Differentiable Manifolds Which Are Homotopy Spheres
J. Milnor

§1. Introduction

This paper will study the problem of classifying differentiable n-manifolds which are homotopy spheres, under the relation of J-equivalence. (See the "dictionary" below.) It is shown that the equivalence classes form an abelian group which is denoted by $\mathfrak{G}^n$. The only groups $\mathfrak{G}^n$ which I have been able to determine completely are the following:

$$\mathfrak{G}^1 = \mathfrak{G}^2 = 0, \quad \mathfrak{G}^5 = 0, \quad \mathfrak{G}^7 = \mathbb{Z}_{28}, \quad \mathfrak{G}^{11} = \mathbb{Z}_{992}.$$ 

However partial information is obtained in many other cases. For example (according to 3.7, 5.8 and 6.9):

**Theorem.** For $k > 1$ the group $\mathfrak{G}^{4k-1}$ is finite but non-trivial.

Section 2 of this paper will study a sum operation for connected manifolds of the same dimension. Section 3 defines an invariant $\lambda'$ for certain $(4k-1)$-manifolds. Section 4 contains examples of homotopy spheres for which the invariant $\lambda'$ takes on all possible values.

Section 5 describes a construction for simplifying manifolds, which was communicated to the author by R. Thom. Using this construction it is shown that the invariant $\lambda'(M)$ determines the J-equivalence class of $M$ uniquely. A corresponding result for dimensions of the form $4k + 1$ is stated without proof. Section 6 studies the following question: Is every homotopy sphere the boundary of a $\pi$-manifold?

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Section 7 contains further discussion and a list of unsolved problems. Operations of "pasting together" manifolds and "straightening angles" are described in an appendix.

Dictionary of terms used. The word manifold will mean a compact, oriented, differentiable manifold, with or without boundaries. (The phrase "topological manifold" will be used in case the differentiable structure has not yet been specified.) The symbol $-M$ will be used for the manifold $M$ with orientation reserved.

Two unbounded manifolds $M_1, M_2$ of the same dimension are **J-equivalent** if there exists a manifold $W$ such that

1) the boundary $\partial W$ is the disjoint union of $M_1$ and $-M_2$, and

2) both $M_1$ and $M_2$ are deformation retracts of $W$.

Thus J-equivalent manifolds belong to the same cobordism class and to the same homotopy type. This concept is due to Thom [3]. It is not known whether J-equivalent manifolds are necessarily diffeomorphic.

By a homotopy sphere we mean a (differentiable) manifold without boundary which has the homotopy type of a sphere. Similarly a homology sphere $M$ must be unbounded and satisfy $H_*(M) \cong E_*(S^n)$. Here $H_*$ denotes homology with integer coefficients, and $S^n$ denotes the unit sphere in Euclidean space $R^{n+1}$. The notation $D^{n+1}$ will be used for the disk bounded by $S^n$.

1.1 **Lemma.** Let $M^n = \partial W^{n+1}$ where $M^n$ is simply connected and $W^{n+1}$ is contractible. Then $M^n$ is J-equivalent to $S^n$.

**Proof.** Choose an imbedding of $D^{n+1}$ in the interior of $W^{n+1}$. Then $(W^{n+1} - \text{interior} (D^{n+1}))$ has* boundary equal to the disjoint union

*Here the symbol $-$ stands for set theoretic subtraction.
of \( M^n \) and \( S^n \). It is not difficult to see that both boundaries are deformation retracts of \( W^{n+1} \)-interior \((U^{n+1})\).

A \( \pi \)-manifold \( W^n \) is characterized by the following property. If \( W^n \) is imbedded in a high dimensional Euclidean space \( R^{n+q} \), then the normal bundle \( V^q \) is trivial. This concept is due to J. H. C. Whitehead [2]. If \( W \) is a \( \pi \)-manifold, then clearly \( \partial W \) is also a \( \pi \)-manifold.

\( W^n \) will be called almost parallelizable if there exists a finite subset \( F \) so that \( W^n - F \) is parallelizable.

1.2 Lemma (J. H. C. Whitehead) Every parallelizable manifold is a \( \pi \)-manifold. Every \( \pi \)-manifold is almost parallelizable.

Proof. A field of tangent \( n \)-frames on \( W^n \subset R^{n+q} \) induces a map \( f \) from \( W^n \) to the Stiefel manifold \( V_{n+q,n} \). Note that \( f \) is covered by a bundle map from \( V^q \) to a corresponding \( SO_q \)-bundle over \( V_{n+q,n} \).

But the space \( V_{n+q,n} \) is \((q-1)\)-connected. (See Steenrod [1] §25.6.) For \( q > n \) this implies that \( f \) is homotopic to a constant; hence that \( V^q \) is trivial.

Similarly a field of normal \( q \)-frames on \( W^n \) induces \( f: W^n \rightarrow V_{n+q,q} \).

Since \( V_{n+q,q} \) is \((n-1)\)-connected, the only obstruction to contracting \( f \) lies in

\[ H^n (W^n; \pi_n (V_{n+q,q})) \]

But this cohomology group can be killed by removing a finite number of points from \( W^n \).

