R-modules which are bounded below and isomorphism classes of projective $R[t]$-modules which are bounded below, whose inverse is given by the functor φ.*

Proof. It is obvious that both functors preserve boundedness condition. Since we have clearly $\varphi(R[t] \otimes_R Q) \simeq Q$ for every projective R-module which is bounded below, it suffices to show that $R[t] \otimes_R \varphi(P) \simeq P$ for every projective $R[t]$-modules which is bounded below. Now consider the quotient map $f : P \to \varphi(P)$ given by the definition of $\varphi(P)$, which can be seen as an epimorphism of graded R-modules. Since $\varphi(P)$ is projective, there exists a right inverse $g : \varphi(P) \to P$, which yields a map of graded $R[t]$-modules $h : R[t] \otimes_R \varphi(P) \to P$. Now clearly $\varphi(h) : \varphi(R[t] \otimes_R \varphi(P)) \xrightarrow{\approx} \varphi(P)$. Hence, since one can easily check that φ is right exact, we have $\varphi(\text{coker } h) = 0$. But since coker h is bounded below, it follows from 1.3.2 that coker h=0, hence h is isomorphic. Now since P is projective, h splits and, by additivity of φ, we have $\varphi(\ker h) \simeq \ker(\varphi(R[t] \otimes_R \varphi(P)) \to \varphi(P)) = 0$. Once again, this implies that $\ker h = 0$, and h is injective. $\square$

Corollary 1.3.4. *The functor $Q \rightsquigarrow R[t,s] \otimes_R Q$ establishes a bijection between isomorphism classes of graded projective R-modules which are bounded below and isomorphism classes of projective $R[t,s]$-modules which are bounded below*

We are now going to construct a left inverse to $K_0(R) \to K_0(R[t])$. In order to see that the functor $Q \rightsquigarrow R \otimes_{R[t]} Q$ yields a map $K_0(R[t]) \to K_0(R)$, we need the following

Proposition 1.3.5. *$R \otimes_{R[t]} -$ is exact on the category of graded $R[t]$-modules.*

Proof. Since the tensor product functor is always right exact, it remains to prove the left exactness. Note that $R \otimes_{R[t]} M$ may also be written as $M/(t-1)M$, so this amounts to proving that for any graded $R[t]$-modules M and any graded submodule M', then $(1-t)M \cap M' = (1-t)M'$. But this follows by an easy induction, since if an element $x = x_0 + x_1 + \ldots \in M$ is such that $(1-t)x = x_0 + (x_1 - tx_0) + \ldots + (x_n - t_{xn-1}) + \ldots \in M'$, then every x_n is actually in M'. $\square$

Theorem 1.3.6 (Grothendieck)**.** *The natural map $K_0(R) \to K_0(R[t])$ is an isomorphism.*

Proof. It is easy to see that the map $K_0(R[t]) \to K_0(R)$ defined above is a left inverse to the map we are considering. Hence it suffices to prove the surjectivity. Let P be a projective $R[t]$-module. We are going to prove that it is of the form $R[t] \otimes_{R[t,s]} N$ for some graded $R[t,s]$-module. To see that, first observe thet $P = R[t]^n/Q$ for some $n \geq 0$ and some module of relations $Q \subset R[t]^n$. Since M is finitely generated and R is noetherian, it follows from the Hilbert Basis Theorem that Q is finitely generated. Now choose a finite set of generators of Q

$$f_i = (f_{j,1}(t), \ldots, f_{j,n}(t)), \quad 1 \leq i \leq m,$$

and define

$$g_i = (g_{j,1}(t,s), \ldots, g_{j,n}(t,s)), \quad 1 \leq i \leq m,$$

by replacing every monomial at^k by $at^k s^{d-k}$, where d is the highest degree of the $f_{i,j}$'s. It is now clear that every $g_{i,j}$ is homogeneous of degree d. Furthermore, if we denote by Q' the submodule of $R[t,s]$ generated by the $g_{i,j}$'s and $N = R[t,s]/Q'$, then $R[t] \otimes_{[t,s]} N \simeq P$.

Since R is regular, $R[t,s]$ is regular by the Syzygy theorem, hence we can choose a finite resolution

$$0 \to P_m \to \ldots \to P_1 \to N \to 0.$$

Using 1.3.5, the following sequence

$$0 \to R[t] \otimes_{R[t,s]} P_m \to \ldots \to R[t] \otimes_{R[t,s]} P_1 \to R[t] \otimes_{R[t,s]} N \simeq P \to 0$$

is exact. Now by 1.3.4, each P_i is (disregarding the grading) a direct sum of modules of the form $R[t,s] \otimes_R Q$, with Q a projective R-module. Since $R[t] \otimes_{R[t,s]} (R[t,s] \otimes_R Q) \simeq R[t] \otimes_R Q$, it follows that every $R[t] \otimes_{R[t,s]} P_i$ is a direct sum of modules of the form $R[t] \otimes Q$. Thus the result follows from the following lemma, which is easily proved by induction.

Lemma 1.3.7. *If*

$$0 \to P'_m \to \ldots P'_0 \to 0$$

is an exact sequence of finitely generated projective R-module, then

$$\sum_i (-1)^i \, [P'_i]_{proj} = 0 \in K_0(R).$$

$\square$

Corollary 1.3.8. *The natural map $K_0(R) \to K_0(R[t,t^{-1}])$ is an isomorphism for every regualr ring R.*

Proof. First note that the map $R[t,t^{-1}] \to R$ sending t to 1 yields a left inverse, by the same reasoning as above. Now since the map $K_0(R) \to K_0(R[t,t^{-1}])$ factors through $K_0(R[t])$ by means of the ring homomorphisms $R \to R[t] \to R[t,t^{-1}]$, it suffice to show that $K_0(R[t]) \to K_0(R[t,t^{-1}])$ is surjective, by **??**. Let P be a finitely generated projective $R[t,t^{-1}]$-module. Then $P = R[t,t^{-1}]^n/Q$ for some module of relations $Q \subset R[t,t^{-1}]^n$. Since P is finitely generated and R is noetherian, it follows from the Hilbert Basis Theorem that Q is finitely generated. Thus we can choose d large enough so that $t^d Q \subset R[t]^n$. Hence

$$P \simeq t^d R[t,t^{-1}]^n / \left(t^d Q\right) \simeq R[t,t^{-1}] \otimes_{R[t]} M$$

for some finitely generated $R[t]$-module M. Using the Syzygy theorem, choose a finite resolution

$$0 \to P_m \to \ldots \to P_1 \to M \to 0$$

by finitely generated projective $R[t]$-modules. Since $R[t,t^{-1}]$ is a flat over $R[t]$, the following sequence is exact

$$0 \to R[t,t^{-1}] \otimes_{R[t]} P_m \to \ldots \to R[t,t^{-1}] \otimes_{R[t]} P_1 \to R[t,t^{-1}] \otimes_{R[t]} M \simeq P \to 0.$$

Thus the result follows from 1.3.7 applied to $R[t,t^{-1}]$. □

Corollary 1.3.9. *The natural map $K_0(\mathbb{Z}) \to K_0(\mathbb{Z}[\mathbb{Z}^n])$ is an isomorphism for all $n \geq 0$.*

1.4 The vanishing of $Wh(\mathbb{Z}^n)$.

In this section, we prove a particular case of the Bass-Heller-Swan theorem, which will appear to be of fundamental importance to study high-dimensional manifolds. The links between this vanishing theorem and manifold topology will be studied in the next section.

The notation $[\ \ ]_{Wh}$ will denote an element in the Whitehead group.

Theorem 1.4.1 (Bass-Heller-Swan)**.** *The Whitehead group $Wh(\mathbb{Z}^n)$ vanishes for all $n \geq 0$.*

We prove the result by induction on n. The vanishing of $Wh(e)$ is a basic fact of linear algebra. Suppose the result has been prove for $n \geq 0$.

Given a class $[P,\alpha]_{Wh} \in Wh(\mathbb{Z}^{n+1})$, there exists a (projective) finitely generated $R[t,t^{-1}]$-module Q such that $P \oplus Q \simeq R[t,t^{-1}]^N$ for some $N \geq 0$. By definition of Wh, we then have

$$[P,\alpha]_{Wh} = [P \oplus Q, \alpha \oplus Id_Q]_{Wh} = [R[t,t^{-1}]^N, \beta]_{Wh}$$

for some automorphism β of the $R[t,t^{-1}]$-module $R[t,t^{-1}]^N$. It is thus enough to prove that $[R[t,t^{-1}]^N, \beta] = 0 \in Wh(\mathbb{Z}^{n+1})$ for any $\beta \in \text{Aut } R[t,t^{-1}]^N$, $N \geq 0$. Since an automorphism of $R[t,t^{-1}]^N$ can be seen as an element of $GL_N(R[t,t^{-1}])$, we start by expressing any matrix in $GL_N(R[t,t^{-1}])$ in a more tractable way.

Proposition 1.4.2. *Any matrix $B \in GL(R[t,t^{-1}])$ can be reduced, modulo $GL(R)$ and $E(R[t,t^{-1}])$, to a matrix of the form*

$$\begin{pmatrix} t^m & 0 \\ 0 & 1 \end{pmatrix} (1 + A(t-1)),$$

where $m \in \mathbb{Z}$, and $A \in M(R)$ with $A(1-A)$ nilpotent.

Proof. Let $m \geq 0$ such that $t^m B$ has entries in $R[t]$, and write

$$t^m B = B_0 + tB_1 + \ldots + t^d B_d,$$

where the B_i have entries in R, and $d \geq 0$. We prove by induction that, modulo $GL(R)$ and $E(R[t])$, we can reduced our study to the case $d \leq 1$. Assume $d > 1$. Then, by writing $M \approx N$ if two matrices $M, N \in M(R[t, t^{-1}])$ are equivalent modulo $GL(R)$ and $E(R[t])$, we have

$$\begin{aligned} t^m B &\approx \begin{pmatrix} t^m B & 0 \\ 0 & 1 \end{pmatrix} \\ &\approx \begin{pmatrix} t^m B & t^{d-1} B_d \\ 0 & 1 \end{pmatrix} \\ &\approx \begin{pmatrix} t^m B - t^d B_d & t^{d-1} B_d \\ -t & 1 \end{pmatrix}, \end{aligned}$$

and the last matrix has entries of degree $\leq d-1$, so we conclude by induction. Thus we have only to deal with the case $t^m B \approx B_0 + tB_1 = (B_0 + B_1) + (t-1)B_1$. Since $t^m B$ must be invertible as a matrix over $R[t, t^{-1}]$, $B_0 + N_1$ is invertible. So factoring out by $B_0 + B_1 \in GL(R)$, we have a matrix of the form $1 + A(t-1) = (1-A) + tA$. Let $C_{-r}, \ldots, C_s \in M(R)$ with $s, r > 0$, such that

$$1 = ((1-A) + tA)(t^{-r} C_{-r} + \ldots + t^s C_s) = (C_0 + tC_1 + \ldots + t^s C_s)((1-A) + tA).$$

It follows that $(1-A)C_i + AC_{i-1} = 0$ for $i \neq 0$. Starting with $AC_s = 0$, we prove by induction that $A^i C_{s-i+1} = 0$ for $1 \leq i \leq s+1$, so in particular $A^{s+1} C_0 = 0$. Similarly, starting with $(1-A)C_{-r} = 0$, we prove by induction that $(1-A)^i C_{-r+i-1} = 0$ for $1 \leq i \leq r$, so in particular $(1-A)^r C_{-1} = 0$. Multiplying the equation $(1-A)C_0 + AC_{-1} = 1$ by $A^s (1-A)^{r-1}$ yields

$$A^s (1-A)^{r-1} = (1-A)^{r-1} A^{s+1} C_0 + A^s (1-A)^r C_{-1} = 0,$$

and the result follows. □

Using 1.4.2, we thus have

$$\begin{aligned} [R[t, t^{-1}]^N, B]_{Wh} = [R[t, t^{-1}]^k, (1-A) + tA]_{Wh} + [R[t, t^{-1}]^k, S]_{Wh} \\ + [R[t, t^{-1}]^k, U]_{Wh} \in Wh(\mathbb{Z}^{n+1}), \end{aligned}$$

with $A(1-A)$ nilpotent, $S \in GL_m(R)$, and $U \in E(R[t])$.
By induction hypothesis, we have $[R[t, t^{-1}]^m, S]_{Wh} = 0$ since $Wh(\mathbb{Z}^n) = 0$ and $S \in GL_k(\mathbb{Z}[\mathbb{Z}^n])$.
Since $E(R[t])$ is generated by unipotent matrices, it is enough to prove the following

Lemma 1.4.3. *Let P be a projective $R[t,t^{-1}]$-module, and α an unipotent automorphism of P. Then $[P,\alpha]_{proj} = 0 \in K_1(R[t,t^{-1}])$.*

Proof. By the Resolution theorem 1.2.5, it is enough to prove the result in $R[t,t^{-1}]$-$\mathbf{mod_{fg}}$. Suppose $\alpha^s = 0$, and let $M_i = \text{Im } (\alpha-1)^{s-i}$. Each M_i is stable under α, and α induces identity on the quotient M_{i+1}/M_i, yielding the following commutative diagram in $R[t,t^{-1}]$-$\mathbf{mod_{fg}}$

$$\begin{tikzcd}
0 \arrow[r] & M_i \arrow[r] \arrow[d, "\alpha_|"'] & M_{i+1} \arrow[r] \arrow[d, "\alpha_|"'] & M_{i+1}/M_i \arrow[r] \arrow[d, "Id"] & 0 \\
0 \arrow[r] & M_i \arrow[r] & M_{i+1} \arrow[r] & M_{i+1}/M_i \arrow[r] & 0.
\end{tikzcd}$$

So, by definition of K_1, $[M_{i+1},\alpha_|]_{fg} = -[M_i,\alpha_|]_{fg}$, and the result follows by induction. $\square$

To complete the proof of 1.4.1 me must show that $[R[t,t^{-1}]^k, 1+(t-1)A]_{Wh} = 0$, with $A \in M(R)$ such that $A(1-A)$ is nilpotent, say $A^s(1-A)^s = 0$. We have $R[t,t^{-1}]^k = M_0 \oplus M_1$, with $M_0 = \ker A^s$, $M_1 = Ker(1-A)^s$, and A stabilizes both submodules. Let A_0 (resp A_1) its restriction to M_0 (resp. M_1).

$$\begin{aligned}
\left[R[t,t^{-1}]^k, 1+(t-1)A\right]_{Wh} &= [M_0, Id_{M_0} + (t-1)A_0]_{Wh} \\
&\quad + [M_1, tId_{M_1} + (Id_{M_1} - tA_1)]_{Wh} \\
\\
&= [M_0, Id_{M_0} + (t-1)A_0]_{Wh} + [M_1,]_{Wh} \\
&\quad + \left[Id_{M_1} + t^{-1}A_1^{-1}(Id_{M_1} - tA_1)\right]_{Wh} \\
\\
&= [M_1, tId_{M_1}]_{Wh} + [M_0, Id_{M_0} + (t-1)A_0]_{Wh} \\
&\quad + [M_1, A_1]_{Wh} + \left[Id_{M_1} + t^{-1}A_1^{-1}(Id_{M_1} - tA_1)\right]_{Wh}
\end{aligned}$$

The last three terms vanish by 1.4.3, since the automorphisms involved are unipotent. If M_1 was free, $[M_1, tId_{M_1}]_{Wh}$ would vanish by definition of the Whitehead group. Here, M_1 is only projective, but the Grothendieck's theorem will allow us to

Lemma 1.4.4. $[M_1, tId_{M_1}]_{Wh} = 0$.

Proof. By the Grothendieck's theorem, let $s \in \mathbb{Z}$ such that $[M_1]_{proj} = [R[t,t^{-1}]^s]_{proj}$. By definition of $K_0(\mathbf{Proj}\ R[t,t^{-1}])$, this implies that M_1 is isomorphic to a projective module Q satisfying

$$Q = R[t,t^{-1}]^s + \sum_i (P^{(i)} - P_1^{(i)} - P_2^{(i)}) \in \Lambda_0(\mathbf{Proj}\ R[t,t^{-1}]),$$

where the $P^{(i)}, P_1^{(i)}, P_2^{(i)}$ are projective modules satisfying an exact sequence

$$0 \to P_1^{(i)} \to P^{(i)} \to P_2^{(i)} \to 0.$$

Now one can deduce the following commutative diagrams

$$\begin{tikzcd}
0 \arrow[r] & P_1^{(i)} \arrow[r] \arrow[d, "tId"'] & P^{(i)} \arrow[r] \arrow[d, "tId"'] & P_2^{(i)} \arrow[r] \arrow[d, "tId"'] & 0 \\
0 \arrow[r] & P_1^{(i)} \arrow[r] & P^{(i)} \arrow[r] & P_2^{(i)} \arrow[r] & 0.
\end{tikzcd}$$

So,

$$(Q, tId) = \big(R[t, t^{-1}]^s, tId\big) + \sum_i \Big((P^{(i)}, tId) - (P_1^{(i)}, tId) - (P_2^{(i)}, tId)\Big) \in \Lambda_1(\mathbf{Proj}\, R[t, t^{-1}]),$$

and hence $[M_1, tId]_{Wh} = [R[t, t^{-1}]^s, tId]_{Wh} = 0.$ $\square$

Chapter 2

From Algebra to Topology: Splitting obstructions

In this chapter, we study a geometric phenomenon which will be of crucial importance in calculating the PL-structure set of the torus: codimension one splitting. Let $f : M \to M'$ be a homotopy equivalence, and N' a two-sided codimension one submanifold of M'. By making f transverse to N', it induces a degree one map $g : N \to N'$, with $N = f^{-1}(N')$. The problem is to decide if we can homotop f to make g a homotopy equivalence. The idea will be to make g highly connected by performing succesive surgeries on N. Note that, instead of classical surgery, everything is done inside M, so we will need different assumptions to make sure we can perform surgery on a class.

In the first section, we prove that this program can be carried out until middle dimension, where a obstruction to perform surgery will appear. In Section 2.2, we prove that the vanishing of this obstruction gives algebraic moves, whose geometric counterparts will be developed in Section 2.3 to achieve the surgery program, yielding the following

Splitting Theorem.
Let $f : M \to \mathbb{T}^n$ be a homotopy equivalence between PL-manifolds of dimension $n \geq 6$, and N' a two-sided codimension submanifold of $\mathbb{T}^n$. Then we can homotop f such that:
- f is transverse to N'.
- the restriction $f_| : N \to N'$ is a homotopy equivalence, where $N = f^{-1}(N')$.

Finally, we relate the algebraic obstruction to the algebraic machinery developed in the previous chapter in Section 2.4. This last section is not necessary for the rest of this thesis, but presents the link between the Whitehead group of a group and the Nil group of its groups ring given by the general Bass-Heller-Swan theorem.

Notation In this chapter, R will denote the ring $\mathbb{Z}[\mathbb{Z}^n]$.

2.1 Surgery below the middle dimension.

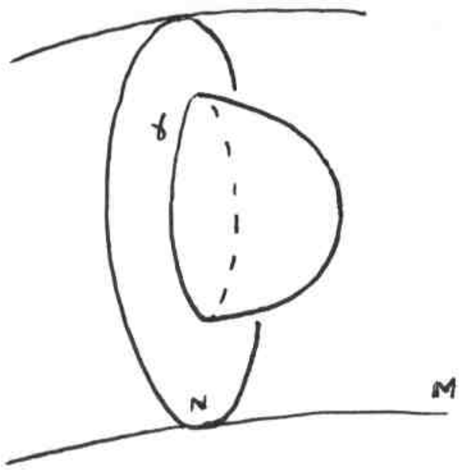
Let M be a PL-manifold of dimension $n \geq 6$, $f : M \to \mathbb{T}^n$ a homotopy equivalence, and N' a two-sided codimension one subtorus of $M' = \mathbb{T}^n$. By first making f transverse regular to N', we can assume it induces a degree one map $g : N \to N'$, with $N = f^{-1}(N')$. We will try to perform sugery on N to make g highly connected.

Lemma 2.1.1. *One can homotop f so that N is connected and $g : N \to N'$ induces an isomorphism on π_1.*

Proof. By performing surgery on a path between two connect components of N, we can first assume that N is connected. Now $g : N \to N'$ is a degree one map between connected compact manifolds, thus induces an epimorphism $g_* : \pi_1(N) \to \pi_1(N')$. Indeed, consider the covering map $Z \to N'$ associated to $g_*\pi_1(N)$. Then by definition g lifts to $\tilde{g} : N \to Z$, and $g_* : H_{n-1}(N) \to H_{n-1}(N')$ factors through $H_{n-1}(N) \xrightarrow{\tilde{g}_*} H_{n-1}(Z) \to H_{n-1}(N')$. Since g is a degree one map, Z is compact (otherwise $H_{n-1}(Z) = 0$), so $g_*\pi_1(N)$ is a subgroup of $\pi_1(N')$ of finite index d, and $H_{n-1}(Z) \to H_{n-1}(N')$ is multiplication by d. Hence $d = 1$ and $g_*\pi_1(N) = \pi_1(N')$. Now consider a loop γ representing an element of $\ker g_*$. Then, since $f : M \to M'$, the commutativity of the following diagram

$$\begin{tikzcd} \pi_1(N) \arrow[r] \arrow[d, "g_*"'] & \pi_1(M') \arrow[d, "\approx"', "f_*"] \\ \pi_1(N') \arrow[r] & \pi_1(M) \end{tikzcd}$$

shows that γ is nullhomotopic in M. Since $n \geq 6$, we can, by a general position argument, assume that γ bounds an embedding $(\mathbb{D}^2, \mathbb{S}^1) \hookrightarrow (M, N)$.



The normal bundle of the embedded (contractible) $\mathbb{D}^2$ is trivial, hence it has a tubular neighboorhood of the form $\mathbb{D}^2 \times \mathbb{D}^{n-2}$. By homotoping f we can assume that f maps this tube in N. Consider the codimension one submanifold N_1 obtained by ambient surgery on γ (namely, $N_1 = N - (\mathbb{S}_1 \times \mathbb{D}^{n-2}) \cup \mathbb{D}^2 \times \mathbb{S}^{n-3}$). f induces a map $g_1 : N_1 \to N'$ and $\ker g_1 \simeq \ker g_* / < [\gamma] >$. Furthermore, by considering a sufficiently small tube $\mathbb{D}^2 \times \mathbb{D}^{n-2}$, we can suppose that the image of the tube is not dense in N'. But since the degree is a local data, this implies that g_1 has degree one. □

We now assume that N is connected and $N \to N'$ induces an isomorphism on π_1. It is not possible to have the same reasoning in higher dimensions since a degree one map does not necessarily induce an epimorphism on $\pi_i, i \geq 2$. However, we have the following

Lemma 2.1.2. *A degree one normal map between compact connected orientable manifolds induces an epimorphism on $H_i, i \geq 1$.*

Proof. Let $h : X \to Y$ be a degree one normal map between compact connected orientable m-manifolds. We have the following commutative diagram

$$\begin{tikzcd} H_i(X) \arrow[r, "f_*"'] \arrow[d, "D"'] & H_i(Y) \arrow[d, "D"'] \\ H^{m-i}(X) & H^{m-i}(Y) \arrow[l, "f^*"'] \end{tikzcd}$$

where D is the inverse of the Poincaré duality isomorphism. Set $f_\sharp = D^{-1} \circ f^* \circ D$. Then, for all $y \in H_i(Y)$,

$$f_* \circ f_\sharp(y) = f_*(f^*(Dy) \cap [X]) = Dy \cap f_*[X] = Dy \cap [Y] = y.$$

Thus $f_* \circ f_\sharp = id$, hence f_* is surjective. $\square$

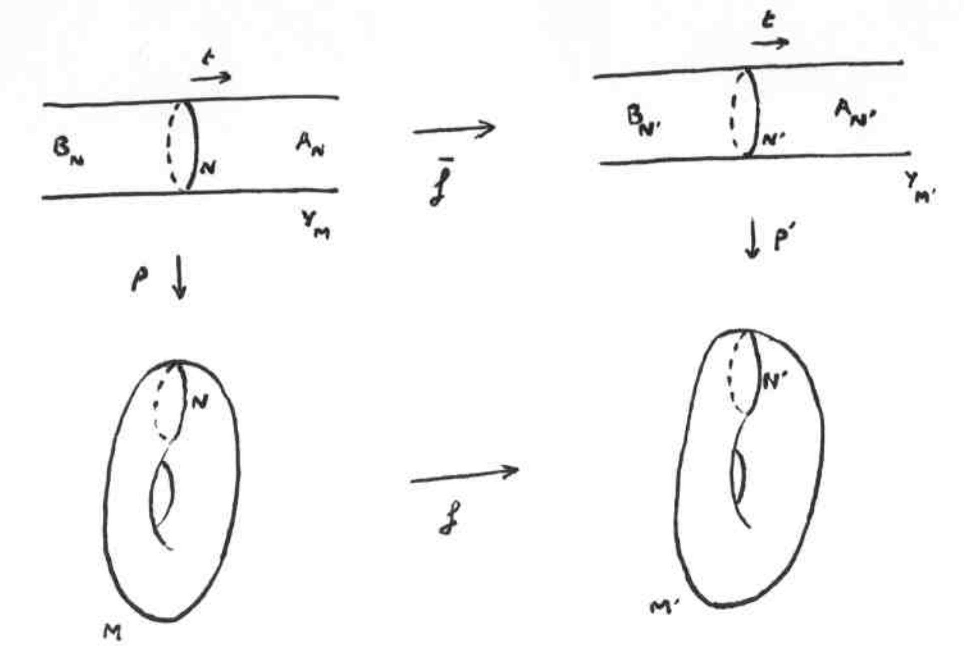
By virtue of the Hurewicz theorem (since f induces an isomorphism on π_1), it is sufficient to prove that we can homotop f so that g induces a homology isomorphism. By Poincaré duality, it is sufficient to prove that it induces a homology equivalence until the middle dimension. Now, if N_1 is obtained from N by surgery, there is no natural map $N \to N_1$. It will be more convenient to consider the following infinite cycling covering.

Definition 2.1.3. Let $p' : Y_{M'} \to M'$ be the infinite cyclic covering associated to the inclusion $\mathbb{Z}^{n-1} = \pi_1(N') \hookrightarrow \pi_1(M') = \mathbb{Z}^n$, and $p : Y_M \to M$ the pullback over f

$$\begin{tikzcd} Y_M \arrow[r, "\overline{f}"] \arrow[d, "p"'] & Y_M \arrow[d, "p'"] \\ M \arrow[r, "f"'] & M'. \end{tikzcd}$$

.

By definition of p', N' lifts to a two-sided codimension one submanifold that we still denote N'. N' divides $Y_{M'}$ into $A_{N'}$ and $B_{N'}$. The group of covering transformations is isomorphic to $\mathbb{Z}$, and we choose a generator t such that $tA_{N'} \subset A_{N'}$.Furthermore, since $\pi_1(N) \to \pi_1(N')$ is an isomorphism, N lifts to a two-sided codimension one submanifold that will still denote by N. We can choose a lift such that $\overline{f}(N) \subset N'$. N divides Y_N into A_N and B_N, and considering the generator t of the infinite cyclic group of covering transformations of p induced by the isomorphism $\pi_1(N) \to \pi_1(N')$, we have $\overline{f}(A_N) \subset A_{N'}$, $\overline{f}(B_N) \subset B_{N'}$ and $tA_N \subset A_N$.



Definition 2.1.4. For any map $h : X \to Y$, define $K_i(h) = \ker H_i(X) \to H_i(Y)$, $K^i(h) = \ker H^i(Y) \to H^i(X)$ for $i \geq 0$ (where the coefficient group is to be specified). If no confusion is possible, we will just denote it by $K_i(X)$ or $K^i(X)$.

We have the following

Lemma 2.1.5. *We have, for $i \geq 1$,*

$$K_i(N) \simeq K_{i-1}(A_N, N) \oplus K_{i-1}(B_N, N)$$

where we consider homology with coefficient in a R-module.

Proof. We have the Mayer-Vietoris exact sequence

$$\to K_{i+1}(Y_M) \to K_i(N) \to K_i(A_N) \oplus K_i(B_N) \to K_i(Y_M) \to$$

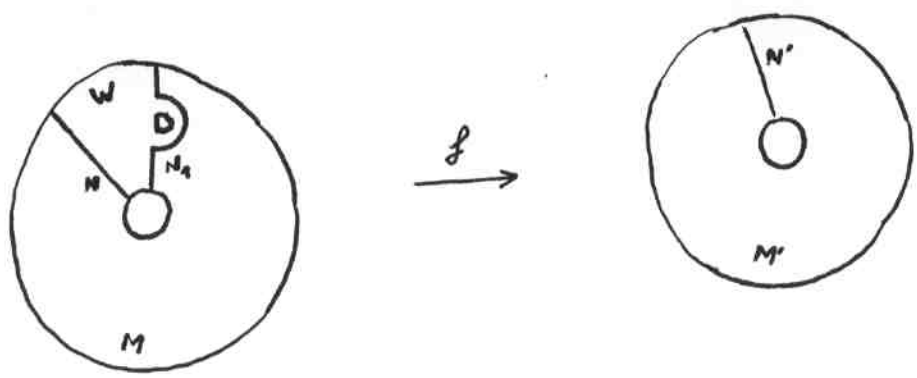
and the exact sequences of pairs

$$\to K_i(Y_M) \to K_i(Y_M, A_N) \to K_{i-1}(A_N) \to K_{i-1}(Y_M) \to$$

$$\to K_i(Y_M) \to K_i(Y_M, B_N) \to K_{i-1}(B_N) \to K_i(Y_M) \to$$

and by excision, $K_i(Y_M, A_N) \simeq K_i(B_N, N)$, $K_i(Y_M, B_N) \simeq K_i(A_N, N)$. Now since f is a homotopy equivalence, $K_i(Y_M) = 0$ for all $i \geq 0$, and the result follows easily. $\square$

We now describe how an ambient surgery (analog to the one done in 2.1.1) will be performed. Suppose we have an embedding $(\mathbb{D}^k, \mathbb{S}^{k-1}) \hookrightarrow (N, \overline{A_N - tA_N})$ ($\overline{A_N - tA_N}$ is a fundamental domain of the infinite cyclic covering). Then the normal bundle of the embedded $\mathbb{D}^k$ is trivial, thus admits a tubular neighboorhood of the form $\mathbb{D}^k \times \mathbb{D}^{n-k}$. By homotoping f we can suppose that f maps this tube in N, and by choosing a sufficiently small tubular neighboorhood we can assume that the image of this tube is not dense in N. Consider the codimension one submanifold N_1 obtained by ambient surgery on this tube (namely, $N_1 = N - (\mathbb{S}_{k-1} \times \mathbb{D}^{n-k}) \cup \mathbb{D}^k \times \mathbb{S}^{n-k-1}$). This submanifold projects to a submanifold N_1 in M. By considering a tubular neighboorhood around N, we can assume that there is a cobordism W between N and N_1, as described in the following picture



and f induces a map $g_1 : N_1 \to N'$, which is of degree one by the same reasoning.

This is the procedure we will use to kill judicious elements in $K_i(A_N, N)$ and $K_i(B_N, N)$. However there is apparently no reason why a homology class should be represented by an embedding. We first prove a lemma which will enable us to eliminate some homology classes, allowing an induction argument. First we need the following lemmas.

Lemma 2.1.6. *If g is k-connected, then $K_k(N), K_{k+1}(A_N, N)$ and $K_{k+1}(B_N, N)$ are all finitely generated R-modules.*

Proof. By 2.1.5, it suffices to prove the result for $K_k(N)$. Since $H_k(N) \to H_k(N')$ is surjective by 2.1.2, we have $K_i(N) \simeq H_{i+1}(C_g)$ for all $i \geq 0$, where C_g is the mapping cone of g. Since g is k-connected, $\pi_i(C_g) = 0 = H_i(C_g)$ for all $i \leq k$, hence $H_{k+1}(C_g)$ is a finitely generated R-module. $\square$

Recall that t denote a covering map generating the covering transformations group of the covering. Consider the inclusion $i : (Y_M, B_N) \to (Y_M, tB_N)$. We have the endomorphism (of R-modules) $(t^{-1})_* : H_i(A_N, N) \to H_i(A_N, N)$ defined by the following commutative diagram:

$$\begin{tikzcd} H_i(Y_M, B_N) \arrow[r, "i_*"] \arrow[d, "\approx"', "\text{excision}"] & H_i(Y_M, tB_N,) \arrow[r, "(t^{-1})_*"] & H_i(Y_M, B_N) \arrow[d, "\approx"', "\text{excision}"] \\ H_i(A_N, N) \arrow[rr, "(t^{-1})_*"] & & H_i(A_N, N), \end{tikzcd}$$

and similarly we can construct an endomorphism $t_* : H_i(B_N, N) \to H_i(B_N, N)$. These endomorphisms induce endomorphisms on $K_i(A_N, N)$ and $(K_i(B_N, N)$ respectively.

Lemma 2.1.7. *If g is k-connected, then $(t^{-1})_*$ (resp. t_*) is nilpotent on $K_{k+1}(A_N, N)$ (resp. $K_{k+1}(B_N, N)$).*

Proof. We prove the lemma for $(t^{-1})_*$, the proof being analog for t_*. Since both modules are finitely generated by 2.1.6, it is enough to prove that any element x is killed by a sufficiently high power of $(t^{-1})_*$. Let c be a cycle representing x. Since c has compact support, there exists a l such that the support of c lies in $\overline{A_N - t^l A_N}$. But then x is killed under the map $H_i(Y_M, B_N) \to H_i(Y_M, t^l B_N)$ induced by inclusion, and so $(t^{-1})^l_* x = 0$. $\square$

The following lemma exhibits homology classes on which surgery will be possible. For a proof, we refer to [FH73].

Lemma 2.1.8. *Suppose g is k-connected, and let $l \geq 1$ be the nilpotence index of $(t^{-1})_*$ on $K_{k+1}(A_N, N)$. Then the image of the composite map*

$$\pi_{k+1}(\overline{A_N - tA_N}, N) \xrightarrow{H} H_{k+1}(\overline{A_N - tA_N}, N) \xrightarrow{j_*} H_{k+1}(A_N, N),$$

where H is the Hurewicz homomorphism and j_* is induced by inclusion, contains $(t^{-1})^{l-1}_*$.

We are now able to prove the main result of this section.

Proposition 2.1.9. *We can homotop f so that there exists a codimension one submanifold $N \subset M$, with $g = f_{|N} : N \to N'$ which is k-connected for every $k < n/2$.*

Proof. We keep the same notations as above. Using 2.1.1, we can suppose that there exists a codimension one connected submanifold $N \subset M$ such that g is a π_1-isomorphism. We prove the proposition by induction on k. Since we just proved it for $k = 0, 1$, let assume g is k-connected for $k < n/2$. If $k + 1 > n/2$, the result follows by Poincaré duality. Otherwise $K_{k+1}(A_N, N)$ is a finitely generated R-module endowed with a nilpotent R-endomorphism $(t^{-1})_*$, whose nilpotence is denoted l. Since $(t^{-1})_* K_{k+1}(A_N, N)$ is also finitely generated, let $x_1, \ldots, x_s$ be a set of generators. By 2.1.8 there are elements in $\pi_{k+1}(\overline{A_N - tA_N}, N)$ whose images under the composite map are $x_1, \ldots, x_s$. Since $k + 1 < n/2$, we can represent them by disjoint embeddings $(\mathbb{D}^{k+1}, \mathbb{S}^k) \hookrightarrow (\overline{A_N - tA_N}, N)$. Using the procedure described above, we perform surgery on these embeddings to obtain a map homotopic to f (for simplicity, we still denote it f) and a codimension one submanifold $N_1 \subset N$. Let A_{N_1}, B_{N_1} be the corresponding sets for N_1. We have the following commutative diagram

$$\begin{tikzcd}
K_{k+1}(A_N, N) \arrow[r] \arrow[d, "\text{excision}"', "\approx"] & K_{k+1}(A_{N_1}, N_1) \arrow[d, "\text{excision}"', "\approx"] & \\
K_{k+1}(Y_M, B_N) \arrow[r] & K_{k+1}(Y_M, B_{N_1}) \arrow[r] & K_k(B_{N_1}, B_N).
\end{tikzcd}$$

Now by excision, $K_k(B_{N_1}, B_N) \simeq K_k(W, N)$, where W is the cobordism between N and N_1 described above. But since W is obtaind from N by adding $k + 1$ handles, $K_k(W, N) = 0$, and so $K_{k+1}(A_N, N) \to K_{k+1}(A_{N_1}, N_1)$ is an epimorphism. But by construction, the kernel of $K_{k+1}(Y_M, B_N) \to K_{k+1}(Y_M, B_{N_1})$ contains $(t^{-1})_*^{l-1}(K_{k+1}(A_N, N))$. Hence using the commutativity of the following diagram

$$\begin{tikzcd}
K_{k+1}(A_N, N) \arrow[r] \arrow[d, "(t^{-1})_*"'] & K_{k+1}(A_{N_1}, N_1) \arrow[r] \arrow[d, "(t^{-1})_*"'] & 0 \\
K_{k+1}(A_N, N) \arrow[r] & K_{k+1}(A_{N_1}, N_1) \arrow[r] & 0,
\end{tikzcd}$$

it follows that $(t^{-1})_* K_{k+1}(A_{N_1}, N_1) = 0$. We can repeat the argument. Thus after finitely many times, $K_{k+1}(A_N, N)$ can be killed. Note that during that procedure, $K_{k+1}(B_N, N)$ has not been affected. Indeed, since W is obtained from N_1 by adding $n - k - 1$ handles and $n - k - 1 > n/2$ since $k + 1 < n/2$, it follows that $K_{k+1}(Y_M, A_{N_1}) \to K_{k+1}(Y_M, A_N)$ is an isomorphism, hence $K_{k+1}(B_N, N) \simeq K_{k+1}(B_{N_1}, N_1)$. So we can apply a similar program to kill $K_{k+1}(B_N, N)$. This completes the induction step. $\square$

2.2 The algebraic obstruction.

Lemma 2.2.1. *(i) Suppose $n = 2k$ is even. Then under the above assumptions, we can homotop f so there exists a codimension one submanifold N such that the induced map $g : N \to N'$ satisfies: $K_i(A_N, N) = 0, K_i(B_N, N) = 0$ for $i < k$, $K_k(A_N, N) = 0$ and $K_k(B_N, N)$ is finitely generated.*

(ii) Suppose $n = 2k + 1$ is odd. Then under the above assumptions, we can homotop f so there exists a codimension one submanifold N such that the induced map $g : N \to N'$ satisfies: $K_i(A_N, N) = 0, K_i(B_N, N) = 0$ for $i \leq k$, $K_{k+1}(A_N, N)$ and $K_{k+1}(B_N, N)$ are finitely generated.

Proof. (i) Using 2.1.9, we can assume that $K_i(A_N, N) = 0, K_i(B_N, N) = 0$ for $i < k$. If we try to carry out the same procedure as in 2.1.9, the only problem arising is the utilisation of the Whitney trick to represent $x_1, \ldots, x_s$ by disjoint embeddings. However, since $\pi_1(N) \to \pi_1(\overline{A_N - tA_N})$ is an isomorphism by definition of the infinite cyclic covering, we can use the Wall's piping out argument to represent them by disjoint embeddings (we refer the reader to the Chapter 4 of [Wal70] for the proof of this lemma). Then the same reasoning aplies and one can kill $K_k(A_N, N)$. By the same argument as in 2.1.5, we have

$$K^i(A_N, N, \mathcal{R}) \oplus K^i(B_N, N, \mathcal{R}) = K^{i-1}(N, \mathcal{R})$$

for any R-module $\mathcal{R}$. Now $K^{i-1}(N, \mathcal{R}) = 0$ for $i > k$ by Poincaré duality, so $K^i(B_N, N, \mathcal{R}) = 0$ for $i > k$. Hence $K_k(B_N, N)$ is a finitely generated projective R-module.
(ii) By 2.1.9, we can assume that $K_i(A_N, N) = 0, K_i(B_N, N) = 0$ for $i \leq k$. Now since $K^{i-1}(N, \mathcal{R}) = 0$ for $i > k+1$ by Poincaré duality, we have $K^i(A_N, N, \mathcal{R}) \oplus K^i(B_N, N, \mathcal{R}) = 0$ for $i > k + 1$, thus $K^i(A_N, N, \mathcal{R}) = 0$ and $K^i(B_N, N, \mathcal{R}) = 0$ for $i > k + 1$. Hence $K_{k+1}(A_N, N, \mathcal{R})$ and $K_{k+1}(B_N, N, \mathcal{R})$ are finitely generated projective R-module. $\square$

Definition 2.2.2. A map $M \to M'$ as in 2.2.1 is called an *almost splitting.*

Recall that the Grothendieck's theorem implies that $K_0(R) \simeq \mathbb{Z}$, hence every finitely generated projective R-module is stably free.

Definition 2.2.3. Define a group $\widetilde{\mathrm{Nil}}\ R$ as follows. Let **Nil** R be the category whose objects are pairs (P, ν), where P is a finitely generated stably free R-module and ν is a nilpotent endomorphism of P, and define $\widetilde{\mathrm{Nil}}\ R = K_0(\mathbf{Nil}\ R)$.

Recall that, since $K_0(R) \simeq \mathbb{Z}$ by the Grothendieck theorem 1.3.8, every finitely generated projective R-module is stably free. Hence, using the usual notations, $(K_{k+1}(B_N, N), t_*)$ (resp. $(K_k(B_N, N), t_*)$) defines an element in $\widetilde{\mathrm{Nil}}\ R$ if $n = 2k+1$ (resp. $n = 2k$). We have the following

Definition 2.2.4. We say that an object (P, ν) in $\widetilde{Nil}\ R$ is *triangular* if there exists a filtration $0 = E_0 \subset E_1 \subset \ldots \subset R_r = P$ such that E_{i+1}/E_i is free and $\nu(E_{i+1}) \subset E_i$ for all i.

An elementary example of a triangular object is given by the following

Lemma 2.2.5. *If P is a free R-module, then $P \otimes_R R[x]/(x^r)$ with $\nu(\sum_i p_i \otimes x^i) = \sum_i p_i \otimes x^{i+1}$ is a triangular object.*

An important property of triangular object is the following

Proposition 2.2.6. *Any triangular object represents the zero element in $\widetilde{Nil}\ R$.*

Proof. We proceed by induction on the length of the filtration by which the object is said to be triangular. If (P, ν) is a triangular object with a filtration of length 1, the result follows immediately from the definition of $\widetilde{Nil}\ R$. Assume we have proved the proposition for filtrations of lengths $m-1$, $m \geq 2$, and consider an object (P, ν) with a filtration $0 = E_0 \subset E_1 \subset \ldots \subset E_m = P$. We have the following exact sequence in $\mathbf{Nil}\ R$

$$0 \to (E_{m-1}, \nu_{E_{m-1}}) \to (P, \nu) \to (P/E_{m-1}, 0) \to 0.$$

Thus $[P, \nu] = [E_{m-1}, \nu_{E_{m-1}}] + [P/E_{m-1}, 0] = [E_{m-1}, \nu_{E_{m-1}}]$ since P/E_{m-1} is free. Now $(E_{m-1}, \nu_{E_{m-1}})$ is a triangular object with a filtration of length $m-1$, hence $[E_{m-1}, \nu_{E_{m-1}}] = 0$ by the induction hypothesis. $\square$

Lemma 2.2.7. *Let (P, ν) be an object in **Nil** R with a filtration $0 = E_0 \subset E_1 \subset \ldots \subset E_m = P$ by finitely generated submodules such that $\nu(E_{i+1}) \subset E_i$. Then there exists an exact sequence in **Nil** R*

$$0 \to (P', \nu') \xrightarrow{u} (P'', \nu'') \xrightarrow{v} (P, \nu) \to 0$$

where (P'', ν'') is a triangular object with respect to a filtration $0 = F_0 \subset F_1 \subset \ldots \subset F_m = P''$ such that $v(F_i) = E_i$.