A similar argument shows the following.

1.3 Lemma. If every component of \( W^n \) has a non-vacuous boundary, then the three concepts: parallelizable, \( \pi \)-manifold, and almost parallelizable, are equivalent.
The \textit{J-homomorphism} of H. Hopf and G. Whitehead will be denoted by

\[ J_n : \pi_n(S^q) \rightarrow \pi_{n+q}(S^q) . \]

(For a definition see Kervaire [4] §1.8. Caution: this homomorphism has nothing to do with J-equivalence.) It will always be assumed that \( q \) is large. This homomorphism will play a fundamental role in what follows.

§2. \textbf{The connected sum of manifolds}

Let \( M_1, M_2 \) be connected differentiable manifolds of the same dimension \( n \). The sum \( M_1 \# M_2 \) is obtained by removing an \( n \)-cell from each, and then pasting the resulting boundaries together. There are three difficulties with this:

1) The pasting must be done in such a way that \( M_1 \# M_2 \) has an orientation compatible with that of both \( M_1 \) and \( M_2 \).

2) Even allowing for orientation, not every diffeomorphism between the boundaries will give rise to the same composite manifold. (According to Milnor [1] it is possible to paste together the boundaries of two 7-cells, obtaining a manifold which is not diffeomorphic to \( S^7 \).)

3) It is necessary to show that the result does not depend on which \( n \)-cell is chosen.

\textbf{Definition.} Choose an orientation preserving imbedding \( h_1 : R^n \rightarrow M_1 \) and an orientation reversing imbedding \( h_2 : R^n \rightarrow M_2 \). Let \( M_1 \# M_2 \) be obtained from the disjoint union of \( M_1 - h_1(0) \) and \( M_2 - h_2(0) \) by identifying \( h_1(x) \) with \( h_2(x/\|x\|^2) \) for each \( x \neq 0 \) in \( R^n \).

\textbf{Remark.} It would be sufficient to specify \( h_1(x) \) and \( h_2(x) \) for \( \|x\| < 1 + \varepsilon \) in order to construct this manifold \( M_1 \# M_2 \). In fact by removing all \( h_1(x) \) with \( \|x\| \leq 1/(1 + \varepsilon) \) from each \( M_1 \), and then
identifying \( h_1(x) \) with \( h_2(x/\|x\|^2) \) for \( 1+\epsilon > x > 1/(1+\epsilon) \), we obtain the identical manifold \( M_1 \# M_2 \).

The following will be proved in a paper by J. Cerf.

2.1 **Theorem of Cerf.** Let \( M \) be a connected \( n \)-manifold. Given two orientation preserving imbeddings \( f, f' : D^n \rightarrow (\text{interior } M) \), there exists a diffeomorphism \( g : M \rightarrow M \) which satisfies \( gf = f' \).

2.2 **Corollary.** The sum \( M_1 \# M_2 \) is well defined up to orientation preserving diffeomorphism.

**Proof of the corollary.** The only choice which occurred in the definition was the choice of imbeddings \( h_1, h_2 \). Given other imbeddings \( h_1', h_2' \), there exist diffeomorphisms \( g_i \) of \( M_i \) so that

\[
g_i h_i(x) = h_i'(x) \text{ for } \|x\| \leq 1 + \epsilon.
\]

These \( g_i \) give rise to a diffeomorphism \( g : M_1 \# M_2 \rightarrow (M_1 \# M_2)' \); which completes the proof.

2.3 **Lemma.** Suppose that the unbounded manifolds \( M_1, M_2 \) are \( J \)-equivalent to \( M'_1 \) and \( M'_2 \) respectively. Then the sum \( M_1 \# M_2 \) is \( J \)-equivalent to \( M'_1 \# M'_2 \).

**Proof.** If the dimension \( n \) is \( \leq 2 \), then the assertion is clear. Hence we may assume that \( n \geq 3 \). Choose manifolds \( W_1 \) so that \( \partial W_1 \) is the disjoint union of the deformation retracts \( M_1 \) and \( -M'_1 \). Choose a differentiable arc \( a_1 \) from \( p_1 \in M_1 \) to \( p'_1 \in M'_1 \) in \( W_1 \), so that the interior of \( a_1 \) lies in the interior of \( W_1 \). We will see that the inclusion map

\[
j : M_1 \rightarrow W_1 - a_1
\]

is a homotopy equivalence.
Since the codimension \( n \) of \( p_1 \) in \( M_1 \) is \( \geq j \), the homomorphisms \( \pi_1(M_1-p_1) \to \pi_1(M_1) \), \( \pi_1(W_1-a_1) \to \pi_1(W_1) \) are isomorphisms. Hence

\[
J_* : \pi_1(M_1-p_1) \to \pi_1(W_1-a_1)
\]

is an isomorphism.