Proof. We proceed by induction on the length m of the filtration of P. If $m = 1$, consider a finitely generated free module F and a surjection $v : F \to P$. Then $0 \to (\ker v, 0) \to (F, 0) \xrightarrow{v} (P, 0) \to 0$ is the desired sequence. Assume the lemma is true for $m-1, m \geq 2$. Applying the lemma to $(E_{m-1}, \nu_{E_{m-1}})$ by the induction hypothesis, there exists a map $v_{m-1} : (F_{m-1}, f_{m-1}) \to (E_{m-1}, \nu_{E_{m-1}})$ satisfying the above conclusions. Since E_m/E_{m-1} is finitely generated, there exists an epimorphism $q : Q \to E_m/E_{m-1}$, where Q is a finitely generated free R-module. q lifts to a map $\bar{q} : Q \to E_m = P$. Let $F = F_{m-1} \oplus Q$ and $v = v_{m-1} \oplus q : F \to P$. We now extend f_{m-1} to an endomorphism of F. Since Q is free, there exists a linear map $\bar{f}$ making the following diagram commutative

$$\begin{tikzcd} Q \arrow[r, dotted, "\bar{f}"] \arrow[d, "v"'] & F_{m-1} \arrow[d, "v_{m-1}"] \\ K \arrow[r, "\nu"'] & E_{m-1}. \end{tikzcd}$$

Let $f = f_{m-1} \oplus \bar{f} : F \to F$. Let $L = \ker v$, $l = f_{|L}$. Then $0 \to (L, l) \hookrightarrow (F, v) \xrightarrow{v} (P, \nu) \to 0$ is the desired sequence. $\square$

Proposition 2.2.8. *$\widetilde{Nil}\ R = 0$.*

Proof. Let $[P, \nu] \in \widetilde{Nil}\ R$, with $\nu^m = 0$, and set $K_i = \text{Im}\ \nu^{m-i}$. We thus have a filtration $0 = K_0 \subset K_1 \subset \ldots \subset K_m = P$, with each K_i finitely generated. Using 2.2.7, let $0 \to (P_1, \nu_1) \xrightarrow{u} (P'', \nu'') \xrightarrow{v} (P, \nu) \to 0$ be an exact sequence in **Nil** R with (P'', ν'') triangular for a filtration $0 = E_0 \subset E_1 \subset \ldots \subset E_m = P''$ with $v(E_i) \subset K_i$. Since (P'', ν'') is triangular, $[P, \nu] = -[P_1, \nu_1]$ by 2.2.6. Let $L_i = u^{-1}(E_i)$. We then have a filtration $0 = L_0 \subset E_1 \subset \ldots \subset L_m = P_1$, and exact sequences $0 \to L_{i+1}/L_i \to E_{i+1}/E_i \to K_{i+1}/K_i \to 0$. Since R is noetherian, L_{i+1}/L_i is finitely generated, and it follows that each L_i is finitely generated by an easy induction. If M is a R-module, let $d(M)$ be the minimal length of a projective resolution of M. Since E_{i+1}/E_i is free, it follows from the exact sequence $0 \to L_{i+1}/L_i \to E_{i+1}/E_i \to K_{i+1}/K_i \to 0$ that $d(L_{i+1}/L_i) = \max\ (1, d(K_{i+1}/K_i) - 1)$. Let $d = max_{0 \leq i \leq m-1} d(K_{i+1}/K_i)$. So after d applications of this procedure, we have an object $(P_d, \nu_d) \in$ **Nil** R such that $[P_d, \nu_d] = (-1)^d [P, \nu]$, and a filtration $0 = F_0 \subset F_1 \subset \ldots \subset F_m = P_d$ such that S_{i+1}/S_i is a finitely generated projective R-module. Hence

$$[P, \nu] = (-1)^d [P_d, \nu_d] = \sum_i [S_{i+1}/S_i, 0] = 0,$$

since each S_{i+1}/S_i is stably free by the Grothendieck theorem 1.3.8. $\square$

Thus the element $(K_k(B_N, N), t_*)$ or $(K_{k+1}(B_N, N), t_*)$ is the zero element in $\widetilde{Nil}\ R = K_0(\mathbf{Nil}\ R)$. This in turn gives information on the structure of the pair, which can be thought as an analog of the fact that an element representing 0 in $K_0(R)$ is stably trivial. Namely, an element representing the zero element in $\widetilde{Nil}\ R = K_0(\mathbf{Nil}\ R)$ is stably trivial in the following sense.

Proposition 2.2.9. *An object $(P, \nu) \in \boldsymbol{Nil}\ R$ represents $0 \in \widetilde{Nil}\ R$ if and only if there exists triangular objects $(T_1, t_1), (T_2, t_2)$ such that*

$$(P, \nu) \oplus (T_1, t_1) \simeq (T_2, t_2).$$

For a proof, we refer to [Ko9]. We will see in the next section the geometric operations corresponding to adding or removing a triangular object, which will allow us to kill the remaining homology kernel.

2.3 The Splitting Theorem.

We are now going to prove the splitting theorem. So far, we have proved that one can homotop f so as to have an almost splitting. Furthermore, we saw in the last section that the remaining homology kernel is stably triangular. In this section, we describe the geometric operations which allow us to add or remove a triangular object, thus proving the Splitting Theorem.

Lemma 2.3.1. (i) *Suppose that $n = 2k \geq 6$, and (N, g) is an almost splitting. If we have an exact sequence in $\boldsymbol{Nil}\ R$*

$$0 \to (P, \nu) \to (P_1, \nu_1) \to (F, f) \to 0$$

with $(P, v) \simeq (K_k(B_N, N), t_*)$ and (F, f) a triangular object, there exists an almost splitting (N_1, g_1) such that $(K_k(B_{N_1}, N_1), t_*) \simeq (P_1, \nu_1)$.

(ii) *Suppose that $n = 2k + 1 \geq 7$, and (N, g) is an almost splitting. If we have an exact sequence in $\boldsymbol{Nil}\ R$*

$$0 \to (P, \nu) \to (P_1, \nu_1) \xrightarrow{v} (F, f) \to 0$$

with $(P, v) \simeq (K_{k+1}(B_N, N), t_*)$ and (F, f) a triangular object, there exists an almost splitting (N_1, g_1) such that $(K_{k+1}(B_{N_1}, N_1), t_*) \simeq (P_1, \nu_1)$.

Proof. The proofs are essentially the same, except for some number changes. We thus restrict to the case where $n = 2k$. Since (F, f) is triangular, it suffices to prove the result for $(F, f) \simeq (R, 0)$ by an easy induction. Let $a \in P_1$ projecting on a generator of R. Then $x = \nu_1(a) \in \ker v$, so we can consider as an element of $(P, \nu) \simeq (K_k(B_N, N), t_*)$. Let $u = \partial x$, where is the boundary map in the exact sequence

$$\ldots \to K_k(t^{-1}B_N, t^{-1}N) \simeq K_k(B_N, \overline{B_N - t^{-1}B_N}) \xrightarrow{\partial}$$

$$\xrightarrow{\partial} K_{k-1}(\overline{B_N - t^{-1}B_N}, N) \to K_{k-1}(B_N, N) \simeq 0 \to \ldots$$

Following 2.1.8, we can show that is representable under the Hurewicz homomorphism by a map $u : (\mathbb{D}^{k-1}, \mathbb{S}^{k-2}) \to (B_N - t^{-1}B_N, N)$. Using the Whitney trick, we can assume that u is an embedding. Hence we can apply the same reasoning and perform surgery on u. Let (N_1, g_1) be the almost splitting obtained, and W be the cobordism between N and N_1. Using the following exact sequence of the triple (B_N, W, N),

$$0 \simeq K_k(W, N) \to K_k(B_N, N) \to K_k(B_N, W) \simeq K_k(B_{N_1}, N_1) \to$$

$$\to K_{k-1}(W, N) \to K_{k-1}(B_N, N) \simeq 0,$$

it follows that $K_k(B_{N_1}, N_1) \simeq K_k(B_N, N) \oplus R$. Furthermore, by construction of u we have that $t_*^{-1}x$ generates the second summand. Now since $t_*(t_*^{-1}x) = x$ and the previous exact sequence preserves the action of t_*, it follows that the two exact sequences

$$0 \to (K_k(B_N, N), t_*) \to (K_k(B_{N_1}, N_1), t_*) \to (K_{k-1}(W, N), t_*) \to 0,$$

$$0 \to (P, \nu) \to (P_1, \nu_1) \to (R, 0) \to 0$$

are isomorphic. Furthermore, one easily checks that the previous construction does not affect $K_k(A_N, N)$ looking at the exact sequence of the triple (A_{N_1}, W, N_1). $\square$

Lemma 2.3.2. *Suppose that $n = 2k \geq 6$, and (N, g) is an almost splitting. If we have an exact sequence in **Nil** R*

$$0 \to (F, f) \to (P, \nu) \to (P_1, \nu_1) \to 0$$

with $(P, v) \simeq (K_k(B_N, N), t_*)$ and (F, f) a triangular object, there exists an almost splitting (N_1, g_1) such that $(K_k(B_{N_1}, N_1), t_*) \simeq (P_1, \nu_1)$.

Proof. Once again we can suppose that $(F, f) \simeq (R, 0)$. Let a be an element of $K_k(B_N, N) \simeq P$ generating F. Let l be the least integer such that $t_*^l K_k(B_N, N) = 0$. Since $t_* a = 0$, $a \in \ker t_* \subset \mathrm{Im}\ t_*^{l-1}$. Thus, using 2.1.8 and Wall's piping out argument, we can represent a by an embedding under the Hurewicz homomorphism. Carrying out again our surgery program, we obtain a new almost splitting (N_1, g_1) and a cobordism W between N and N_1. Using the exact sequence of the triple (B_N, W, N)

$$0 \to K_k(W, N) \to K_k(B_N, N) \to K_k(B_N, W) \simeq K_k(B_{N_1}, N_1) \to 0,$$

and using the fact that it preserves the action of t_*, it follows that this sequence is isomorphic to

$$0 \to (R, 0) \to (P, \nu) \to (P_1, \nu_1) \to 0.$$

Furthermore, by looking at the exact sequence of the triple (A_{N_1}, W, N_1), one can check that $K_k(A_N, N)$ is not affected during the procedure. □

Lemma 2.3.3. *Suppose that $n = 2k + 1 \geq 7$, and (N, g) is an almost splitting. If we have an exact sequence in* ***Nil*** *R*

$$0 \to (F, f) \to (P, \nu) \to (P_1, \nu_1) \to 0$$

with $(P, v) \simeq (K_{k+1}(B_N, N), t_*)$ and (F, f) a triangular object, there exists an almost splitting (N_1, g_1) such that $(K_{k+1}(B_{N_1}, N_1), t_*) \simeq (P_1, \nu_1)$.

This lemma is quite technical and uses ideas from the Chapter 4 of [Wal70]. We refer the reader to [FH73] for the details.

Proof of the Splitting Theorem By 2.2.8 and 2.2.9, the homology kernel $(K_k(B_N, N), t_*)$ (resp. $(K_{k+1}(B_N, N), t_*)$) is stably triangular. Now, using 2.3.1, 2.3.2 and 2.3.3, we can find an almost splitting (N_1, g_1) such that $K_k(B_N, N) = 0$ (resp. $K_{k+1}(B_N, N) = 0$). This in turn implies by Poincaré duality and 2.1.5 that g_1 is an homotopy equivalence.

2.4 In light of the general Bass-Heller-Swan Theorem...

This last section presents without proof the links between the algebraic obstruction constructed in Section 2.2 and the Whitehead torsion of the associated homotopy equivalence. It is not needed for the rest of the thesis.

First we have the generalisation of the Bass-Heller-Swan theorem to an arbitrary ring.

Theorem 2.4.1. *For any ring R,*

$$K_1(R[t,t^{-1}]) \simeq K_1(R) \oplus K_0(R) \oplus \widetilde{Nil}(R) \oplus \widetilde{Nil}(R).$$

Corollary 2.4.2. *For any group π,*

$$Wh(\pi \times \mathbb{Z}) \simeq Wh(\pi) \oplus \widetilde{K}_0(\mathbb{Z}\pi) \oplus \widetilde{Nil}(\mathbb{Z}\pi) \oplus \widetilde{Nil}(\mathbb{Z}\pi).$$

The algebraic obstruction previously defined in Section 2.2 is related to the Whitehead torsion of the associated homotopy equivalence by the following

Theorem 2.4.3. *Let $f : M \to M'$ be a homotopy equivalence between compact m-dimensional manifolds ($m \geq 6$) with fundamental group of the form $\pi \times \mathbb{Z}$, and N a two-sided codimension one submanifold of M. The obstruction to splitting f along N is given by $\phi(\tau)$, where τ is the Whitehead torsion of f, and*

$$\phi : Wh(\pi \times \mathbb{Z}) \to \widetilde{K}_0(\pi) \oplus \widetilde{Nil}(R)$$

is the projection[1] *given by the decomposition of $Wh(\pi \times \mathbb{Z})$.*

What we computed in Section 2.2 was the projection of $\phi(\tau)$ on $\widetilde{Nil}(\mathbb{Z}\pi)$. Because of the Grothendieck theorem 1.3.8, it was sufficient to prove its vanishing to deduce that the map is splittable along N. For further details, we refer the reader to [FH73]

[1] The copy of $\widetilde{Nil}(\mathbb{Z}\pi)$ chosen in the decomposition of $Wh(\pi \times \mathbb{Z})$ is actually not relevant, see [HS70] for further details.

Chapter 3

A crash course in surgery theory.

This chapter is intended to present in a very concise way the necessary background on surgery theory and homotopy theory, which will be used to prove the topological rigidity of the torus in high dimensions. For a (incredibly) more detailed exposition of this material, we refer to [Ran02].

For sake of simplicity, and since this the only case we will encounter in this thesis, we will assume that **all the spaces we study have a fundamental group with vanishing Whitehead group**. That will allow us to avoid complications in the exposition of surgery theory due to so called *decorations* of algebraic L-groups. Furthermore, since we will have to consider possibly *TOP*-, *PL*-, or *DIFF*-manifolds, when a manifold comes with a given structure and that the discussion applies to equally to all the above structures, we will just call it a CAT-manifold (*CAT = TOP, PL, DIFF*).

3.1 Surgery obstructions.

3.1.1 Degree one normal maps, surgery obstructions, and L-groups.

The basic question of surgery theory is to know whether a finite CW-complex is homotopy equivalent to a compact CAT-manifold (we will only consider *oriented* manifolds). We outline here the main steps of the so called *surgery program*, and explain how the so called *surgery obstructions* arise.

The first obstruction for a finite CW-complex to be homotopy equivalent to an orientd compact manifold is given by Poincaré duality. We will restrict ourselves to the case of *oriented Poincaré spaces*, that is to say, spaces with a class $[X] \in H_m(X;\mathbb{Z})$ such that

$$\cap[X] : H^n(X;\mathbb{Z}) \to H_{m-n}(X;\mathbb{Z})$$

is an isomorphism for all n. Since for any oriented Poincaré space X there exists a degree one map $f : M \to X$ with M a compact oriented manifold, the idea is to modify f in such a way that it becomes a homotopy equivalence (or equivalently that it induces isomorphisms on homotopy groups). Since a degree one map induces an epimorphism on π_1 by 2.1.1, one can try to perform surgery on a set of generators of $\ker(f_* : \pi_1(M) \to \pi_1(X))$ to obtain a new map $f_1 : M_1 \to X$ which is a π_1-isomorphism. The situation is not that simple in higher dimensions since a degree one map does not necessarily induce epimorphisms on $\pi_i,\ i \geq 2$. However, it induces epimorphisms on $H_i, i \geq 2$ by 2.1.2. Thus, it suffices to modify f_1 so that it induces an isomorphism on π_1 and $H_i, i \geq 2$ (or equivalently until middle dimension by Poincaré duality). It appears that to carry out such a program, one needs some additional bundle data. Hence we are considering instead *degree one normal maps* (see the definition below). Below middle dimension, an element in the kernel of $H_i(N) \xrightarrow{f_*} H_i(X)$ is representable by a framed embedding with trivial normal bundle (mainly because of the Whitney embedding theorem), on which one might perform surgery. Thus for a degree one normal map $f : M \to X$ with M a m-dimensional CAT-manifold ($m = 2n$ or $2n+1$), one can construct a bordant n-connected degree one normal map $f_n : M_n \to X$. However there is an obstruction to kill $\ker f_* : H_n(M) \to H_n(X)$ living in the algebraic L-group $L_m(\mathbb{Z}[\pi_1(X)])$. If the surgery obstruction vanishes, one can find a bordant degree one $n+1$-connected degree one normal map $f_{n+1} : M_{n+1} \to X$, which is a homotopy equivalence by Poincaré duality.

3.1.2 Simply-connected obstructions.

A fundamental case is the case of degree one normal maps over simply-connected manifolds.

Proposition 3.1.1. *The simply-connected L-groups are given by*

$$L_n(e) = \begin{cases} \mathbb{Z} & \textit{if } n \equiv 0 \ (mod \ \ 4), \\ \mathbb{Z}_2 & \textit{if } n \equiv 2 \ (mod \ 4) \\ 0 & \textit{if } n \equiv 1,3 \ (mod \ 4) \end{cases}$$

Proposition 3.1.2. *Given a degree one normal map $(f,b) : N \to M$ with M simply-connected, we have*

$$S(f,b) = \begin{cases} \frac{1}{8}(\sigma(N) - \sigma(M)) & \textit{if } n \equiv 0 \ (mod \ \ 4), \\ \textit{Arf invariant of the intersection form} & \textit{if } n \equiv 2 \ (mod \ 4) \\ 0 & \textit{if } n \equiv 1,3 \ (mod \ 4). \end{cases}$$

3.1.3 Codimension-one splitting.

Let M be a PL-manifold of dimension $n \geq 5$ such that $Wh(\pi_1(M)) = 0$, and let $\xi \in L_{n+1}(\pi_1(M))$. We recall the following theorem

Theorem 3.1.3 (Realization Theorem)**.** *Let M be a compact oriented PL-manifold of dimension $n \geq 5$, and $\xi \in L_{n+1}(\mathbb{Z}[\pi_1(M)])$. Then there exists a degree one normal map $(F,b) : N \to M \times [0,1]$ covering a map of triads*

$$F = (F;_0 F, \partial_1 F) : (N; \partial_0 N, \partial_1 N) \to (M \times [0,1]; M \times 0 \cup \partial M \times [0,1], M \times 1)$$

such that
- *$\partial_0 F$ is a PL-homotopy equivalence,*
- *$\partial_1 F$ is a homotopy equivalence,*
- *$\xi = S(F,b)$.*

Thus, using the Realization Theorem, let

$$(\psi, b) : (N, \partial N) \to (M \times [0,1]), \partial(M \times [0,1])$$

be a degree one normal map with surgery obstruction ξ. Let L be a codimension one submanifold of M without boundary. By the Splitting Theorem, we can split ψ along $L \times [0,1]$ to obtain a degree one normal map

$$(\psi_|, b_|) : \psi^{-1}(L \times [0,1]) \to L \times [0,1].$$

The surgery obstruction of this new map yields an element in $L_n(G)$. Let

$$\alpha(L) : L_{n+1}(G \times \mathbb{Z}) \to L_n(G)$$

be the induced map.

Proposition 3.1.4. *$\alpha(L)$ is a well-defined homoprhism, and the following sequence is exact and splits:*

$$L_{n+1}(G) \xrightarrow{L_{n+1}(i_*)} L_{n+1}(G \times \mathbb{Z}) \xrightarrow{\alpha(L)} L_n(G)$$

and a splitting is given by crossing with $\mathbb{S}^1$

Corollary 3.1.5. *We have*

$$L_n(\mathbb{Z}^k) \simeq \sum_{0\leq l\leq k} \binom{k}{l} L_{n-l}(e).$$

Let us give a geometric interpretation of the map $L_{n+1}(i_*)$.
Let $\xi \in L_{n+1}(G)$. By the Relaization Theorem 3.1.3, let

$$(\varphi, b) : N \to L \times [0,1] \times [0,1]$$

be a degree one normal map with surgery obstruction ξ and such that, with $\partial N = \partial_- N \cup \partial_+ N$, $\varphi_| : \partial_- N \to (L \times [0,1] \times \{0\}) \cup (L \times \{0\} \times [0,1]) \cup (L \times \{1\} \times [0,1])$ a PL-homeomorphism. We can use it to identify the copy $L \times \{0\} \times [0,1]$ and $L \times \{1\} \times [0,1]$ in ∂N. By glueing them together, we obtain a normal map

$$(g, b') : N' \to L \times \mathbb{S}^1 \times [0,1].$$

Let $\iota(L) : L_{n+1}(G) \to L_{n+1}(G \times \mathbb{S}^1)$ be the induced map.

Proposition 3.1.6. *$\iota(L)$ is a well-defined homomorphism, and $\iota(L) = L_{n+1}(i_*)$.*

3.1.4 Products

Suppose we are given a degree one normal map

$$(\varphi, b) : M \to N$$

and a PL-manifold X. We can construct the normal map

$$(\varphi \times Id_X, b \times Id) : M \times X \to N \times X.$$

We would like a formula for $S(\varphi \times Id_N)$. The answer has been given by Morgan [Mor78] for X simply-connected, and by Ranicki [Ran80] in the general case. There exist "symmetric L-groups" $L^n(\pi')$, and a pairing

$$L_i(\pi) \otimes L^j(\pi') \to L_{i+j}(\pi \times \pi')$$

such that $S(\varphi \times Id_X) = S(\varphi) \otimes \sigma^*(X)$, where

$$\sigma^* : \Omega_* B\pi' \to L^*(\pi')$$

is the *Mischenko-Ranicki symmetric signature*. The symmetric L-groups are hard to compute. However, we have the following particular case:

Proposition 3.1.7. *(i) The symmetric L-groups $L^*(e)$ are 4-periodic, and given by*

$$L^n(e) = \begin{cases} \mathbb{Z} & \text{if } n \equiv 0 \ (mod \ \ 4), \\ \mathbb{Z}_2 & \text{if } n \equiv 1 \ (mod \ 4) \\ 0 & \text{if } n \equiv 1,2 \ (mod \ 4). \end{cases}$$

(ii) If N is simply connected, then

$$S(\varphi \times Id_N) = \begin{cases} S(\varphi).\sigma(N) & \text{if dim } N \equiv 0 \ (mod \ 4), \\ 0 & \text{if dim } N \equiv 1 \ (mod \ 4) \text{ and } N \text{ has a zero de Rham invariant}, \\ 0 & \text{if dim } N \equiv 2,3 \ (mod \ 4). \end{cases}$$

Proposition 3.1.8. *We have the following commutative diagram:*

$$\begin{tikzcd} L_n(\pi) \arrow[r, "\otimes\ \sigma^*(N)"] \arrow[d, "L_n(pr_\pi)"'] & L_{n+m}(\pi \times \pi') \arrow[d, "L_{n+m}(pr_{\pi \times \pi'})"] \\ L_n(e) \arrow[r] & L_{n+m}(e), \end{tikzcd}$$

where the lower horizontal map is given by considering N as a simply-connected manifold (c.f. [Ran80]).

3.2 The surgery exact sequence.

Given a compact m-dimensional CAT-manifold M without boundary, we have the following exact sequence:

$$\cdots \to \mathcal{S}_{CAT}\left(M \times \mathbb{D}^{k+1}, M \times \mathbb{S}^k\right) \to \left[\left(M \times \mathbb{D}^k, M \times \mathbb{S}^k\right), (G/CAT, *)\right] \xrightarrow{S} L_{m+k}\left(\mathbb{Z}\left(\pi_1(M)\right)\right) \to \cdots$$

$$\cdots \to L_{m+1}\left(\mathbb{Z}\left[\pi_1(M)\right]\right) \to \mathcal{S}_{CAT}(M) \to [M, G/CAT] \xrightarrow{S} L_m\left(\mathbb{Z}\left[\pi_1(M)\right]\right).$$

3.2.1 The sets

The structure set

It is the object that reflects the various CAT-manifolds homotopy equivalent to M, and is consequently what one generally tries to compute. If M is a m-dimensional CAT-manifold with boundary (possibly empty), $\mathcal{S}_{CAT}(M, \partial M)$ is defined as the set of equivalence classes of pairs (N, f), with N a m-dimensional closed CAT-manifold and $f : N \to M$ an homotopy equivalence such that $\partial f : \partial N \to \partial M$ is a CAT-isomorphism, with (N_1, f_1) and (N_2, f_2) equivalent if there exists a CAT-isomorphism $h : N_1 \to N_2$ such that the following diagram commutes up to homotopy:

$$\begin{tikzcd} N_1 \arrow[r, "f_1"] \arrow[d, "h"'] & M \\ N_2. \arrow[ur, "f_2"'] & \end{tikzcd}$$

The normal invariants

Roughly speaking, the degree one normal maps are the candidates on which one might try to perform surgery to obtain a homotopy equivalence. There are two equivalent ways to define a normal invariant:

- The degree one normal maps: Define a degree one normal map as a pair as a commutative diagram

$$\begin{tikzcd} \nu_N \arrow[r, "b"] \arrow[d] & \eta \arrow[d] \\ N \arrow[r, "f"] & M, \end{tikzcd}$$

  with $f : N \to M$ a degree one map, $\eta : M \to BCAT$ a stable $CAT-$bundle, and $b : \nu_N \to \eta$ a bundle stable isomorphism. We will often write $(f, b) : M \to N$ for a degree one normal map, or even just $f : N \to M$ if the framing is not relevant or obvious.

- $[M, G/CAT]$: It is the set of stable CAT-bundles over M (up to stable isomorphism) such that the associated spherical fibration is strongly fiber homotopically trivial. In other words, it is the set of homotopy classes of lifts

$$\begin{tikzcd} & BCAT \arrow[d] \\ M \arrow[r, "0"'] \arrow[ur, dotted] & BG \end{tikzcd}$$

The algebraic L-groups

It is where surgery obstruction lie. They are defined in terms of forms and formations. For a precise account on this subject, we refer to [Ran02].

3.2.2 The maps

The surgery obstruction map S

Given a degree one normal map $f : N \to M$ between two m-dimensional manifolds, one can perform surgery on it to obtain a $\left[\frac{m}{2}\right]$-connected degree one normal map $f' : N' \to M$. Define $S_M(f,b)$ (or simply $S(f,b)$) as the obstruction to make this map a homotopy equivalence. This obstruction lives in $L_m\left(\mathbb{Z}\left[\pi_1(M)\right]\right)$.

The forgetful map $\mathcal{S}(M) \to [M, G/CAT]$

A homotopy equivalence $f : N \to M$ naturally gives rise to a degree one normal map

$$\begin{tikzcd}
\nu_N \arrow[r] \arrow[d] & (f^{-1})^*\nu_N \arrow[d] \\
N \arrow[r, "f"] & M.
\end{tikzcd}$$

The action of $L_{m+1}\left(\mathbb{Z}\left[\pi_1(M)\right]\right)$ on $\mathcal{S}(M)$

Let $f : N \to M$ be a homotopy equivalence between m-oriented compact PL-manifold, and $\xi \in L_{m+1}\left(\mathbb{Z}\left[\pi_1(M)\right]\right)$ By the realization theorem 3.1.3, consider a degree one normal map $(F,b) : (W, \partial W_0, \partial W_1) \to (N\times[0,1], N\times\{0\}, N\times\{1\})$ with obstruction ξ, and such that $F_{|\partial W_0} : \partial W_0 \to N$ is a CAT-isomorphism, and set
$\xi.\,(f : N \to M) = \left(f \circ F^{-1}_{|\partial W_1} : \partial W_1 \to N\right)$.

3.2.3 An exact sequence of what ?

One has to be careful with the surgery exact sequence. Although $[M, G/CAT]$ and $L_m\left(\mathbb{Z}\left[\pi_1(M)\right]\right)$ both carry a natural group structure, the surgery obstruction map S is generally NOT a group morphism. However, S is in some special cases a group morphism. This is the case for example if M is a suspension, or for $M \times \mathbb{D}^k$ ($k \geq 1$) appearing in the surgery obstruction map.

3.3 Classifying spaces and their homotopy groups.

We give here some homotopy groups of the classifying spaces BG, BPL, BO, and their associated homotopy fibers G/PL and PL/O, needed for the rest of this thesis.

3.3.1 $\pi_i(PL/O)$ and smoothing theory.

The problem here is to decide if a PL-manifold can be given a smooth structure. First recall the following

Theorem 3.3.1 (Cairns-Hirsch)**.** *A PL-manifold M is smoothable if and only if its tangent bundle $M \to BPL$ admits a lift to BO.*

Since we have the fibration $PL/O \to BO \to BPL$, we need to understand the space PL/O. We have the following theorem.

Theorem 3.3.2. *PL/O is 7-connected.*

Corollary 3.3.3 (Smoothing Theorem)**.** *Every PL-manifold of dimension ≤ 5 is smoothable.*

Proof. By obstruction theory, a map $M \to BPL$ admits a lift to BO if and only some obstructions $\omega_i \in H^{i+1}(M; \pi_i(PL/O))$ vanish. But since M is of dimension ≤ 5, $\omega_i = 0$ for $i \geq 5$. Furthermore, since $\pi_i(PL/O) = 0$ for $i \leq 4$, $\omega_i = 0$ for $i \leq 4$. The result then follows from the Cairns-Hirsch theorem. □

3.3.2 BO, BG and the J-homomorphism.

We present some low dimensional computations of the J-homomorphism $J : \pi_i(BO) \to \pi_i(BG)$ given by associating to a stable vector bundle over $\mathbb{S}^i$ its spherical bundle.

i	$\pi_i(BO)$	$\pi_i(BG)$	$\pi_i(BO) \xrightarrow{J} \pi_i(BG)$
1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \xrightarrow{\approx} \mathbb{Z}_2$
2	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \xrightarrow{\approx} \mathbb{Z}_2$
3	0	$\mathbb{Z}_2$	$0 \xrightarrow{0} \mathbb{Z}_2$
4	$\mathbb{Z}$	$\mathbb{Z}_{24}$	$\mathbb{Z} \xrightarrow{pr} \mathbb{Z}_{24}$
5	0	0	$0 \xrightarrow{0} 0$
6	0	0	$0 \xrightarrow{0} 0$

3.3.3 $\pi_i(G/PL)$ and the surgery obstruction map of spheres.

We want to understand the surgery obstruction map $\pi_i(G/PL) \to L_i(e)$ arising in the surgery exact sequence of a PL-sphere. Recall the following theorem of Stallings.

Theorem 3.3.4 (Stallings). *For $i \geq 6$, $\mathcal{S}_{PL}(\mathbb{S}^i) = 0$.*

Corollary 3.3.5. *The surgery obstruction map $\pi_i(G/PL) \to L_i(e)$ is an isomorphism for $i \geq 6$.*

Proof. Immediate from the surgery exact sequence. □

In low dimensions, the same reasoning does not apply. Instead, we start by showing that $\pi_i(G/PL)$ and $L_i(e)$ are abstractly isomorphic, then by proving that the surgery obstruction map is an isomorphism. We already know $L_i(e), i \leq 5$ by **??**. Thus we compute the homotopy groups of G/PL.

Recall that PL/O is 7-connected. Thus the exact sequence

$$\pi_n(PL/O) \to \pi_n(G/O) \to \pi_n(G/PL) \to \pi_{n-1}(PL/O)$$

yields $\pi_n(G/PL) \simeq \pi_n(G/O)$, for $1 \leq n \leq 5$. We now use the following long exact sequence

$$\ldots \to \pi_{n+1}(BO) \xrightarrow{J} \pi_{n+1}(BG) \to \pi_n(G/O) \to \pi_n(BO) \xrightarrow{J} \pi_n(BG) \to \ldots$$

to calucate the homotopy groups of G/PL in low dimensions.

Proposition 3.3.6. $\pi_1(G/PL) \simeq 0$.

Proof. Since $J : \pi_2(BO) \to \pi_2(BG)$ and $J : \pi_1(BO) \to \pi_1(BG)$ are isomorphisms, we have

$$0 \to \pi_1(G/O) \to \pi_1(BO) \xrightarrow{\simeq} \pi_1(BG) \to 0$$

hence $\pi_1(G/O) \simeq 0$. □

Proposition 3.3.7. $\pi_2(G/PL) \simeq \mathbb{Z}_2$.

Proof. Since $J : \pi_2(BO) \to \pi_2(BG)$ is an isomorhism, we have the following exact sequence

$$0 \to \mathbb{Z}_2 \to \pi_2(G/O) \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{J} \mathbb{Z}_2 \to 0$$

hence $\pi_2(G/O) \simeq \mathbb{Z}_2$. □

Proposition 3.3.8. *$\pi_3(G/PL) \simeq 0$.*

Proof. Since $J : \pi_4(BO) \to \pi_4(BG)$ is surjective, we have the following exact sequence

$$\mathbb{Z} \xrightarrow{J} \mathbb{Z}_{24} \xrightarrow{0} \pi_3(G/O) \to 0$$

hence $\pi_3(G/O) \simeq 0$. □

Proposition 3.3.9. *$\pi_4(G/PL) \simeq \mathbb{Z}$.*

Proof. Since $J : \pi_4(BO) \to \pi_4(BG)$ is surjective, we have the following exact sequence

$$0 \to \pi_3(G/O) \to \mathbb{Z} \xrightarrow{J} \mathbb{Z}_{24} \to 0$$

hence $\pi_4(G/O) \simeq \mathbb{Z}$. □

Proposition 3.3.10. *$\pi_5(G/PL) = 0$.*

Proof. Immediate from $\pi_5(BO) = 0$ and $\pi_6(BG) = 0$. □

We now use these calculations to study the surgery obstruction map $\pi_n(G/PL) \to L_n(e)$ in dimension ≤ 5. We just proved that these groups are abstractly isomorphic, and we want to prove that the surgery obstruction map realizes an isomorphism in dimension other than 4, and is a monomorphism in any case.

In odd dimensions, both groups are zero, so we restrict to dimensions 2 and 4. Since $\pi_2(G/PL) \simeq L_2(e) \simeq \mathbb{Z}_2$ and $\pi_4(G/PL) \simeq L_4(e) \simeq \mathbb{Z}$, the maps are either zero or injective.
We recall two classical facts.

Proposition 3.3.11. *There exists an almost parallelizable 4-manifold with signature 16.*

Corollary 3.3.12. *$\pi_4(G/PL) \to L_4(e)$ is injective.*

Proposition 3.3.13. *There exists a degree one normal map $(\mathbb{T}^2, \varepsilon^2) \to (\mathbb{S}^2, \varepsilon^2)$ with Arf invariant 1.*

Corollary 3.3.14. *$\pi_2(G/PL) \to L_2(e)$ in an isomorphism.*

Finally, we have proved the following

Theorem 3.3.15. *The surgery obstruction map $\pi_n(G/PL) \to L_n(e)$ is an isomorphism in dimension $n \neq 4$, and a monorphism in all dimensions.*

Chapter 4

The classification of PL-homotopy tori.

We now start our program to prove the topological rigidity of the torus in high dimensions. According to the definition given in the previous chapter, this amounts to proving that $\mathcal{S}_{TOP}(\mathbb{T}^n)$ consists of a single element for $n \geq 5$.

We start by computing $\mathcal{S}_{PL}(\mathbb{T}^n)$. This might seem quite surprising since the question of the topological rigidity of the torus is merely formulated in terms of topological manifolds. However, this will appear to be more tractable, mainly because of the following fundamental theorem:

Rokhlin's Theorem. *A* 4*-dimensional* PL*-manifold with vanishing first and second Stiefel-Whiteney classes has a signature dividible by* 16.

In Section 4.1 we show that the action of the L-group on the PL-structure set is transitive, thus reducing the computation of the $\mathcal{S}_{PL}$ to the computation of the stabilizer of a given element. This in turn will be done in Section 4.2, by means of the Splitting Theorem. This chapter follows the proof of [HS70].

4.1 Normal invariants of PL-homotopy tori.

Our aim is to compute the structure set $\mathcal{S}_{PL}(\mathbb{T}^k), k \geq 5$, and more generally $\mathcal{S}_{PL}(\mathbb{T}^k \times \mathbb{D}^n), n+k \geq 5$. Our main tool is the surgery exact sequence

$$L_{n+k+1}\left(\mathbb{Z}\left[\pi_1(\mathbb{T}^k \times \mathbb{D}^n)\right]\right) \to \mathcal{S}_{PL}(\mathbb{T}^k \times \mathbb{D}^n) \xrightarrow{\eta_{\mathbb{T}^k \times \mathbb{D}^n}}$$

$$\xrightarrow{\eta_{\mathbb{T}^k\times\mathbb{D}^n}} \left[\left(\mathbb{T}^k\times\mathbb{D}^n,\partial\right),(G/PL,*)\right] \xrightarrow{S_{\mathbb{T}^k\times\mathbb{D}^n}} L_{n+k}\left(\mathbb{Z}\left[\pi_1(\mathbb{T}^k\times\mathbb{D}^n)\right]\right).$$

We first have to understand the maps involved.

Proposition 4.1.1. *Let n,k such that $n+k\geq 5$. Then $\eta_{\mathbb{T}^k\times\mathbb{D}^n}=0$.*

Proof. It amounts to proving that $S_{\mathbb{T}^k\times\mathbb{D}^n}$ is injective. Recall that G/PL is a loop space, say $G/PL=\Omega Y$, so

$$[(\mathbb{T}^k\times\mathbb{D}^n,\partial),(G/PL,*)]=[S^n\mathbb{T}^k,\Omega Y]=[S^{n+1}\mathbb{T}^k,Y].$$

Now we have the following

Lemma 4.1.2. *$S\mathbb{T}^k$ has the homotopy type of a wedge of spheres.*

Proof. We prove by induction on l that each attaching map of a l-cell of $\mathbb{T}^k$ has trivial suspension. It is obvious for $l=1$. Let $l\geq 2$ and suppose it is true until $l-1$. We are looking at the attaching map of the top cell of a certain subtorus $\mathbb{T}^l\subset\mathbb{T}^k$. Identify $\mathbb{T}^l$ and $[0,1]^l$ with faces identifies in the natural way, and let $\phi:\mathbb{S}^{l-1}\approx\partial([0,1]^l)\hookrightarrow[0,1]^l\xrightarrow{proj}\mathbb{T}^l$ the attaching map of the top l-cell. For every $1\leq i\leq l$, $\phi_{i,0}=\phi_{|\partial([0,1]\times...\times\{0\}\times[0,1])}$ and $\phi_{i,1}=\phi_{|\partial([0,1]\times...\times\{1\}\times[0,1])}$ are attaching maps of some $(l-1)$-cells, and $\phi_{i,0}=-\phi_{i,1}\in\pi_{l-2}(\mathbb{T}^l)$ because of the identification. Now by the induction hypothesis, we can suppose that $S\phi$ factors through

$$S\phi:S(\partial([0,1]^l))\approx\mathbb{S}^l\xrightarrow{C}\bigvee\mathbb{S}^l\to\mathbb{T}^l,$$

where the collapsing map $C:\mathbb{S}^l\to\bigvee\mathbb{S}^l$ is obtained by collapsing the subsets $\partial([0,1]\times...\times\{1\}\times[0,1]),\partial([0,1]\times...\times\{1\}\times[0,1]),1\leq i\leq l$. But by definition of the addition in $\pi_l,l\geq 2$, this is exactly $\sum_i(S\phi_{i,0}+S\phi_{i,1})=0$. □

In particular, $S^{n+1}\mathbb{T}^k$ has the homotopy type of a wedge of spheres, so

$$[S^{n+1}\mathbb{T}^k,Y]=[\bigvee\mathbb{S}^{i+n+1},Y]=\bigoplus\pi_{i+n+1}(Y)=\bigoplus\pi_{i+n}(G/PL)$$

with $\binom{k}{i}$ summands π_{i+n} for each i. But $L_{n+k}(\mathbb{Z}^n)=\oplus\binom{k}{i}L_{n+i}(e)$ by (a result which will appear in the crash course in surgery theory), and the following diagram is commutative

$$\begin{tikzcd}
{[S^n\mathbb{T}^k,G/PL]} \arrow[r, "S_{\mathbb{T}^k\times\mathbb{D}^n}"] \arrow[d, "\approx"'] & L_{n+k}(\mathbb{Z}^k) \arrow[d, "\approx"] \\
\oplus\binom{k}{i}\pi_{n+i}(G/PL) \arrow[r, "\oplus S_{\mathbb{S}^{n+i}}"'] & \oplus\binom{k}{i}L_{n+i}(e)
\end{tikzcd}$$

Hence $S_{\mathbb{T}^k\times\mathbb{D}^n}$ is injective, since every $S_{\mathbb{S}^{n+i}}$ is injective by 3.3.15. $\square$

Corollary 4.1.3. *Every PL-manifold of dimension ≥ 5 which is homotopy equivalent to the torus is stably parallelizable.*

Proof. $\eta_{\mathbb{T}^k}=0$, so given any homotopy equivalence $h: M\to\mathbb{T}^k$, we have $h_*\tau_N = \tau_{\mathbb{T}^k}=0$, hence $\tau_N=0$. $\square$

Corollary 4.1.4. *For every n,k such that $n+k\geq 5$,*

$$\mathcal{S}_{PL}(\mathbb{T}^k\times\mathbb{D}^n)\simeq L_{n+k+1}(\mathbb{Z}^k)/Stab[Id_{\mathbb{T}^k\times\mathbb{D}^n}].$$

Proof. Immediate from the surgery exact sequence. $\square$

4.2 Computation of $\mathcal{S}_{PL}(\mathbb{T}^k\times\mathbb{D}^n)$, $n+k\geq 5$.

By the results of the previous section, it is now necessary to describe precisely the action of $L_{n+k+1}(\mathbb{Z}^k)$ on $\mathcal{S}_{PL}(\mathbb{T}^k\times\mathbb{D}^n)$. We start by expliciting a set of generators.

For every $J\subset\{1,\dots,k\}$, let $|J|$ denote its cardinal, J^c its complementary, and set $T(J)=\{(x_1,\dots,x_k)\in\mathbb{T}^k=(\mathbb{S}^1)^k | x_i=*\text{ if } i\notin J\}$. For each J with $|J|+n\equiv 1(\text{mod } 2)$, we associate an element $\xi(J)\in L_{n+k+1}(\mathbb{Z}^k)$ in the following way (for simplicity, set $m=|J|$):

- If $m+n\geq 5$, by the Realization Theorem 3.1.3, choose a degree one normal map (M,h,F) over $([0,1]^{m+n+1},\varepsilon^{m+n+1})$ such that $S_{[0,1]^{m+n+1}}(M,h,F)$ is a generator of $L_{m+n+1}(e)$. Let K be obtained from $T(J)\times[0,1]^n\times[0,1]$ by taking the connected sum with M along $T(J)\times[0,1]^n\times\{1\}$. This yields the degree one normal

$$\begin{array}{c} K=(T(J)\times[0,1]^n\times[0,1])\,\sharp\, M \\ \Big\downarrow{\scriptstyle Id\,\sharp\, h} \\ (T(J)\times[0,1]^n\times[0,1])\,\sharp\,[0,1]^{m+n+1}=T(J)\times[0,1]^n\times[0,1] \end{array}$$

  with framing the connected sum of the framings. Denote (K,f,E) this degree one normal map. We write $(K,f,E)=(T(J)\times[0,1]^n\times[0,1])\sharp(M,h,F)$. By definition of the surgery obstructions,

$$S_{T(J)\times[0,1]^n\times[0,1]}(K,f,E)=L_{m+n+1}i(S_{[0,1]^{m+n+1}}(M,h,F)),$$

where $i : e \hookrightarrow \mathbb{Z}^m$ is the inclusion. Define

$$\xi(J) = S_{\mathbb{T}^k \times \mathbb{D}^n \times [0,1]}(K \times T(J^c), f \times Id, E \times D) \in L_{n+k+1}(\mathbb{Z}^k)$$

- If $m+n=3$, the previous construction does not work. Indeed, if there was a degree one normal map (W, φ, F) over $(\mathbb{D}^4, \varepsilon^4)$ with surgery obstruction $1 \in L_4(e)$, one would obtain, by glueing a copy of $\mathbb{D}^4$ with W along their boundaries using φ, a 4-dimensional PL-manifold with vanishing w_1 and w_2, contradicting Rokhlin's theorem (see the proof of 4.2.5 for more details). To overcome this problem, we use the periodicity of surgery obstructions. Namely, by the Realization Theorem, let (M, h, F) a degree one normal map over $(\mathbb{D}^8, \varepsilon^8)$ with surgery obstruction $1 \in L_8(e)$. Let $(K, f, E) = (T(J) \times \mathbb{D}^n \times \mathbb{CP}^2 \times [0,1]) \coprod (M, h, F)$, and $\xi(J) = S(K \times T(J^c), f \times Id, E \times D) \in L_{m+n+5}(\mathbb{Z}^k) = L_{m+n+1}(\mathbb{Z}^k)$.

- If $m+n=1$, choose a degree one normal map $(h, F) : (\mathbb{S}^1 \times \mathbb{S}^1, \varepsilon^2) \to (\mathbb{S}^2, \varepsilon^2)$ with nonzero Arf invariant. Let K be obtained from $T(J) \times [0,1]^n \times [0,1]$ by taking the connected sum with $\mathbb{S}^1 \times \mathbb{S}^1$ in the interior. This yields the degree one normal

$$\begin{array}{c} K = (T(J) \times [0,1]^n \times [0,1]) \sharp (\mathbb{S}^1 \times \mathbb{S}^1) \\ \Big\downarrow {\scriptstyle Id \,\sharp\, h} \\ (T(J) \times [0,1]^n \times [0,1]) \sharp \mathbb{S}^2 = T(J) \times [0,1]^n \times [0,1] \end{array}$$

with framing the connected sum of the framings. Denote (K, f, E) this degree one normal map. By definition of the surgery obstructions,

$$S_{T(J) \times [0,1]^n \times [0,1]}(K, f, E) = L_2 i(S_{\mathbb{S}^2}(\mathbb{S}^1 \times \mathbb{S}^1, h, F)),$$

where $i : e \hookrightarrow \mathbb{Z}$ is the inclusion. Define

$$\xi(J) = S_{\mathbb{T}^k \times \mathbb{D}^n \times [0,1]}(K \times T(J^c), f \times Id, E \times D) \in L_{n+k+1}(\mathbb{Z}^k)$$

Note that we could have crossed with $\mathbb{CP}^2$ to define all the $\xi(J)$. The periodicity in surgery obstructions ensures that this does not change anything. This computation will be useful while studying the action of an element $\xi(J)$ on $\mathcal{S}_{PL}(\mathbb{T}^k \times \mathbb{D}^n)$. We now prove that we have constructed a set of generators.

Proposition 4.2.1. *Every element of $L_{n+k+1}(\mathbb{Z}^k)$ has a unique expression $\sum a(J)\xi(J)$, where J is a non empty subset of $\{1,\dots,k\}$ such that $|J|+n \equiv 1$ (mod 2), and $a(J)\in\mathbb{Z}$ if $|J|+n\equiv 3$ (mod 4) (resp. $a(J)\in\mathbb{Z}_2$ if $|J|+n\equiv 1$ (mod 4).*

Proof. We first define a map on L-groups which corresponds to the geometric operation of splitting the degree one normal map representing an element $\xi(J)$ along subtori.
Let $J\subset H\subset\{1,\dots,n\}$ (and denote $m=|J|$, $l=|H|$), with $m+n\equiv 1$ (*mod* 2). Just as before we define elements $\xi(H,J)\in L_{l+n+1}(\mathbb{Z}^l)$ by crossing with $T(H-J)$ instead of $T(J^c)$. Let suppose first that $m=l-1$. We define a map $\alpha(J,H)$: $L_{m+n+2}(\pi_1(T(H)))\to L_{m+n+1}(\pi_1(T(J)))$ as follows. By the Realization Theorem, consider a degree one normal map (W,h,F) over $(T(H)\times\mathbb{D}^n\times\mathbb{CP}^2\times[0,1],\varepsilon^{m+n+6})$. Now apply the Splitting Theorem to make $h_{|\partial W\times\mathbb{CP}^2}$ transverse to $T(J)\times\mathbb{D}^n\times\mathbb{CP}^2\times\{0\}$ and $T(J)\times\mathbb{D}^n\times\mathbb{CP}^2\times\{1\}$. This gives a degree one normal map (K,f,E) over $(T(J)\times\mathbb{D}^n\times\mathbb{CP}^2\times[0,1],\varepsilon^{m+n+5})$, and define $\alpha(J,H)x=S(K,f,E)$. This yields a map $L_{m+n+6}(\pi_1(T(H)))\to L_{m+n+5}(\pi_1(T(J)))$. By periodicity of surgery obstruction, this defines the desired map. Note that if $m+n\geq 5$, it is not necessary to cross with $\mathbb{CP}^2$ to use the Splitting Principle, and the periodicity of surgery obstructions implies we would define the same element.
For arbitrary $J\subset H\subset\{1,\dots,n\}$, we define $\alpha(J,H)$ as follows. Choose the unique sequence $J=J_0\subset\cdots\subset J_s=H$ with $|J_{i+1}|=|J_i|+1$ and $max(J_i-J)<max(J_{i+1}-J)$, and set $\alpha(J,H)=\alpha(J_0,J_1)\circ\cdots\circ\alpha(J_{s-1},J_s)$ (with the convention $\alpha(J,J)=Id$). The choice of the filtration is actually irrelevant, we only fix it to have maps defined with no ambiguity.

By definition of $\xi(H,J)$ and $\alpha(J,H)$, we have immediately

Lemma 4.2.2. *For every $J\subset K\subset L$ with $|J|+n$ odd,*

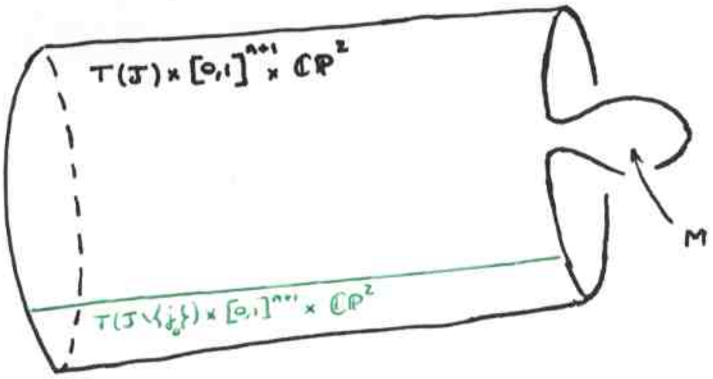
$$\alpha(K,L)\xi(L,J)=\xi(K,J).$$

Moreover, we have

Lemma 4.2.3. *For every $J\subset L, H\subset L, J\subsetneq K$ with $|J|+n$ odd,*

$$\alpha(K,L)\xi(L,J)=0.$$

Proof. Let $j_0 \in J - K$. Recall that we defined $\xi(J)$ as $S(W, \varphi, E)$, where $(W, \varphi, E) = (M, h, F) \coprod (T(J) \times [0,1]^{n+1} \times \mathbb{CP}^2)$, (M, h, F) representing an element of $B_{|J|+n+5}([0,1]^{|J|+n+5}, \varepsilon)$ with $S(M, h, F)$ the chosen generator of $L_{|J|+n+5}(e)$. We can take the boundary connected sum along a disk that misses $T(J - \{j_0\}) \times [0,1]^{n+1} \times \mathbb{CP}^2$,



so φ restricts to a PL-homeomorphism on $T(J - \{j_0\}) \times [0,1]^{n+1} \times \mathbb{CP}^2$. We then obtain $\xi(L, J)$ by crossing with $T(L - J)$. But now, since $T(K) \subset T(J - \{j_0\})$, we can assume $\varphi \times Id_{T(L-J)}$ restricts to a PL-homeomorphism on $T(K) \times [0,1]^{n+1} \times \mathbb{CP}^2 \times T(L - J)$, and the surgery obstrution of such a restriction is precisely $\alpha(K, L)\xi(L, J)$ by definition of $\alpha(K, L)$. Hence $\alpha(K, L)\xi(L, J) = 0$. $\square$

Let $w(J) : L_{|J|+n+1}(\pi_1(T(J))) \to L_{|J|+n+1}(e)$ be the natural projection, and let $\delta_{H,J}$ be the chosen generator of $L_{|J|+n+1}(e)$ if $J = H$ and $|J| + n$ is odd, 0 otherwise. Then

Lemma 4.2.4. *For every subsets J, H such that $|J| + n$ is odd,*

$$w(H)\alpha(H)\xi(J) = \delta_{H,J}.$$

Proof. The case $J = H$ is obvious. If $J \not\subset H$, $\alpha(H)\xi(J) = 0$. Finally, if $J \subsetneq H$, $\alpha(H)\xi(J) = \xi(H, J)$ by 4.2.2. But $\xi(H, J)$ is obtained by crossing a degree one normal map with a torus, whose signature and De Rham invariant vanish. Hence, by the formula for simply connected surgery obstructions 3.1.7, $w(J)\alpha(H, J) = 0$. $\square$

We are now able to prove 4.2.1. Let A the abelian group with genetors the subsets $J \subset \{1, \dots, n\}$ with $|J| + n$ odd, and relations $2J = 0$ for $|J| + n \equiv 1\ (mod\ 4)$. Define a map $\rho : L_{n+k+1}(\mathbb{Z}^k) \to A, \xi \mapsto \sum (w(J)\alpha(J)\xi)J$. ρ is surjective since $\rho(\sum a(J)\xi(J)) = \sum a(J)J$. But by 3.1.5, these two groups are isomorphic. Hence ρ is an isomorphism, which conludes 4.2.1. $\square$

We now have to understand which elements of $L_{n+k+1}(\mathbb{Z}^k)$ stabilize $[Id_{\mathbb{T}^k\times\mathbb{D}^n}]$. First of all, note that if $|J|+n$ is an odd integer ≥ 5 the construction of $\xi(J)$ directly shows that $\xi(J)$ stabilizes $[Id_{\mathbb{T}^k\times\mathbb{D}^n}]$. Recall that we constructed two degree one normal maps

$$h : M \to [0,1]^{m+n+1}$$

$$Id \sharp h : (T(J)\times[0,1]^n\times[0,1]) \sharp M \to (T(J)\times[0,1]^n\times[0,1]) \sharp [0,1]^{m+n+1}$$

and constructed $\xi(J)$ by crossing with $T(J^c)$ and taking the associated surgery obstruction. Now, by definition of the action of $L_{n+k+1}(\mathbb{Z}^k)$ on $\mathcal{S}_{PL}(\mathbb{T}^k\times\mathbb{D}^n)$, we have

$$\xi(J).[Id_{\mathbb{T}^k\times\mathbb{D}^n}] = \left[\begin{array}{c} (T(J)\times[0,1]^n\times\{1\}\sharp\partial M)\times T(J^c) \\ \\ \downarrow (Id_{T(J)\times[0,1]^n\times\{1\}}\sharp\partial h)\times Id_{T(J^c)} \\ \\ (T(J)\times[0,1]^n\times\{1\}\sharp\partial[0,1]^{m+n+1})\times T(J^c) \end{array}\right]$$

But $(Id_{T(J)\times[0,1]^n\times\{1\}}\sharp\partial h)\times Id_{T(J^c)}$ is clearly a PL-homeomorphism, hence $\xi(J).[Id_{\mathbb{T}^k\times\mathbb{D}^n}] = [Id_{\mathbb{T}^k\times\mathbb{D}^n}]$.

The situation for $|J|+n=1$ is even easier. Since the connected sum of $T(J)\times[0,1]^n\times[0,1]$ with $\mathbb{S}^1\times\mathbb{S}^1$ was taken in the interior, it has no effect on the boundary, so the restriction of the degree one normal map obtained after taking the connected sum is a PL-homeomorphism when restricted to the boundary, and the degree one normal map obtained after crossing with $T(J^c)$ restricts to a PL-homeomorphism on the boundary, hence $\xi(J).[Id_{\mathbb{T}^k\times\mathbb{D}^n}] = [Id_{\mathbb{T}^k\times\mathbb{D}^n}]$.

The only remaining case is $|J|+n=3$. Recall that since there exists no 4-dimensional PL-manifold with vanishing w_1 and w_2 and signature 8 by Rokhlin's theorem, we were forced to cross with $\mathbb{CP}^2$, so the previous argument does not apply. However, there exists a 4-dimensional PL-manifold with vanishing w_1 and w_2 and signature 16 (references ?), which yields a degree one normal map (W,h,F) over $(\mathbb{D}^4,\varepsilon^4)$ with obstruction twice the chosen generator of $L_4(e)$. Once again the periodicity of surgery obstructions shows that we could have defined $2\xi(J)$ starting with this normal map, taking the boundary connected sum with $T(J)\times\mathbb{D}^n\times[0,1]$, crossing with $T(J^c)$ and evaluating the surgery obstruction of the resulting degree one normal map. Hence the same reasoning as for $|J| + n \geq 5$ implies $2\xi(J).[Id_{\mathbb{T}^k\times\mathbb{D}^n}] = [Id_{\mathbb{T}^k\times\mathbb{D}^n}]$.

We are now going to use Rokhlin's theorem to prove that there are essentially the only elements of $L_{n+k+1}(\mathbb{Z}^k)$ acting trivially on $\mathcal{S}_{PL}(\mathbb{T}^k\times\mathbb{D}^n)$. Namely we have

Proposition 4.2.5. *Let $\xi = \sum a(J)\xi(J) \in L_{n+k+1}(\mathbb{Z}^k)$. Then ξ acts trivially on $[Id_{\mathbb{T}^k\times\mathbb{D}^n}]$ if and only if $a(J)$ is even whenever $|J| + n = 3$.*

Proof. Suppose we have an element $\xi = \sum a(J)\xi(J) \in L_{n+k+1}(\mathbb{Z}^k)$ acting trivially on $[Id_{\mathbb{T}^k\times\mathbb{D}^n}]$ and a subset H with $|H| + n = 3$ and $a(H)$ odd. The idea here will be, given a degree one normal map representing ξ, to lower the dimension using splitting ideas and derive a contradiction to Rokhlin's theorem. Since even multiples of $\xi(H)$ act trivially, we can assume $a(H) = 1$. By the Realization Theorem, let (W, φ, F) be a degree one normal map over $(\mathbb{T}^k \times \mathbb{D}^n \times [0,1], \varepsilon^{n+k+1})$

$$\varphi : (W, \partial_0 W, \partial_1 W) \to (\mathbb{T}^k\times\mathbb{D}^n\times[0,1], \mathbb{T}^k\times\mathbb{D}^n\times\{0\}\cup\mathbb{T}^k\times\mathbb{S}^{n-1}\times[0,1], \mathbb{T}^k\times\mathbb{D}^n\times\{1\})$$

with $\varphi_{|\partial_0 W}$ a PL-homeomorphism. Since ξ acts trivially, we can assume $\varphi_{|\partial_1 W}$ is also a PL-homeomorphism, hence $\varphi_{|\partial W}$ is a PL-homeomorphism. So we can homotop φ rel ∂W to make it transverse to $T(H) \times \mathbb{D}^n \times [0,1]$, yielding a degree one normal map (P, f, E) over $(T(H) \times \mathbb{D}^n \times [0,1], \varepsilon^4)$, with $f_| : \partial P \to \partial(T(H) \times \mathbb{D}^n \times [0,1])$ a PL-homeomorphism. Note that this coincides with the definition of $\alpha(H)$, namely $S((P, f, E) \times \mathbb{CP}^2) = \alpha(H)\xi$. The important fact here is that we did not use the splitting lemma since φ was already a PL-homeomorphism when restricted to the boundary, hence we did not have to cross with $\mathbb{CP}^2$, which allows us to work in dimension 4. We thus have

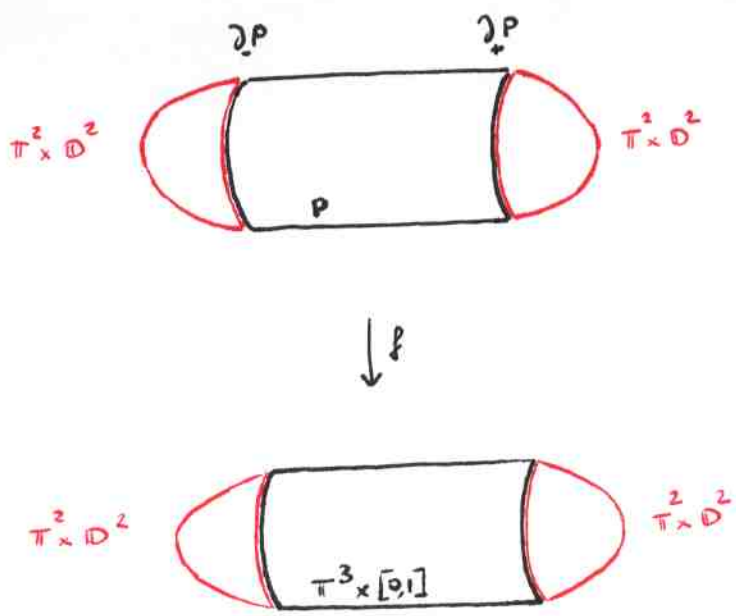
$$I((P, f, E) \times \mathbb{CP}^2) = w(H)S((P, f, E) \times \mathbb{CP}^2) = w(H)\alpha(H)\xi = a(H) = 1.$$

By periodicity of simply connected surgery obstructions, this implies $I(P, f, E) = 1$. After performing surgeries on it, we can assume that f induces an isomorphism on π_1.

Let us work out the cas $n = 0$. We will explain later how to adapt the proof in the other cases.
Since $\mathbb{T}^3 = \mathbb{T}^2 \times \mathbb{S}^1$, we can glue two copies of $\mathbb{T}^2 \times \mathbb{D}^2$ along $\partial_- P$ and $\partial_+ P$ using

f. Denote by W the PL-manifold obtained, and let Q denote the image of the two copies of $\mathbb{T}^2 \times \mathbb{D}^2$ in W. We then have a degree one normal map

$$g : W \to (\mathbb{T}^2 \times \mathbb{D}^2) \cup (\mathbb{T}^3 \times [0,1]) \cup (\mathbb{T}^2 \times \mathbb{D}^2) = \mathbb{T}^2 \times \mathbb{S}^2.$$



Let $K_2(W)$ (resp. $K_2(P)$) denote the kernel of $f_* : H_2(W) \to H_2(\mathbb{T}^2 \times \mathbb{S}^2)$ (resp $f_* : H_2(P) \to H_2(\mathbb{T}^3 \times [0,1])$).

Lemma 4.2.6. $S_{\mathbb{T}^2\times\mathbb{S}^2}(W, g, D) = 1$.

Proof. By definition of surgery obstructions, it is clearly sufficient to prove that the inclusion $i : P \hookrightarrow W$ induces an isomorphism between $K_2(P)$ and $K_2(W)$. For simplicity, let $P' = \mathbb{T}^3 \times [0,1]$, $W' = \mathbb{T}^2 \times \mathbb{S}^2$, and Q' the two copies of $\mathbb{T}^2 \times \mathbb{D}^2$ in W'.

- $\iota_* : K_2(P) \to K_2(W)$ is surjective: By the Mayer-Vietoris exact sequence, we have

$$\begin{tikzcd}
H_2(P\cap Q) \arrow[r, "\alpha"] \arrow[d, "\approx"] & H_2(P)\oplus H_2(Q) \arrow[r, "i_*\oplus j_*"] \arrow[d] & H_2(W) \arrow[r, "\beta"] \arrow[d, "f_*"] & H_3(P\cap Q) \arrow[d, "\approx"] \\
H_2(P'\cap Q') \arrow[r, "\alpha'"] & H_2(P)\oplus H_2(Q') \arrow[r, "i'_*\oplus j'_*"] & H_2(W') \arrow[r, "\beta'"] & H_3(P'\cap Q')
\end{tikzcd}$$

where the vertical isomorphisms come form the fact that f restricts to a PL-homeomorphism on Q and $P \cap Q$. Let $x \in K_2(W)$. The commutativity of the right square and the fact that the map $H_3(P \cap Q) \to H_3(P' \cap Q')$ is an isomorphism imply that x is sent to 0 under the map $H_2(W) \to H_3(P \cap Q)$. By exactness, let $u \in H_2(P), v \in H_2(Q)$ such that $i_*(u) - j_*(v) = x$. The commutativity of the middle square and the fact that $x \in K_2(W)$ imply $(i'_* \oplus j'_*)(f_*(u), f_*(v)) = 0$. Hence by exactness there exists $s \in H_2(P \cap Q)$ such that $\alpha'(s) = (f_*(u), f_*(v))$. Let t denote the antecedent of s under the left vertical isomorphism. By commutativity of the left square and the fact that $H_2(Q) \to H_2(Q')$ is an isomorphism, $\alpha(s) = (u+y, v)$, with $y \in K_2(P)$. But by exactness, $(i'_* \oplus j'_*)(u+y, v) = 0$, and $x = -i_*(y)$.

- $\iota_* : K_2(P) \to K_2(W)$ is injective: By the Mayer-Vietoris exact sequence, we have

$$\begin{tikzcd}
H_1(P \cap Q) \arrow[r, "\alpha"] \arrow[d, "(1)"', "\approx"] & H_1(P) \oplus H_1(Q) \arrow[r, "i_* \oplus j_*"] \arrow[d, "(2)"', "\approx"] & H_1(W) \arrow[r, "\beta"] \arrow[d, "(3)"', "f_*"] & H_2(P \cap Q) \arrow[r, "\gamma"] \arrow[d, "(4)"', "\approx"] & H_2(P) \oplus H_2(Q) \arrow[d, "(5)"'] \\
H_1(P' \cap Q') \arrow[r, "\alpha'"] & H_1(P) \oplus H_2(Q') \arrow[r, "i'_* \oplus j'_*"] & H_1(W') \arrow[r, "\beta'"] & H_2(P' \cap Q') \arrow[r, "\gamma'"] & H_2(P) \oplus H_2(Q)
\end{tikzcd}$$

where the vertical isomorphisms come form the fact that f restricts to a PL-homeomorphism on Q and $P \cap Q$, and the assumption that f induces a π_1 isomorphism. Let $x \in K_2(P)$ such that $\iota_*(x) = 0$. By exactness, there exists $y \in H_2(P \cap Q)$ such that $\gamma(y) = (x, 0)$. By commutativity of the right square, the image z of y under the right vertical isomorphism is sent to 0, so there exists $v \in H_1(W')$ such that $\beta'(v) = z$. Now we use the following classical fact on exact sequences: In the previous situation, if the vertical maps (1), (2) and (4) are surjective, the map (3) is injective. Thus the map $H_1(W) \to H_1(W')$ is injective. But this argument works equally for homology with coefficients in a finite field. Thus the map $H_1(W) \otimes \mathbb{Z}_p \xrightarrow{f_* \otimes Id} H_1(W') \otimes \mathbb{Z}_p$ is injective, hence surjective, for every prime p. Now this implies that the map $H_1(W) \to H_1(W')$ is surjective. Let $u \in H_1(W)$ be an antecedent of v under this map. By commutativity of the second square from the right and the fact that $H_2(P \cap Q) \to H_2(P' \cap Q')$ is an isomorphism, we have $\beta(u) = y$, so $(x, 0) = \gamma(y) = \gamma \circ \beta(u) = 0$ by exactness.

□

Since $\sigma(\mathbb{T}^2 \times \mathbb{S}^2) = 0$, the formula for simply connected surgery obstructions gives $\sigma(W) = 8$. Now in order to apply Rokhlin's theorem, we want to prove that $w_1(W) = 0, w_2(W) = 0$. Since g is a degree one normal map and $\tau_{\mathbb{T}^3 \times [0,1]} = 0$, τ_P is trivial, hence its Stiefel-Whitney classes vanish. Thus, by the long exact sequence of the pair (W, P), it is sufficient to prove the following

Lemma 4.2.7. *(i)* $H^1(W, P; \mathbb{Z}_2) = 0$;

(ii) $\delta : H^1(P; \mathbb{Z}_2) \to H^1(W, P; \mathbb{Z}_2)$ *is onto.*

Proof. (i) We have

$$\begin{aligned} H^1(W, P; \mathbb{Z}_2) &= H^1(Q, P \cap Q; \mathbb{Z}_2) && \text{(by excision)} \\ &= H_3(Q; \mathbb{Z}_2) && \text{(by Lefschetz duality)} \\ &= 0 \end{aligned}$$

(ii) We have the following commutative diagram

$$\begin{array}{ccc} H^1(P; \mathbb{Z}_2) & \xrightarrow{\delta} & H^2(W, P; \mathbb{Z}_2) \\ {\scriptstyle f_*}\big\downarrow & & \big\downarrow{\scriptstyle f_*} \\ H^1(P'; \mathbb{Z}_2) & \xrightarrow{\delta} & H^2(W', P'; \mathbb{Z}_2). \end{array}$$

The left vertical arrow is an isomorphism by hypothesis on f. The right vertical arrow is an isomorphism by excision and the fact that f is an PL-homeomorphism when restricted to Q and $P \cap Q$. Hence it is equivalent to proving the surjectivity of the lower horizontal map. Now, by the exact sequence of the pair (W', P'), we have

$$H^1(W', P'; \mathbb{Z}_2) \longrightarrow H^1(W'; \mathbb{Z}_2) \longrightarrow H^1(P'; \mathbb{Z}_2) \xrightarrow{\delta} H^2(W', P'; \mathbb{Z}_2)$$

Furthermore,

$$\begin{aligned} H^2(W', P'; \mathbb{Z}_2) &= H^2(Q', P' \cap Q; \mathbb{Z}_2) && \text{(by excision)} \\ &= H_2(Q'; \mathbb{Z}_2) && \text{(by Lefschetz duality)} \\ &= H_2(\mathbb{T}^2 \times \mathbb{D}^2; \mathbb{Z}_2) \oplus H_2(\mathbb{T}^2 \times \mathbb{D}^2; \mathbb{Z}_2) \\ &= \mathbb{Z}_2 \oplus \mathbb{Z}_2 \end{aligned}$$

Hence, the symmetry of the problem implies that δ is either surjective or 0. If it was 0, then we would have the surjection $H^1(W';\mathbb{Z}_2) \simeq \mathbb{Z}_2^2 \to \mathbb{Z}_2^3 \simeq H^1(P';\mathbb{Z}_2)$, contradiction.

$\square$

Thus we have constructed a 4-dimensional PL-manifold with vanishing w_1 and w_2 and signature 8. It admits a PL-structure by the Smoothing Theorem 3.3.3. This in turn contradicts the Rokhlin's theorem.

We now explain how to obtain such a contraction for the other values of n. The explicit constructions involved are used only to prove:
- $H_3(Q',\mathbb{Z}_2) = 0$,
- the computation of the various groups in the sequence $H^1(W',P';\mathbb{Z}_2) \to H^1(W';\mathbb{Z}_2) \to H^1(P';\mathbb{Z}_2) \xrightarrow{\delta} H^2(W',P';\mathbb{Z}_2)$ to prove the surjectivity of δ,
- $\sigma(W') = 0$.

First remark that the Splitting Theorem applies in the other cases, even if we are considering manifolds with boundaries. We then start with a degree one normal map $f : P \to \mathbb{T}^{3-n} \times \mathbb{D}^n, n = 0, 1, 2$.

- $n = 1$:
  Glue $P' = \mathbb{T}^2 \times \mathbb{D}^2$ and $Q' = \mathbb{T}^2 \times \mathbb{D}^2$ along their common boundary $\mathbb{T}^2 \times \mathbb{D}^2$, to obtain $W' = \mathbb{T}^2 \times \mathbb{S}^2$. Nothing is changed, except in proving that δ is onto. it is now immediate that δ is either surjective or zero since $H_2(Q';\mathbb{Z}_2) = \mathbb{Z}_2$. If it was zero, we would have

  $$\xrightarrow{\delta=0} H^2(W',P';\mathbb{Z}_2) \xrightarrow{\approx} H^2(P';\mathbb{Z}_2) \xrightarrow{0} H^2(W';\mathbb{Z}_2) \hookrightarrow H^3(W',P';\mathbb{Z}_2)$$

  but the last injection is impossible by cardinality.

- $n = 2$:
  Glue $\mathbb{T}^1 \times \mathbb{D}^3$ and $Q' = \mathbb{D}^2 \times \mathbb{S}^2$ along their common boundary $\mathbb{S}^1 \times \mathbb{S}^2$ to obtain $W' = \mathbb{S}^4$. $\sigma(\mathbb{S}^4) = 0$, and $H_3(Q';\mathbb{Z}_2) = 0$. Furthermore $H_2(Q';\mathbb{Z}_2) = \mathbb{Z}_2$, so δ is again either surjective or zero. If it was zero, we would have a surjection $H^1(W';\mathbb{Z}_2) \approx 0 \to \mathbb{Z}_2 \approx H^1(P';\mathbb{Z}_2)$.

- $n = 3$:
  Glue $P' = \mathbb{D}^4$ and $Q' = P'$ along their common boundary. This time all groups involved are zero and the result follows.

Finally for $n \geq 4$ the previous construction of $\xi(J)$ shows that every element of $L_{n+k+1}(\mathbb{Z}^k)$ acts trivially on $\mathcal{S}_{PL}(\mathbb{T}^k \times \mathbb{D}^n)$, which conludes the proof of 4.2.5. $\square$

We are now able to give a simple description of $\mathcal{S}_{PL}(\mathbb{T}^k \times \mathbb{D}^n)$.

Definition 4.2.8. For $|J| + n = 3$, let

$$\lambda_J(\xi) = w(J)\alpha(J)\xi(mod 2)$$

called a geometric coordinate of ξ.

Consider a basis $t_1, \ldots, t_k$ of $H^1(\mathbb{T}^k; \mathbb{Z}_2)$. For a subset $J = \{i_1, \ldots, i_{|J|} | i_1 < \ldots < i_{|J|}\} \subset \{1, \ldots, k\}$ with $|J| + n = 3$, let $t_J = t_1 \wedge \ldots \wedge t_{|J|}$. Then the (t_J) form a basis of $H^{3-n}(\mathbb{T}^k \times \mathbb{D}^n; \mathbb{Z}_2)$. We have:

Theorem 4.2.9. *Let*

$$\lambda^* : \mathcal{S}_{PL}(\mathbb{T}^k \times \mathbb{D}^n) \to H^{3-n}(\mathbb{T}^k; \mathbb{Z}_2), \quad x \mapsto \sum_{|J|+n=3} \lambda_J(\xi) t_J$$

where $\xi \in L_{n+k+1}(\mathbb{Z}^k)$ *is such that* $\xi.[Id_{\mathbb{T}^k \times \mathbb{D}^n}] = x$. *Then* λ^* *is a well-defined bijection.*

Proof. Immediate from the above discussion. $\square$

Chapter 5

PL-structures on topological manifolds.

Considering a homotopy equivalence $f : M \to \mathbb{T}^k$, we want to prove that M is actually homeomorphic to $\mathbb{T}^k$. Such a homeomorphism naturally endowes M with a PL-structure. Our first task is then to prove that M indeed admits a PL-structure. Note that a PL-structure on M gives rise to a PL-structure on its (topological) tangent bundle $\tau_M : M \to BTOP$. In other words, there exists a lift

$$\begin{tikzcd} & BPL \arrow[d] \\ M \arrow[r, "\tau_M"'] \arrow[ur, dotted] & BTOP. \end{tikzcd}$$

By considering the fibration $TOP/PL \to BPL \to BTOP \to BTOP/PL$, we see that this is the case if and only if the map $M \xrightarrow{\tau_M} BTOP \to BTOP/PL$ is null-homotopic. Furthermore, in the case where such a lift exists, the various bundle reductions are then classified by $[M, TOP/PL]$. Hence it is necessary to study the homotopy properties of TOP/PL to carry out our program.
More precisely, we are going to prove the following

Theorem. *TOP/PL has the homotopy type of an Einlenberg-Mac Lane space $K(\mathbb{Z}_2, 3)$.*

Note that this theorem will allow us to answer our previous question. Namely, we will prove the following

Obstruction Theorem. *A topological manifold admits a PL-structure if and only if a certain obstruction $\kappa(M) \in H^4(M, \mathbb{Z}_2)$ vanishes. In that case, the different PL-structures are in (unnatural) bijection with $H^3(M, \mathbb{Z}_2)$.*

We will see in Chapter 6 how to compute this obstruction in the situation relevant to our study. Note that the Obstruction Theorem, which can be thought as a first step in our attempt to classify topological manifolds homotopy equivalent to the torus, relies heavily on the PL-classification of PL-manifolds homotopy equivalent to the torus made in the previous section. The fundamental ingredient which made possible such an intermediary classification was Rokhlin's theorem, which roughly speaking states that there is a factor 2 between TOP- and PL-manifolds. The theorem giving the homotopy type of TOP/PL asserts that this is essentially the only difference. Furthermore, it will in turn allow us to deduce a topological classification. Following [HS70], we describe in Section 5.1 some properties of finite coverings of PL-homotopy tori needed to compute the homotopy type of TOP/PL, which will be done in Section 5.2. The Obstruction Theorem will finally be proven in Section 5.3. The last two sections are greatly inspired by [Rud01].

5.1 Finite coverings of PL-homotopy tori

We are now going to study the effect of a finite covering on the PL-structure set. Namely, we want to prove that a PL-manifold homotopy equivalent to $\mathbb{T}^k \times \mathbb{D}^n$ ($n+k \geq 5$, $n \neq 3$) has a finite cover PL-homeomorphic to $\mathbb{T}^k \times \mathbb{D}^n$. We first define what coverings we will be interested in.

Definition 5.1.1. Let $p : \mathbb{T}^k \times \mathbb{D}^n \to \mathbb{T}^k \times \mathbb{D}^n$ be a finite covering map. We say that p is *nice* if there exists integers $d_1, \ldots, d_k$ such that $p(x_1, \ldots, x_k, y_1, \ldots, y_n) = (x_1^{d_1}, \ldots, x_k^{d_k}, y_1, \ldots, y_n)$.

Let $p : \mathbb{T}^k \times \mathbb{D}^n \to \mathbb{T}^k \times \mathbb{D}^n$ be a nice finite covering. It can be used to pullback several objects:

- There is the naturel pullback map $p^* : H^{3-n}(\mathbb{T}^k \times \mathbb{D}^n; \mathbb{Z}_2) \to H^{3-n}(\mathbb{T}^k \times \mathbb{D}^n; \mathbb{Z}_2)$.
- Consider $f : M \to \mathbb{T}^k \times \mathbb{D}^n$ representing an element of $\mathcal{S}_{PL}(\mathbb{T}^k \times \mathbb{D}^n)$. Since every nice finite covering of $\mathbb{T}^k \times \mathbb{D}^n$ is PL-homeomorphic to $\mathbb{T}^k \times \mathbb{D}^n$, we

have the following pullback diagram

$$\begin{tikzcd}
\bar{M} \arrow[r, "\bar{f}"] \arrow[d, "f^*p"'] & \mathbb{T}^k \times \mathbb{D}^n \arrow[d, "p"] \\
M \arrow[r, "f"'] & \mathbb{T}^k \times \mathbb{D}^n
\end{tikzcd}$$

and $\bar{f} : \bar{M} \to \mathbb{T}^k \times \mathbb{D}^n$ is a homotopy equivalence. Let $p^t : \mathcal{S}_{PL}(\mathbb{T}^k \times \mathbb{D}^n) \to \mathcal{S}_{PL}(\mathbb{T}^k \times \mathbb{D}^n)$ the induced map.

- Let $x \in L_{n+k+1}(\mathbb{Z}^k)$, realized by a degree one normal map $(h, F) : (M, \nu_M) \to (\mathbb{T}^k \times \mathbb{D}^n \times [0,1], \varepsilon^{n+k+1})$. We then the have comutative diagram

$$\begin{tikzcd}
\widetilde{M} \arrow[r, "\tilde{h}"] \arrow[d, "h^*(p\times Id)"'] & \mathbb{T}^k \times \mathbb{D}^n \times [0,1] \arrow[d, "p\times Id"] \\
M \arrow[r, "h"'] & \mathbb{T}^k \times \mathbb{D}^n \times [0,1]
\end{tikzcd}$$

Let h^*F the induced framing of $\tau_{\widetilde{M}} \oplus \nu_{\mathbb{T}^k\times\mathbb{D}^n\times[0,1]}$. We then have a degree one normal map $(\tilde{h}, h^*F) : (\widetilde{M}, \nu_M) \to (\mathbb{T}^k \times \mathbb{D}^n \times [0,1], \varepsilon^{n+k+1})$. One easily checks that this defines maps

$$p^! : L_{n+k+1}(\mathbb{Z}^k) \to L_{n+k+1}(\mathbb{Z}^k)$$

$$p^\sharp : B_{n+k+1}(\mathbb{T}^k \times \mathbb{D}^n \times [0,1], \varepsilon^{n+k+1}) \to B_{n+k+1}(\mathbb{T}^k \times \mathbb{D}^n \times [0,1], \varepsilon^{n+k+1}).$$

Furthermore, the following diagram is clearly commutative

$$\begin{tikzcd}
B_{n+k+1}(\mathbb{T}^k \times \mathbb{D}^n \times [0,1], \varepsilon^{n+k+1}) \arrow[r, "S"] \arrow[d, "p^\sharp"] & L_{n+k+1}(\mathbb{Z}^k) \arrow[r] \arrow[d, "p^!"] & \mathcal{S}_{PL}(\mathbb{T}^k \times \mathbb{D}^n) \arrow[d, "p^t"] \\
B_{n+k+1}(\mathbb{T}^k \times \mathbb{D}^n \times [0,1], \varepsilon^{n+k+1}) \arrow[r, "S"] & L_{n+k+1}(\mathbb{Z}^k) \arrow[r] & \mathcal{S}_{PL}(\mathbb{T}^k \times \mathbb{D}^n)
\end{tikzcd}$$

We want to understand the map p^t. In order to do that, we study the effect of a pullback on the geometric coordinates of an obstruction. By the previous computation of $\mathcal{S}_{PL}(\mathbb{T}^k \times \mathbb{D}^n)$, we can suppose from now on that $0 \le n \le 2$.

We start with the simplest nice covering map, namely

$$p : \mathbb{T}^k \times \mathbb{D}^n \to \mathbb{T}^k \times \mathbb{D}^n, (x_1, \ldots, x_k, y_1 \ldots, y_n) \mapsto (x_1, \ldots, x_i^d, \ldots, x_k, y_1, \ldots, y_n).$$

Proposition 5.1.2. *Let $J \subset \{1, \ldots, k\}$ with $|J| + n = 3$. If $i \notin J$, then $p^! \xi(J) = \xi(J)$.*

Proof. Recall that by definition, $\xi(J) = S(K \times T(J^c), f \times Id, E \times D) \in L_{n+k+1}(\mathbb{Z}^k)$, with $f : K \to T(J) \times [0,1]$ and $Id : T(J^c) \to T(J^c)$. It is now clear that $p^\sharp(K \times T(J^c), f \times Id, E \times D) = (K \times T(J^c), f \times g_p, E \times D_p)$ where $g_p : T(J^c) \to T(J^c)$ take the i-th coordinate to the power of d, and D_p is the natural framing. But (g_p, D_p) and (Id, D) are clearly framed cobordant, so $p^\sharp(K \times T(J^c), f \times Id, E \times D)$ and $(K \times T(J^c), f \times Id, E \times D)$ are framed cobordant, hence their surgery obstructions coincide. $\square$

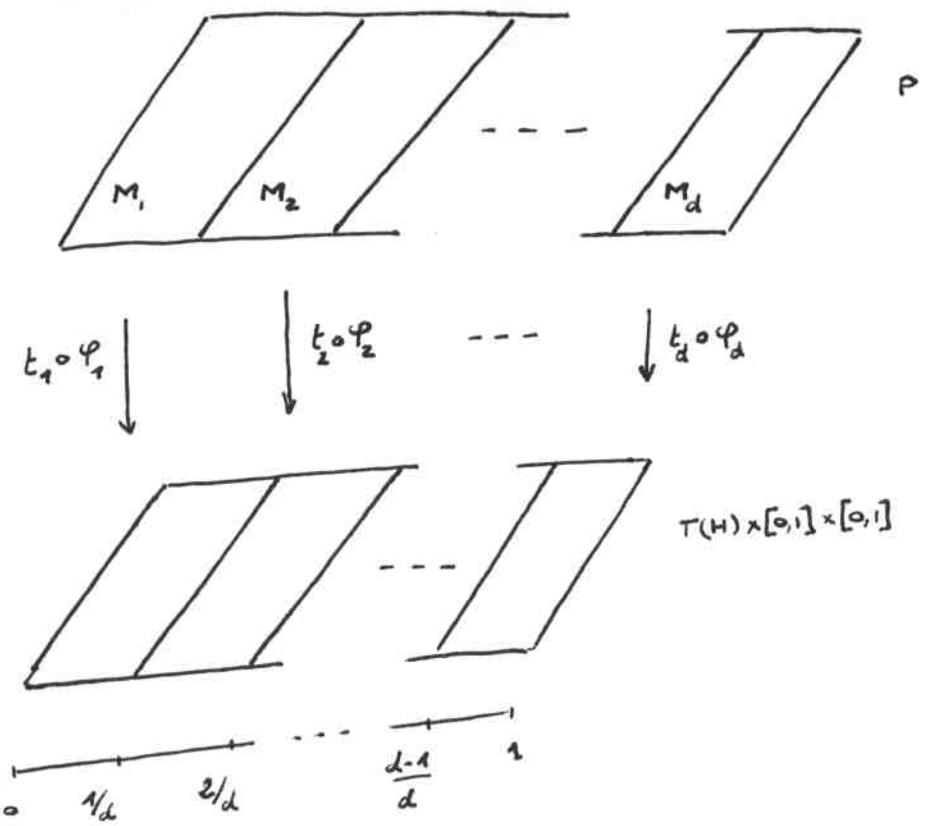
Proposition 5.1.3. *Let $J \subset \{1, \ldots, k\}$ with $|J| + n = 3$. If $i \in J$, then $p^! \xi(J) = d\xi(J)$.*

Proof. Let $H = 1, \ldots, n - i$. By 4.2.3, $\alpha(H)\xi(J) = 0$, hence by exactness of the following sequence

$$0 \to L_{n+k+1}(\pi_1(T(H) \times \mathbb{D}^n)) \xrightarrow{j*} L_{n+k+1}(\pi_1(\mathbb{T}^k \times \mathbb{D}^n)) \xrightarrow{\alpha(H)} L_{n+k}(\pi_1(T(H) \times \mathbb{D}^n)) \to 0$$

there exists $\xi \in L_{n+k+1}(\pi_1(T(H) \times \mathbb{D}^n))$ such that $\xi(J) = j_*(\xi)$. Let us describe geometrically the effect of $p^!$ on such an element.

Let $(h, F) : M \to (T(H) \times [0,1] \times [0,1], \varepsilon)$ a degree one normal map with surgery obstruction ξ and such that, with $\partial M = \partial_- M \cup \partial_+ M$, $h_| : \partial_- M \to (T(H) \times [0,1] \times \{0\}) \cup (T(H) \times \{0\} \times [0,1]) \cup (T(H) \times \{1\} \times [0,1])$ is a PL-homeomorphism. We can use it to identify the copy $T(H) \times \{0\} \times [0,1]$ and $T(H) \times \{1\} \times [0,1]$ in ∂M. By glueing them together, we obtain a normal map $(f, E) : (N, \nu_N) \to (T(H) \times \mathbb{S}^1 \times [0,1], \nu)$ whose surgery obstruction is $j_*(\xi) = \xi(J)$. Now it is clear that $p^\sharp(N, f, E)$ can be obtained as follows: Consider d copies (M_i, h_i, F_i) of (M, h, F). Glue together the copy $T(H) \times \{1\} \times [0,1]$ in ∂M_i and $T(H) \times \{0\} \times [0,1]$ in ∂M_{i+1}, $1 \leq i \leq d-1$, and denote P the space obtained. Let $t_i : T(H) \times [0,1] \times [0,1] \to T(H) \times [0,1] \times [0,1], (x, y, t) \mapsto (x, y, (t + i - 1)/d)$. Let $\varphi : (t_1 \circ f_1) \cup \ldots \cup (t_d \circ f_d)$ and $D = F_1 \cup \ldots \cup F_d$.



It is now not hard to see that $p^\sharp(N, f, E)$ is obtained by glueing together the the copy $T(H) \times \{0\} \times [0, 1]$ in ∂M_1 and $T(H) \times \{1\} \times [0, 1]$ in ∂M_d, i.e $p^\sharp(N, f, E) = j_*(S(P, \varphi, D))$. But by additivity of surgery obstructions, $S(P, \varphi, D) = dS(M, h, F)$, hence $p^! \xi(J) = p^! j_* S(M, h, F) = dj_* S(M, h, F) = d\xi(J)$. $\square$

We thus have proved the following

Proposition 5.1.4. *Let $J \subset \{1, \dots, k\}$ with $|J| + n = 3$. Then*

$$\lambda_H(p^t \partial \xi(J)) = \begin{cases} 1 & \text{if } H = J \text{ and } i \notin J \\ d & \text{if } H = J \text{ and } i \in J \\ 0 & \text{otherwise} \end{cases}$$

Recall that $p^* : H^{3-n}(\mathbb{T}^k; \mathbb{Z}_2) \to H^{3-n}(\mathbb{T}^k; \mathbb{Z}_2)$ is given by $p^*(t_J) = dt_J$ if $i \in J$, $p^*(t_J) = t_J$ sinon. Furthermore, every nice covering map $p(x_1, \dots, x_k, y_1, \dots, y_n) = (x_1^{d_1}, \dots, x_k^{d_k}, y_1, \dots, y_n)$ can be written as a composite of coverings we have been considering previously. Thus we have proved the following

Theorem 5.1.5. *Let $q : \mathbb{T}^k \times \mathbb{D}^n \to \mathbb{T}^k \times \mathbb{D}^n$ be a nice covering map, with $n+k \geq 5$ and $n \leq 2$. Then the following diagram commutes:*

$$\begin{tikzcd} \mathcal{S}_{PL}(\mathbb{T}^k \times \mathbb{D}^n) \arrow[r, "\lambda^*", "\simeq"'] \arrow[d, "q^t"'] & H^{3-n}(\mathbb{T}^k; \mathbb{Z}_2) \arrow[d, "q^*"] \\ \mathcal{S}_{PL}(\mathbb{T}^k \times \mathbb{D}^n) \arrow[r, "\simeq", "\lambda^*"'] & H^{3-n}(\mathbb{T}^k; \mathbb{Z}_2). \end{tikzcd}$$

Corollary 5.1.6. *Every PL-manifold homotopy equivalent to $\mathbb{T}^k \times \mathbb{D}^n$ ($n+k \geq 5$ and $n \neq 3$) has a finite covering PL-homeomorphic to $\mathbb{T}^k \times \mathbb{D}^n$.*

Proof. If $n \leq 2$, we have $q^* = 0$ for the nice covering map $q(x_1, \ldots, x_k, y_1, \ldots, y_n) = (x_1^2, \ldots, x_k^2, y_1, \ldots, y_n)$. If $n \geq 4$, every PL-manifold homotopy equivalent to $\mathbb{T}^k \times \mathbb{D}^n$ is actually PL-homeomorphic to $\mathbb{T}^k \times \mathbb{D}^n$ by 4.2.9. $\square$

5.2 The homotopy type of TOP/PL

First we give a bit of structure on the different PL-structures a topological manifold might carry.

Definition 5.2.1. Let M be a topological manifold whose boundary is a PL-manifold. A *PL-structuralization* is a homeomorphism $h : N \to M$ with N a PL-manifold and such that $h_{|\partial N} : \partial N \to \partial M$ is a PL-homeomorphism. Two PL-structuralizations $h_i : N_i \to M, i = 0, 1$ are called *concordant* if there exists a PL-homeomorphism $\varphi : N_0 \to N_1$ and a homeomorphism $H : N_0 \times [0,1] \to M \times [0,1]$ such that:
- $H_{N_0 \times 0} = h_0$
- $H_{N_0 \times 1} = h_1 \circ \varphi$
- $H : \partial N_0 \times [0,1] \to \partial M \times [0,1]$ coincides with $h_0 \times Id_{0,1}$.
A *PL-structure* on M is an equivalence class of PL-structuralizations. Let $\mathcal{T}_{PL}(M)$ denote the set of all PL-structures on M.

As we mentionned it earlier, a PL-structure on M yields a PL-structure on the topological stable tangent bundle. The converse is true, but requires the difficult theorem

Theorem 5.2.2 (Product Structure Theorem)**.** *For every $n \geq 5$ and every $k \geq 0$, the natural map $\mathcal{T}_{PL}(M) \to \mathcal{T}_{PL}(M \times \mathbb{R}^k)$ obtained by associating to any PL-structuralization $h : N \to M$ the PL-structuralization $h \times Id : N \times \mathbb{R}^k \to M \times \mathbb{R}^k$ is a bijection.*

The classical proof of the Structure Theorem uses the Stable Homeomorphism Theorem of Kirby [KS77] which in turn relies on the properties of PL-homotopy tori under finite converings. Thus surgery theory is the key ingredient in proving the Product Structure Theorem, which explains the dimension restiction $n \geq 5$. It fails in dimension 3. There is a corresponding theorem for DIFF-structures on PL-manifolds known as the Cairns-Hirsch theorem, which holds in any dimension.

A fundamental consequence of the Product Structure Theorem is the following

Theorem 5.2.3 (Classification Theorem)**.** *Let M be a topological manifold of dimension ≥ 5 which admits a PL-structure. Then there exists a bijection*

$$\sigma : [(M, \partial M), (TOP/PL, *)] \to \mathcal{T}_{PL}(M)$$

The proof can be found in [KS77]. Here is at least the construction of the maps in both senses.

Consider from now M as a PL-manifold. A PL-structuralization $h : N \to M$ yields an isomorphism of topological stable tangent bundles, since h is a homeomorphism. It also yields a new PL-structure on the stable tangent bundle on M. The situation is resumed in the following diagram, in which the upper triangle and both lower triangles are commutative.

$$\begin{tikzcd}
 & TOP/PL \arrow[d] \\
N \arrow[r] \arrow[d, "h"'] & BPL \arrow[d] \\
M \arrow[r, "\tau^{TOP}"'] \arrow[ur, "\tau^{PL}"'] \arrow[ur, dotted, bend left, "\tilde{\tau}^{PL}"] & BTOP
\end{tikzcd}$$

Thus we have two PL-structures on τ^{TOP}. Hence their difference yields an element $\tilde{\tau}^{PL} - \tau^{PL} \in [(M, \partial M), (TOP/PL, *)]$. We then define the map $\pi : g \mapsto \tilde{\tau}^{PL} - \tau^{PL}$.

Now given an element $\alpha \in [(M, \partial M), (TOP/PL, *)]$, we thus have a PL-bundle over M which is trivial, when seen as a TOP-bundle. Thus we have the following commutative diagram

$$\begin{tikzcd} M \times \mathbb{R}^p \arrow[rr, "H"] \arrow[dr] & & E(\alpha) \arrow[dl] \\ & M & \end{tikzcd}$$

Since $E(\alpha)$ is a PL-manifold, the homeomorphism H endowes $M \times \mathbb{R}^p$ with a PL-structure. But now by the Product Structure Theorem, this yields a PL-struture on M. More precisely, there is a homeomorphism $g : N \to M$ with N a PL-manifold such that H and $g \times Id_{\mathbb{R}^p}$ are concordant. We then define the map $\sigma : \alpha \mapsto g$.

Note that a PL-structuralization of M can be seen as representing an element of the structure set. In other words, there is a forgetful map $\beta : \mathcal{T}_{PL}(M) \to \mathcal{S}_{PL}(M)$. We have the following

Proposition 5.2.4. *Let $k, n \geq 0$ such that $n + k \geq 5$. Let $x, y \in \mathcal{T}_{PL}(\mathbb{T}^k \times \mathbb{D}^n)$ such that $\beta(x) = \beta(y) \in \mathcal{S}_{PL}(\mathbb{T}^k \times \mathbb{D}^n)$. Then there exists a finite covering $p : \mathbb{T}^k \times \mathbb{D}^n \to \mathbb{T}^k \times \mathbb{D}^n$ such that $p^*(x) = p^*(y)$.*

This amounts to proving the following lemma.

Lemma 5.2.5. *Let $k, n \geq 0$ such that $n + k \geq 5$. For any homeomorphismn $h : \mathbb{T}^k \times \mathbb{D}^n \to \mathbb{T}^k \times \mathbb{D}^n$ which is isotopic rel $\partial(\mathbb{T}^k \times \mathbb{D}^n)$ to the identity, there exists a nice finite covering $p : \mathbb{T}^k \times \mathbb{D}^n \to \mathbb{T}^k \times \mathbb{D}^n$ and a lift $\tilde{h} : \mathbb{T}^k \times \mathbb{D}^n \to \mathbb{T}^k \times \mathbb{D}^n$ isotopic rel $\partial(\mathbb{T}^k \times \mathbb{D}^n)$ to the identity, such that the following diagram commutes*

$$\begin{tikzcd} \mathbb{T}^k \times \mathbb{D}^n \arrow[r, "\tilde{h}"] \arrow[d, "p"] & \mathbb{T}^k \times \mathbb{D}^n \arrow[d, "p"] \\ \mathbb{T}^k \times \mathbb{D}^n \arrow[r, "h"'] & \mathbb{T}^k \times \mathbb{D}^n \end{tikzcd}$$

In other words, the proposition asserts that $\beta : \mathcal{T}_{PL}(\mathbb{T}^k \times \mathbb{D}^n) \to \mathcal{S}_{PL}(\mathbb{T}^k \times \mathbb{D}^n)$ is injective up to finite coverings.

Proof. Identify $\mathbb{T}^k$ with $(\mathbb{S}^1)^n$ and denote e its basepoint, and endow it with its invariant metric. For an integer $\lambda > 0$, consider the nice covering map $p_\lambda : \mathbb{T}^k \times \mathbb{D}^n \to \mathbb{T}^k \times \mathbb{D}^n, (x, y) \mapsto (x^\lambda, y)$. Let $h_t : \mathbb{T}^k \times \mathbb{D}^n \to \mathbb{T}^k \times \mathbb{D}^n$ be a homotopy rel $\partial(\mathbb{T}^k \times \mathbb{D}^n)$ between $h = h_0$ and $Id = h_1$ (for the special case $n = 0$, note that since $\pi_1(\mathbb{T}^k)$ acts trivially on $[\mathbb{T}^k, \mathbb{T}^k]$, we can assume that the h_t are basepoint preserving). The pullback covering $h^* p_\lambda$ is isomorphic to p_λ. Thus we have the following commutative diagram

$$\begin{tikzcd}
\mathbb{T}^k \times \mathbb{D}^n \arrow[r, "\approx"'] \arrow[drr, "p_\lambda"'] \arrow[rrr, dotted, bend left, "\tilde{h}"] & & E(h^* p_\lambda) \arrow[r, "\approx"'] \arrow[d, "h^* p_\lambda"] & \mathbb{T}^k \times \mathbb{D}^n \arrow[d, "p_\lambda"] \\
& & \mathbb{T}^k \times \mathbb{D}^n \arrow[r, "h"'] & \mathbb{T}^k \times \mathbb{D}^n
\end{tikzcd}$$

Similarly, we can construct a continuous family of maps $\tilde{h}_t : \mathbb{T}^k \times \mathbb{D}^n, \quad t \in [0, 1]$ with $\tilde{h}_0 = \tilde{h}$ and such that $p_\lambda \circ \tilde{h}_t = h_t \circ p_\lambda$.

Consider first the case $n = 0$. We can assume that $\tilde{h}(e) = e$. Thus $\tilde{h}_1$ is a deck transformation. But since $t \mapsto \tilde{h}_t(e)$ is a continuous path in the discret set $p_\lambda^{-1}(e)$, $\tilde{h}_1(e) = \tilde{h}_0(e) = e$, hence $\tilde{h}_1 = Id$. Now for all $x \in p_\lambda^{-1}(e)$, $t \mapsto \tilde{h}_t(x)$ is a continuous path in $p_\lambda^{-1}(e)$, hence $\tilde{h}(x) = \tilde{h}_1(x) = x$. Now choose $\varepsilon > 0$. Choose $0 < \delta < \varepsilon$ such that $d(\tilde{h}(x), \tilde{h}(y)) < \varepsilon$ whenever $d(x, y) < \delta$, and choose λ large enough so the diameter of any closed (isometric) fundamental domain is less than δ. Now given $x \in \mathbb{T}^k$, choose $x_0 \in p_\lambda^{-1}(e)$ such that x and x_0 are in the same fundamental domain. We have,

$$d(x, \tilde{h}(x)) \leq d(x, x_0) + d(x_0, \tilde{h}(x)) = d(x, x_0) + d(\tilde{h}(x_0), \tilde{h}(x)) \leq 2\varepsilon$$

So for every $\varepsilon > 0$ there exists an integer $\lambda \geq 0$ such that $d(\tilde{h}, Id_{\mathbb{T}^k}) \leq \varepsilon$, where $d(\tilde{h}, Id_{\mathbb{T}^k}) = sup_{x \in \mathbb{T}^k} d(\tilde{h}(x), x)$. Now, by local contractibility of the space of homeomorphisms of a compact topological manifold, this implies that there exists a nice finite covering such that h lifts to a homeomorphism $\tilde{h}$ isotopic to $Id_{\mathbb{T}^k}$.

For $n > 0$, the previous argument is not sufficient since the isotopy has no reason to be rel $\partial(\mathbb{T}^k \times \mathbb{D}^n)$. To overcome that, we need to do the previous procedure "far from the boundary". More precisely, for $0 < \eta < 1$, let $D_\eta \subset \mathbb{D}^n$ the disk of radius η. This time, we can assume that $\tilde{h}$ stabilizes every element of $p_\lambda^{-1}(\{e\} \times \partial(\mathbb{D}^n))$. Now choose $\varepsilon > 0$. Choose $0 < \delta < \varepsilon$ such that $d(\tilde{h}(x), \tilde{h}(y)) < \varepsilon$ whenever $d(x, y) < \delta$, choose η close enough to 1 so that given an element $x \in \mathbb{T}^k \times D_\eta$, one can choose an element $x_0 \in p_\lambda^{-1}(\{e\} \times \partial(\mathbb{D}^n))$ such that $d(x, x_0) \leq \delta$, and choose λ large enough so that any closed (isometric) foundamental domain has diameter less than δ. The same reasoning as above shows that $d(\tilde{h}_{|\mathbb{T}^k \times D_\eta}, Id_{\mathbb{T}^k \times D_\eta}) \leq 2\varepsilon$. Thus there exists a nice covering such that there exists an isotopy $\varphi : \mathbb{T}^k \times D_\eta \times [0, 1] \to \mathbb{T}^k \times D_\eta \times [0, 1]$ between $Id_{\mathbb{T}^k \times D_\eta}$ and $\tilde{h}_{|\mathbb{T}^k \times D_\eta}$. One then construct the desired isotopy rel $\partial(\mathbb{T}^k \times \mathbb{D}^n)$ by setting:

$$\psi(a, t) = \begin{cases} \varphi(a, t) & \text{if } |a| \leq \eta \\ \varphi(a, \frac{1-|a|}{1-\eta}) & \text{if } |a| \geq \eta \end{cases}$$

$\square$

Now consider the map

$$\Phi : \pi_n(TOP/PL) = [(\mathbb{D}^n, \partial), (TOP/PL, *)] \xrightarrow{pr^*}$$
$$\xrightarrow{pr^*} [(\mathbb{T}^k \times \mathbb{D}^n, \partial), (TOP/PL, *)] \xrightarrow{\sigma} \mathcal{T}_{PL}(\mathbb{T}^k \times \mathbb{D}^n) \xrightarrow{\beta} \mathcal{S}_{PL}(\mathbb{T}^k \times \mathbb{D}^n)$$

with $k \geq 0$ such that $n + k \geq 5$, and $pr : \mathbb{T}^k \times \mathbb{D}^n \to \mathbb{D}^n$ the natural projection.

Proposition 5.2.6. *The map Φ is injective. Furthermore, if for some $\lambda \geq 0$, $p_\lambda^* \Phi(x) = p_\lambda^* \Phi(y)$, then $x = y$.*

Proof. Let $x, y \in \pi_n(TOP/PL)$ such that $\Phi(x) = \Phi(y)$. By 5.2.4, there exists a nice covering $p : \mathbb{T}^k \times \mathbb{D}^n \to \mathbb{T}^k \times \mathbb{D}^n$ such that $p^*(\sigma \circ pr^*(x)) = p^*(\sigma \circ pr^*(y))$. Since p is a nice covering, $p^* \circ \sigma \circ pr^* = \sigma \circ pr^*$. Thus $\sigma \circ pr^*(x) = \sigma \circ pr^*(y)$. Now this clearly implies $x = y$, since both σ and pr^* are injective.

Let $p : \mathbb{T}^k \times \mathbb{D}^n \to \mathbb{T}^k \times \mathbb{D}^n$ be a finite covering, and let $x, y \in \pi_n(TOP/PL)$ such that $p^*(\Phi(x)) = p^*(\Phi(y))$, that is $p^*(\beta \circ \sigma \circ pr^*(x)) = p^*(\beta \circ \sigma \circ pr^*(y))$. β is clearly natural for finite coverings, so $\beta(\ p^*(\sigma \circ pr^*(x))\ ) = \beta(\ p^*(\sigma \circ pr^*(y))\ )$. By 5.2.4, there exists a finite covering q such that $q^* p^*(\sigma \circ pr^*(x)) = q^* p^*(\sigma \circ pr^*(y))$, or in other words $(p \circ q)^*(\sigma \circ pr^*(x)) = (p \circ q)^*(\sigma \circ pr^*(y))$. Now again this implies $x = y$. $\square$

Corollary 5.2.7. *(i)* $\pi_n(TOP/PL) = 0$ *for every* $n \neq 3$.

(ii) $\pi_3(TOP/PL)$ *has at most two elements.*

Proof. This is an immediate consequence of 5.1.6, 5.2.6 and 4.2.9. □

The homotopy type of TOP/PL is thus almost determined. Namely, TOP/PL is either contractible or a $K(\mathbb{Z}_2, 3)$. If TOP/PL was contractible, any map $M \xrightarrow{\tau_M} BTOP \to BTOP/PL$ would be nullhomotopic. In particular, any topological manifold of dimension ≥ 5 would admit a PL-structure, by the Classification Theorem 5.2.3.

Let us recall that there exists a closed manifold F of dimension 4 which satisfies $w_1(F) = 0, w_2(F) = 0$, and $\sigma(F) = 8$. The Rokhlin's theorem implies that F does not admit a PL-structure, otherwise it would admit a DIFF-structure by the Smoothing Theorem. This will allow us to construct a topological manifold of dimension 5 which admits no PL-structure. Note that by the Smoothing Theorem, this amounts to giving an example of a 5-dimensional topological manifold without any smooth structure.

Proposition 5.2.8. $F \times \mathbb{R}$ *admits no smooth structure.*

Proof. Suppose it is the case. We can find an C^0-approximation f of $pr : R \times \mathbb{R} \to \mathbb{R}$ which equals pr on $]-\infty, 0]$ and is smooth on $[1, \infty[$. By Sard theorem, choose a regular value a of f, and set $F' = f^{-1}(a), W = f^{-1}([0, a])$. Then F' is a smooth 4-dimension manifold whose tangent bundle is stably isomorphic to $i^*\tau_W$ (with $i : F' \hookrightarrow W$ the inclusion), by the tubular neighborhood theorem. But $\tau_W^{TOP} = \tau_F^{TOP} \oplus \varepsilon^1$, so $w_1(W) = 0, w_2(W) = 0$. Thus, by naturality of the Stiefel-Whitney classes, $w_1(F') = i^*w_1(W) = 0, w_2(F') = i^*w_2(F') = 0$. Furthermore, W is a topological cobordism between F and F', so $\sigma(F') = 8$. So F' is a smooth 4-dimensional manifold with $w_1(F') = 0, w_2(F') = 0$, and $\sigma(F') = 8$, which contradicts Rokhlin's theorem. □

Note that, by the Product Structure Theorem, $F \times \mathbb{R}^k$ gives an example of a topological manifold with no PL-structure in any dimension ≥ 4.

5.3 The Kirby-Siebenmann obstruction

We can now turn back to our problem of determining the PL-structures a topological manifold might carry. We have just proved that we have the following fibration

$$K(\mathbb{Z}_2, 3) \to BPL \to BTOP$$

Since TOP/PL is a H-group, it has a classifying space, yielding a fibration

$$BPL \to BTOP \to BTOP/PL \simeq K(\mathbb{Z}_2, 4).$$

Since $BTOP/PL$ is 3-connected, it has a fundamental class

$$\alpha \in H^4(BTOP/PL; \pi_4(BTOP/PL))$$

representing the inverse of the Hurewicz isomorphism $\pi_4(BTOP/PL) \to H_4(BTOP/PL; \mathbb{Z})$ under the identification

$$H^4(BTOP/PL; \pi_4(BTOP/PL)) \simeq Hom(H_4(BTOP/PL; \mathbb{Z}), \pi_4(BTOP/PL)).$$

Definition 5.3.1. (i) Under the natural map $BTOP \to BTOP/PL$, α pullbacks to a class

$$\kappa \in H^4(BTOP; \mathbb{Z}_2),$$

called the *universal Kirby-Siebenmann class.*
(ii) Let M be a topological manifold, and $f : M \to BTOP$ the classifying map for its topological tangent bundle. We define the *Kirby-Siebenmann obstruction* $\kappa(M)$ by

$$\kappa(M) = f^*\kappa \in H^4(M; \mathbb{Z}_2).$$

Theorem 5.3.2. *Let M be a topological manifold of dimension ≥ 5. Then M admits a PL-structure if and olny if $\kappa(M) = 0$. Furthermore, if $\kappa(M) = 0$, the various PL-structures are in (unnatural) correspondence with $[M, TOP/PL] \simeq H^3(M; \mathbb{Z}_2)$.*

Proof. Let $f : M \to BTOP$ the classifying map of the topological tangent bundle of M. By the Classification Theorem 5.2.3, M admits a PL-structure if and only if f lifts to BPL. Since, we have a fibration $BPL \to BTOP \xrightarrow{\phi} BTOP/PL$, f lifts to BPL if and only if the composition $M \xrightarrow{f} BTOP \xrightarrow{\phi} BTOP/PL$ is nullhomotopic. Now, since $BTOP/PL$ is a $K(\mathbb{Z}_2, 4)$, $\phi \circ f \in [M, BTOP/PL] \simeq H^4(M, \mathbb{Z}_2)$, with isomorphism given by

$$[M, K(\mathbb{Z}_2, 4)] \to H^4(M, \mathbb{Z}_2), \quad g \mapsto g^*\alpha.$$

Hence, M admits a PL-structure if and only if

$$(\phi \circ f)^*\alpha = f^*(\phi^*\alpha) = f^*\kappa = \kappa(M) = 0.$$

The second assertion follows immediately from the properties of the fibration $TOP/PL \to BPL \to BTOP$. □

Chapter 6

Topological rigidity of the torus

We are now going to prove the main theorem of this thesis. Recall that we found in Chapter 5 a necessary and sufficient condition for a manifold of dimension ≥ 5 to admit a PL-structure. In Chapter 4, we were able to compute the PL-structure set $\mathcal{S}_{PL}(\mathbb{T}^n), n \geq 5$. We now combine these results to deduce the rigidity theorem.

There is still one obstacle left. In order to prove that a high-dimensional homotopy torus admits a PL-structure, we need to prove the vanishing of its Kirby-Siebenmann obstruction. But since κ lies in the cohomology of $BTOP$, this obstruction is generally not preserved under homotopy equivalence. Section 5.1 is thus devoted to constructing an intermediary charactristic class $\omega_G(M)$ carrying enough data on $\kappa(M)$ to derive the existence of a PL-structure on a homotopy torus from its vanishing, and such that $\omega_G(M) = \omega_G(\mathbb{T}^n)$ for a manifold homotopy equivalent to $\mathbb{T}^n$. In section 5.2, we finally carry out our program and prove the rigidity theorem. This chapter follows the strategy of [HW69].

6.1 An intermediary characteristic class.

Definition 6.1.1. Let $STOP$ be the subgroup of TOP consisting of orientation preserving maps, *Spin-Top* its (double) universal cover, and B*STop*, B*Spin-Top* their classifying spaces.

Proposition 6.1.2. *BSpin-Top is* 3*-connected, and* $\pi_4(BSpin\text{-}Top) \simeq \mathbb{Z} \oplus \mathbb{Z}_2$.

Proof. It amounts to proving that *Spin-Top* is 2-connected. By definition, it is simply-connected. Furthermore, $\pi_2(Spin\text{-}Top) \simeq \pi_2(Top) = \pi_3(BTOP)$. But the exact sequence of homotopy groups of the fibration $TOP/PLBPL \to BTOP$ yields

$$\pi_3(BPL) \to \pi_3(BTOP) \to \pi_2(TOP/PL).$$

Now, since PL/O is 4-connected, $\pi_3(BPL) \simeq \pi_3(BO) \simeq 0$, and $\pi_2(TOP/PL) \simeq 0$ by the theorem on the homotopy type of TOP/PL.

As for $\pi_4(BSpin\text{-}Top) \simeq \pi_4(BSpin\text{-}Top)$, the exact sequence of homotopy groups of the fibration $TOP/PLBPL \to BTOP$ yields

$$\pi_4(TOP/PL) \to \pi_4(BPL) \to \pi_4(BTOP) \to \pi_3(TOP/PL) \to \pi_3(BPL).$$

Now, since PL/O is 4-connected, $\pi_3(BPL) \simeq \pi_3(BO) = 0$ and $\pi_4(BPL) \simeq \pi_4(BO) \simeq \mathbb{Z}$. By the theorem on the homotopy type of TOP/PL, $\pi_4(TOP/PL) = 0$, and $\pi_3(TOP/PL) \simeq \mathbb{Z}_2$, and the result follows. $\square$

It follows from $BSpin\text{-}Top$ has a fundamental class $\omega \in H^4(B(Spin\text{-}Top); \pi_4(B(Spin\text{-}Top)))$, which has two components $\omega_{free} \in H^4(BSpin\text{-}Top; \mathbb{Z})$, $\omega_{tors} \in H^4(BSpin\text{-}Top; \mathbb{Z}_2)$.

Proposition 6.1.3. *$H^4(BSpin\text{-}Top; \mathbb{Z}) \simeq \mathbb{Z}$, generated by ω_{free}.*

Proof. We have, since $BSpin\text{-}Top$ is 3-connected

$$\begin{aligned} H^4(BSpin\text{-}Top; \mathbb{Z}) &\simeq Hom(H_4(BSpin\text{-}Top; \mathbb{Z}), \mathbb{Z}) \\ &\simeq Hom(\pi_4(BSpin\text{-}Top); \mathbb{Z}) \qquad \text{(Hurewicz)} \\ &\simeq Hom(\mathbb{Z} \oplus \mathbb{Z}_2; \mathbb{Z}) \\ &\simeq \mathbb{Z}. \end{aligned}$$

Now, by the universal coefficient theorem,

$$H^4(BSpin\text{-}Top, \pi_4(BSpin\text{-}Top)) \simeq Hom(H_4(BSpin\text{-}Top, \mathbb{Z})\pi_4(BSpin\text{-}Top)),$$

and under this isomorphism, ω represents h^{-1}, where

$$h : \pi_4(BSpin\text{-}Top) \to H_4(BSpin\text{-}Top, \mathbb{Z})$$

is the Hurewicz isomorphism. So $h^{-1} : \mathbb{Z} \oplus \mathbb{Z}_2 \xrightarrow{(\pm Id)\oplus Id} \mathbb{Z} \oplus \mathbb{Z}_2$, hence $\omega_{free} \in H^4(BSpin\text{-}Top; \mathbb{Z}) \simeq Hom(H_4(BSpin\text{-}Top; \mathbb{Z}), \mathbb{Z})$ represents $\mathbb{Z} \oplus \mathbb{Z}_2 \xrightarrow{(\pm Id)\oplus Id} \mathbb{Z} \oplus \mathbb{Z}_2 \xrightarrow{(\pm Id)\oplus 0} \mathbb{Z}$, which is a generator of $Hom(\mathbb{Z} \oplus \mathbb{Z}_2, \mathbb{Z})$. $\square$

Proposition 6.1.4. *(i) $H^4(BS\ Top; \mathbb{Z})_{free} \simeq \mathbb{Z}$.*

(ii) The natural map $H^4(BSTop; \mathbb{Z})_{free} \to H^4(BSpin\text{-}Top\ ; \mathbb{Z})$ is injective.

(iii) The natural map $H^4(BS\ Top; \mathbb{Z})_{free} \to H^4(BSO; \mathbb{Z})$ is injective [1].

[1] Actually, Wall asserts in [HW69] that the natural map $H^4(BS\ Top; \mathbb{Z}) \to H^4(BSO; \mathbb{Z})$ is an isomoprhism.

Proof. Since *Spin-Top* is the universal double cover of *STop*, the natural map *Spin-Top* $\to$ *S Top* yields an isomorphism $\pi_*(Spin\text{-}Top) \otimes \mathbb{Q} \to \pi_*(S\ Top) \otimes \mathbb{Q}$, and thus an isomorphism $\pi_n(BSpin\text{-}Top) \otimes \mathbb{Q} \to \pi_n(BS\ Top) \otimes \mathbb{Q}$ for $n \geq 1$, by commutativity of the following diagram

$$\begin{tikzcd}
\pi_n(BSpin\text{-}Top) \otimes \mathbb{Q} \arrow[r] \arrow[d, "\approx"'] & \pi_n(BS\ Top) \otimes \mathbb{Q} \arrow[d, "\approx"] \\
\pi_{n-1}(Spin\text{-}Top) \otimes \mathbb{Q} \arrow[r, "\approx"] & \pi_{n-1}(S\ Top) \otimes \mathbb{Q}.
\end{tikzcd}$$

There is also an isomorphism on $\pi_0 \otimes \mathbb{Q} \simeq 0$, so by the generalized Whitehead theorem, *BSpin-Top* $\to$ *BSTop* induces an isomorphism in cohomology with $\mathbb{Q}$-coefficients, which proves (ii).

Recall that we have a fibration $BPL \to BTOP \to K(\mathbb{Z}_2, 4)$. Once again, the generalized Whitehead theorem yields an isomorphism $H^4(BPL;\mathbb{Q}) \to H^4(BTOP;\mathbb{Q})$. Since PL/O is 4-connected, the natural map $BPL \to BO$ yields an isomorphism $\pi_n(BPL) \to \pi_n(BO), n \leq 4$, and so an isomorphism $H^n(BPL,\mathbb{Z}) \to H^n(BO,\mathbb{Z}), n \leq 4$, by the Hurewicz theorem. Thus $H^4(BTop;\mathbb{Q}) \to H^4(BO;\mathbb{Q})$ is an isomorphism. By the commutativity of the following diagram

$$\begin{tikzcd}
H^4(BS\ Top;\mathbb{Q}) \arrow[r] & H^4(BSO;\mathbb{Q}) \\
H^4(BTop;\mathbb{Q}) \arrow[r, "\approx"] \arrow[u] & H^4(BSO;\mathbb{Q}), \arrow[u]
\end{tikzcd}$$

it is sufficient to prove that we have isomorphism $H^*(BS\ Top;\mathbb{Q}) \simeq H^*(BTOP;\mathbb{Q})$, $H^*(BSO;\mathbb{Q}) \simeq H^*(BO;\mathbb{Q})$. The natural maps $SO \to O$, $S\ Top \to TOP$ yield isomorphisms

$$\pi_n(SO) \otimes \mathbb{Q} \to \pi_n(O) \otimes \mathbb{Q}, \qquad \pi_n(S\ Top) \otimes \mathbb{Q} \to \pi_n(Top) \otimes \mathbb{Q}$$

for $n \geq 1$, by commutativity of the following diagrams

$$\begin{tikzcd}
\pi_n(BSO) \otimes \mathbb{Q} \arrow[r] \arrow[d, "\approx"'] & \pi_n(BO) \otimes \mathbb{Q} \arrow[d, "\approx"] \\
\pi_{n-1}(SO) \otimes \mathbb{Q} \arrow[r, "\approx"] & \pi_{n-1}(O) \otimes \mathbb{Q}
\end{tikzcd}
\qquad
\begin{tikzcd}
\pi_n(BS\ Top) \otimes \mathbb{Q} \arrow[r] \arrow[d, "\approx"'] & \pi_n(BTOP) \otimes \mathbb{Q} \arrow[d, "\approx"] \\
\pi_{n-1}(S\ Top) \otimes \mathbb{Q} \arrow[r, "\approx"] & \pi_{n-1}(TOP) \otimes \mathbb{Q}.
\end{tikzcd}$$

It also yields an isomorphism on $\pi_0 \otimes \mathbb{Q} \simeq 0$, so (iii) then follows form the generalized Whitehead theorem, and (i) follows from the fact that $H^4(BSO;\mathbb{Z}) \simeq \mathbb{Z}$. $\square$

Definition 6.1.5. Let p_{top} be a generator of $H^4(BS\ Top, \mathbb{Z})_{free}$, called a *topological universal Pontryagin class.*

Corollary 6.1.6. *There exists an integer $d \neq 0$ such that $d\omega_{free}$ is induced from $p_{top} \in H^4(B(STop);\mathbb{Z})$.*

Proposition 6.1.7. *The natural map $H^4(BTOP;\mathbb{Z}_2) \to H^4(BSpin\text{-}Top;\mathbb{Z}_2)$ maps the Kirby-Siebenmann class κ to ω_{tors}.*

Proof. Since both $BSpin\text{-}Top$ and $BTOP/PL \simeq K(\mathbb{Z}_2, 4)$ are 3-connected, the naturality of the Hurewicz homomorphism yields the following commutative diagram

$$\begin{tikzcd}
H^4(BTOP/PL;\mathbb{Z}_2) \arrow[r, "\simeq"] \arrow[d] & Hom(\pi_4(BTOP/PL), \mathbb{Z}_2) \arrow[d] \\
H^4(BSpin\text{-}Top;\mathbb{Z}_2) \arrow[r, "\simeq"] & Hom(\pi_4(BSpin\text{-}Top), \mathbb{Z}_2)
\end{tikzcd}$$

obtained by dualizing the natural map

$$\pi_4(BSpin\text{-}Top) \simeq \pi_4(BTOP) \simeq \mathbb{Z} \oplus \mathbb{Z}_2 \xrightarrow{0 \oplus Id} \mathbb{Z}_2 \simeq \pi_4(BTOP/PL).$$

Thus the fundamental class $\alpha \in H^4(BTOP/PL;\mathbb{Z}_2)$ corresponding to the inverse of the Hurewicz homomorphism $h^{-1}_{BTOP/PL} : \mathbb{Z}_2 \xrightarrow{Id} \mathbb{Z}$ maps to the element $(\mathbb{Z} \oplus \mathbb{Z}_2 \xrightarrow{0 \oplus Id} \mathbb{Z}_2) \in H^4(BSpin\text{-}Top;\mathbb{Z}_2)$.
Now the fundamental class $\omega \in H^4(BSpin\text{-}Top;\pi_4(BSpin\text{-}Top))$ corresponds, under the canonic identification, to the inverse of the Hurewicz map $h^{-1}_{BSpin\text{-}Top} : \mathbb{Z} \oplus \mathbb{Z}_2 \xrightarrow{(\pm Id) \oplus Id} \mathbb{Z} \oplus \mathbb{Z}_2$, so ω_{tors} corresponds to $\mathbb{Z} \oplus \mathbb{Z}_2 \xrightarrow{0 \oplus Id} \mathbb{Z} \oplus \mathbb{Z}_2$. Hence The natural map $H^4(BTOP/PL;\mathbb{Z}_2) \to H^4(BSpin\text{-}Top;\mathbb{Z}_2)$ maps α to ω_{tors}. But by defintion, κ is induced from α under the natural map $H^4(BTOP/PL;\mathbb{Z}_2) \to H^4(BTOP;\mathbb{Z}_2)$. Thus the result follows from the commutativity of the following diagram, where all maps are the natural ones

$$\begin{tikzcd}
H^4(BTOP/PL;\mathbb{Z}_2) \arrow[rr] \arrow[dr] & & H^4(BSpin\text{-}Top;\mathbb{Z}_2) \\
& H^4(BTOP;\mathbb{Z}_2). \arrow[ur] &
\end{tikzcd}$$

$\square$

Definition 6.1.8. (i) Let ω_G be the image of ω under the map

$$H^4(B(\mathit{Spin}\text{-}\mathit{Top}); \pi_3(\mathit{TOP})) \to H^4(B(\mathit{Spin}\text{-}\mathit{Top}); \pi_3(G)).$$

(ii) For a topological manifold M with vanishing w_1 and w_2, and with classifying space of its topological spin tangent bundle $f : M \to \mathit{BSpin}\text{-}\mathit{Top}$, let

$$\omega_G(M) = f^*\omega_G.$$

Theorem 6.1.9. *Let M a topological manifold homotopy equivalent to a torus. Then M admits a topological spin-structure, for which $\omega_G(M) = 0$.*

Proof. Consider the Whitehead tower of BSG:

$$\begin{tikzcd}
\vdots \arrow[d] & \\
X_2 \arrow[d] & \\
\mathit{BSpin}\text{-}G \arrow[r] \arrow[d] & K(\pi_3(BSG), 3) \\
BSG \arrow[r] & K(\pi_2(BSG), 2).
\end{tikzcd}$$

We have the following commutative diagram

$$\begin{tikzcd}
& X_2 \arrow[d] & \\
\mathit{BSpin}\text{-}\mathit{Top} \arrow[r] \arrow[d] \arrow[ur, dotted] & \mathit{BSpin}\text{-}G \arrow[r] \arrow[d] & K(\pi_3(BSG), 3) \\
BS\ \mathit{Top} \arrow[r] & BSG \arrow[r] & K(\pi_2(BSG), 2)
\end{tikzcd}$$

and the map $\mathit{BSpin}\text{-}\mathit{Top} \to \mathit{BSpin}\text{-}G$ lifts to X_2 if and only if the composite

$$\mathit{BSpin}\text{-}\mathit{Top} \to \mathit{BSpin}\text{-}G \to K(\pi_3(BSG), 3)$$

is nullhomotopic, by obstruction theory. Since $H^3(\mathit{BSpin}\text{-}\mathit{Top}; \pi_3(BSG)) = 0$, such a lift exists. The induced map $\mathit{BSpin}\text{-}\mathit{Top} \to X_2$ yields the map

$$\pi_4(\mathit{BSpin}\text{-}\mathit{Top}) \simeq \pi_4(BTOP) \to \pi_4(BG) \simeq \pi_4(X_2)$$

We have the following lemma:

Lemma 6.1.10. *The map $H^4(X_2; \pi_4(BG)) \to H^4(BSpin\text{-}Top; \pi_4(BG))$ maps the fundamental class α_{X_2} of X_2 to ω_G.*

Proof. Since X_2 is 3-connected, we have

$$H^4(X_2; \pi_4(BG)) \simeq Hom(\pi_4(X_2), \pi_4(BG)) \simeq Hom(\pi_4(BG), \pi_4(BG))$$

and, under this isomorphism, α_{X_2} represents $\mathbb{Z}_{24} \xrightarrow{Id} \mathbb{Z}_{24}$. Furthemore, since $BSpin\text{-}Top$ is 3-connected,

$$H^4(BSpin\text{-}Top; \pi_4(BG)) \simeq Hom(\pi_4(BTOP), \pi_4(BG))$$

and the map $H^4(X_2; \pi_4(BG)) \to H^4(BSpin\text{-}Top; \pi_4(BG))$ is obtained by dualizing the natural map $\pi_4(BTOP) \to \pi_4(BG)$. Hence α_{X_2} maps to $\mathbb{Z} \oplus \mathbb{Z}_2 \xrightarrow{pr \oplus i} \mathbb{Z}_{24}$.

Now recall that

$$\omega \in H^4(BSpin\text{-}Top; \pi_4(BSpin\text{-}Top)) \simeq Hom(\pi_4(BTOP), \pi_4(BTOP))$$

is represented by $\mathbb{Z} \oplus \mathbb{Z}_2 \xrightarrow{(\pm Id) \oplus Id} \mathbb{Z} \oplus \mathbb{Z}_2$, hence $\omega_G \in H^4(BSpin\text{-}Top; \pi_4(BG)) \simeq Hom(\pi_4(BTOP), \pi_4(BG))$ is represented by $\mathbb{Z} \oplus \mathbb{Z}_2 \xrightarrow{pr \oplus i} \mathbb{Z}_{24}$, and the result follows. □

Lemma 6.1.11. *Let M a topological oriented manifold whose Spivak normal fibration admits a spin structure $M \to BSpin\text{-}G$. Then there exists a topological spin structure on M*

$$\begin{tikzcd} & BSpin\text{-}Top \arrow[d] \\ M \arrow[ur] \arrow[r] & BTOP \end{tikzcd}$$

such that the induced spin structure on the Spivak normal fibration $M \to BSpin\text{-}Top \to BSpin\text{-}G$ agrees with the given structure $M \to BSpin\text{-}G$.

Proof. First consider the case of an oriented manifold $M \to BS\ Top$ with a lift of its oriented Spivak fibration $M \to BS\ G$ to $BSpin\text{-}G$. Since $Spin\text{-}Top$ (resp. $Spin\text{-}G$) is the universal double cover of $S\ Top$ (resp. SG), we have fibrations

$$\begin{tikzcd} K(\mathbb{Z}_2, 1) \simeq F \arrow[r] \arrow[d] & F' \simeq K(\mathbb{Z}_2, 1) \arrow[d] \\ BSpin\text{-}Top \arrow[r] \arrow[d] & BSpin\text{-}G \arrow[d] \\ BS\ Top \arrow[r] & BS\ G. \end{tikzcd}$$

Now, by the result of Boardman and Vogt on homotopy-everything H-spaces [BV68], there are fibrations

$$\begin{tikzcd}
BSpin\text{-}Top \arrow[r] \arrow[d] & BSpin\text{-}G \arrow[d] \\
BS\ Top \arrow[r] \arrow[d] & BS\ G \arrow[d] \\
K(\mathbb{Z}_2, 2) \simeq BF \arrow[r] & BF' \simeq K(\mathbb{Z}_2, 2).
\end{tikzcd}$$

Since $\pi_1(TOP) \to \pi_1(G)$ is an isomorphism, $F \to F'$ and $BF \to BF'$ are homotopy equivalences. Let α (resp. α') be the fundamental class of BF (resp. BF'). Then $BF \to BF'$ maps α to α' since $H^2(K(\mathbb{Z}_2, 2), \mathbb{Z}_2) \simeq \mathbb{Z}_2$. We thus have the following commutative diagram

$$\begin{tikzcd}
 & BSpin\text{-}Top \arrow[r] \arrow[d] & BSpin\text{-}G \arrow[d] \\
M \arrow[r, "f"] & BS\ Top \arrow[r, "g"] \arrow[d, "\varphi"] & BS\ G \arrow[d, "\psi"] \\
 & K(\mathbb{Z}_2, 2) \simeq BF \arrow[r] & BF' \simeq K(\mathbb{Z}_2, 2).
\end{tikzcd}$$

By obstruction theory, since $M \xrightarrow{g \circ f} BS\ G$ lifts to $BSpin\text{-}G$, $(g \circ f)^* \psi^* \alpha' = 0$. But

$$(g \circ f)^* \psi^* \alpha' = f^* \varphi^* \alpha = 0$$

hence $M \to BS\ Top$ lifts to $BSpin\text{-}Top$. Furthermore, the various lifts are in bijective correspondence with $[M, F]$. But $F \xrightarrow{\approx} F'$ yields a bijection $[M, F] \to [M, F]$, and $[M, F]$ classifies the lifts of $M \to BS\ G$. Thus there exists a lift $M \to BSpin\text{-}Top$ compatible with both the orientation of the tangent bundle $M \to BS\ Top$ and the spin structure on the Spivak fibration $M \to BSpin\text{-}G$.

The same reasoning applies to the fibrations

$$\begin{tikzcd}
K(\mathbb{Z}_2, 0) \simeq F \arrow[r, "\approx"] \arrow[d] & F' \simeq K(\mathbb{Z}_2, 0) \arrow[d] \\
BS\ Top \arrow[r] \arrow[d] & BS\ G \arrow[d] \\
BTOP \arrow[r] & BG,
\end{tikzcd}$$

so given a topological manifold with a lift $M \to BS\ G$ of its Spivak fibration, there exists an orientation of the topological tangent bundle $M \to BS\ Top$ compatible with both the topological structure $M \to BTOP$ and the orientation of the Spivak fibration $M \to BS\ G$.

The result now follows by combining these two assertions. $\square$

Let M be a topological manifold homotopy equivalent to a torus. Then M is orientable and has a trivial Spivak fibration, and so admits the trivial lift $M \to X_2$. By 0.0.3, let $M \to BSpin\text{-}Top$ be a spin-structure on M such that $M \to BSpin\text{-}Top \to BSpin\text{-}G$ is nullhomotopic. We thus have the following commutative diagram

$$\begin{tikzcd}
 & & X_2 \arrow[d] \\
 & BSpin\text{-}Top \arrow[r] \arrow[ur] \arrow[d] & BSpin\text{-}G \arrow[d] \\
M \arrow[r] \arrow[ur] & BS\ Top \arrow[r] & BS\ G.
\end{tikzcd}$$

In particular, we have the following commutative triangle

$$\begin{tikzcd}
H^4(X_2;\mathbb{Z}_{24}) \arrow[rr] \arrow[dr, "0"'] & & H^4(BSpin\text{-}Top;\mathbb{Z}_{24}) \arrow[dl] \\
 & H^4(M;\mathbb{Z}_{24}). &
\end{tikzcd}$$

Since the horizontal map sends α_{X_2} to ω_G by 0.0.2, $\omega_G(M) = 0$. $\square$

6.2 The Rigidity Theorem.

Lemma 6.2.1. *The topological Pontryagin class p_{top} is additive modulo torsion with respect to direct sums of vector bundles.*

Proof. Let $f_\xi, f_\eta : M \to BS\ Top$. We have the following commutative diagram

$$\begin{tikzcd}
M \arrow[r, "f_\xi \times f_\eta"] & BS\ Top \times BS\ Top \arrow[r, "\oplus"] & BS\ Top \\
& BSO \times BSO \arrow[u] \arrow[r, "\oplus"] & BSO \arrow[u]
\end{tikzcd}$$

Let $m \neq 0$ such that p_{top} is mapped to $mp_1 \in H^4(BSO, \mathbb{Z})$ under the map $BSO \to BS\ Top$. As before, we can prove that $BSO \times BSO \to BS\ Top \times BS\ Top$ induces an isomorphism on $H^4(\ ;\mathbb{Q})$, so that $H^4(BS\ Top{\times}BS\ Top; \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z} \oplus$ (torsion). Now, by additivity of the first Pontryagin class,

$$H^4(BSO;\mathbb{Z}) \simeq \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \simeq H^4(BSO{\times}BSO;\mathbb{Z})$$

is the diagonal map, so p_{top} is mapped to (mp_1, mp_1) under

$$H^4(BS\ Top;\mathbb{Z}) \to H^4(BSO;\mathbb{Z}) \to H^4(BSO{\times}BSO;\mathbb{Z}).$$

Thus p_{top} is mapped to (p_{top}, p_{top}) + torsion under the map $H^4(BS\ Top;\mathbb{Z}) \to H^4(BS\ Top{\times}BS\ Top;\mathbb{Z})$, and the result follows. $\square$

Proposition 6.2.2. *Let M be a topological manifold homotopy equivalent to $\mathbb{T}^n, n \geq 5$. Then M admits a PL-structure.*

Proof. Let $p : \mathbb{T}^n \to \mathbb{T}^n$ be the 2^n-sheeted covering map given by $p\,(x_1, \ldots, x_n) = (x_1^2, \ldots, x_n^2)$, and $\pi : \widetilde{M} \to M$ its pullback over f

$$\begin{tikzcd}
\widetilde{M} \arrow[r, "F"] \arrow[d, "\pi"'] & \mathbb{T}^n \arrow[d, "p"] \\
M \arrow[r, "f"'] & \mathbb{T}^n
\end{tikzcd}$$

π being a local homeomorphism, the tangent bundle of $\widetilde{M}$ is induced from that of M, so $\kappa(\widetilde{M}) = \pi^*\kappa\,(M)$. But $p^* : H^4(\mathbb{T}^n, \mathbb{Z}_2) \to H^4(\mathbb{T}^n, \mathbb{Z}_2)$ is the zero map, since $p^* : H^1(\mathbb{T}^n, \mathbb{Z}) \to H^1(\mathbb{T}^n, \mathbb{Z})$ is multiplication by 2. Furthermore, f being a homotopy equivalence, F is also a homotopy equivalence. Hence $\pi^* : H^4(M, \mathbb{Z}_2) \to$

$H^4(M, \mathbb{Z}_2)$ is the zero map, so $\kappa(\widetilde{M}) = 0$ and $\widetilde{M}$ admits a PL-structure by the Obstruction Theorem.

Now $\widetilde{M}$ being a PL-homotopy torus, it is stably parallelizable by 4.1.3. Since $\widetilde{M}$ has torsion-free cohomology, it follows from 6.2.1 that $p_{top}(\widetilde{M}) = 0 = \pi^* p_{top}(M)$. But $H^4(M, \mathbb{Z})$ has no torsion, so $\pi^* : H^4(M, \mathbb{Z}) \to H^4(\widetilde{M}, \mathbb{Z})$ is injective, which implies $p_{top}(M) = 0$. Thus, by 6.1.6, there exists $d \neq 0$ such that $d\omega_{free}(M) = 0$, hence $\omega_{free}(M) = 0$ since M has torsion-free cohomology. It follows that $\omega_G(M) = i_* \kappa(M)$, where $i : \mathbb{Z}_2 \hookrightarrow \mathbb{Z}_{24}$ is the canonical injection. Recall that, M being a homotopy torus, $\omega_G(M) = 0$ by 6.1.9, hence $i_* \kappa(M) = 0$. But $i_* : H^4(M, \mathbb{Z}_2) \to H^4(\widetilde{M}, \mathbb{Z}_{24})$ is clearly injective, so $\kappa(M) = 0$, and M admits a PL-structure by the Obstruction Theorem. $\square$

Theorem 6.2.3. *Let M a topological manifold of dimension $n \geq 5$, and $f : M \to \mathbb{T}^n$ a homotopy equivalence. Then f is homotopic to a homeomorphism.*

Proof. Recall that the PL-structures on $\mathbb{T}^n$ are classified by $H^3(\mathbb{T}^n, \mathbb{Z}_2)$. Now the same group also classifies the homotopy PL-structures on $\mathbb{T}^n$, by 4.2.9. Thus we have the maps

$$H^3(\mathbb{T}^n, \mathbb{Z}_2) \cong [\mathbb{T}^n, TOP/PL] \xrightarrow[\approx]{j_{TOP}} \mathcal{T}_{PL}(\mathbb{T}^n) \xhookrightarrow{\beta} \mathcal{S}_{PL}(\mathbb{T}^n) \cong H^3(\mathbb{T}^n, \mathbb{Z}_2) \,.$$

Since $H^3(\mathbb{T}^n, \mathbb{Z}_2)$ is finite, we get

$$\mathcal{T}_{PL}(\mathbb{T}^n) \xrightarrow[\approx]{\beta} \mathcal{S}_{PL}(\mathbb{T}^n) \,.$$

Thus, to the homotopy equivalence $f : M \to \mathbb{T}^n$ representing an element of $\mathcal{S}_{PL}(\mathbb{T}^n)$ there corresponds a homeomorphism $g : N \to \mathbb{T}^n$ and a homeomorphism $H : N \to M$ such that the following triangle

$$\begin{tikzcd}
M \arrow[r, "f"] & \mathbb{T}^n \\
N \arrow[u, "H"] \arrow[ur, "g"'] &
\end{tikzcd}$$

commutes up to homotopy. In particular, f is homotopic to a homeomorphism. $\square$

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