Let \( \hat{M}_1 \subset \hat{W}_1 \) denote the universal covering spaces, and let \( \hat{p}_1 \subset \hat{a}_1 \) denote the inverse images of \( p_1, a_1 \). The inclusion

\[
(\hat{M}_1, \hat{M}_1-\hat{p}_1) \to (\hat{W}_1, \hat{W}_1-\hat{a}_1)
\]

gives rise to a homomorphism between exact sequences of homology groups. Using the Five Lemma it follows that

\[
J_* : H_k(\hat{M}_1-\hat{p}_1) \to H_k(\hat{W}_1-\hat{a}_1)
\]

is an isomorphism for all \( k \). Therefore \( J \) is a homotopy equivalence. (Compare J.H.C. Whitehead [3].)

Choose tubular neighborhoods \( N_1 \) of \( a_1 \), and let \( W \) be a manifold obtained from \( W_1-N_1 \) and \( W_2-N_2 \) by pasting together the boundaries in such a way that \( \partial W \) is the disjoint union of \( M_1 \# M_2 \) and \( -(M_1' \# M_2') \).

Since the inclusions

\[
M_1- (M_1 \cap N_1) \to W_1-N_1
\]

are homotopy equivalences, it follows easily that the inclusion

\[
M_1 \# M_2 \to W
\]

is a homotopy equivalence. A corresponding argument takes care of the inclusion \( (M_1' \# M_2') \to W \). This completes the proof of 2.3.

It is clear that the operation \( \# \) is associative and commutative, providing that we do not distinguish between diffeomorphic manifolds.

Furthermore the sphere acts as a zero element: \( M \# S^n \simeq M \).
2.4 **Lemma.** Suppose that $M$ is a homotopy n-sphere. Then $M \# (-M)$ is $J$-equivalent to $S^n$.

**Proof.** Let $U$ denote the interior of a disk $D^2 \subset M$. Consider the topological manifold $(M-U) \times [0,1]$. This is differentiable, except along the "angles" $\partial U \times [0]$ and $\partial U \times [1]$. Let $W$ be a differentiable manifold obtained from $(M-U) \times [0,1]$ by straightening these angles. (See the Appendix.) Then $W$ is a contractible manifold with boundary $M \# (-M)$. Together with 1.1 this completes the proof.

Now combining 2.3 and 2.4 this proves:

2.5 **Theorem.** The set of all $J$-equivalence classes of homotopy n-spheres forms an abelian group under the operation $\#$.

This group will be denoted by $\Theta^n$. It is clear that $\Theta^1 = 0$.

Since Munkres [1] has shown that a 2-manifold has an essentially unique differentiable structure, it follows that $\Theta^2 = 0$.

[Two subgroups of $\Theta^n$ will also be studied. $\Theta^p(\pi)$ will denote the subgroup formed by all $\pi$-manifolds in $\Theta^n$, and $\Theta^p(\partial \pi)$ will denote the subgroup formed by all boundaries of $\pi$-manifolds.]

§3. **The invariant** $\lambda^*(M^{4k-1})$

Let $M$ be a $(4k-1)$-manifold which is (1) a homology sphere, and (2) the boundary of some $\pi$-manifold $W$. The intersection number of two homology class $\alpha, \beta$ of $W$ will be denoted by $\langle \alpha, \beta \rangle$. Let $I(W)$ denote the index of the quadratic form

$$\alpha \rightarrow \langle \alpha, \alpha \rangle,$$

where $\alpha$ varies over the Betti group $H_{2k}(W)/(\text{torsion})$. Integer coefficients are to be understood.
Define \( I_k \) as the greatest common divisor of \( I(M) \) where \( M \) ranges over all almost parallelizable manifolds of dimension \( 4k \) which have no boundary. This number has been studied by Kervaire and Milnor [1]. (See 3.7.)

3.1 Lemma. The residue class of \( I(W) \) modulo \( I_k \) is an invariant of the boundary \( M \).

Proof. If \( M \) is the boundary of two parallelizable manifolds \( W_1 \) and \( W_2 \), let \( N \) be the unbounded \( 4k \)-manifold obtained from \( W_1 \) and \( -W_2 \) by pasting together the common boundary. Clearly

\[
I(N) = I(W_1) - I(W_2).
\]

Let \( p \) be a point of \( M \). Then the complement \( N \setminus p \) is parallelizable. In fact \( N \setminus p \) is the union of parallelizable manifolds \( W_1 \setminus p \) and \( W_2 \setminus p \), having an intersection \( M \setminus p \) which is acyclic. Given a field of \( 4k \)-frames on \( W_1 \setminus p \) and on \( W_2 \setminus p \), it is possible to deform one of the two so that they coincide along \( M \setminus p \). Therefore \( N \) is almost parallelizable; and

\[
I(N) \equiv 0 \pmod{I_k}.
\]

This completes the proof.

Not every residue class can occur:

3.2 Lemma. The index \( I(W) \) of an almost parallelizable manifold is always divisible by \( 8 \); providing that \( \partial W \) is a homology sphere.

Proof. First observe that the intersection number \( \langle \alpha, \alpha \rangle \) is always an even integer. This is the homology translation of the statement that

\[
\text{Sq}^{2k} : H^{2k}(W, \partial W ; \mathbb{Z}) \longrightarrow H^{4k}(W, \partial W ; \mathbb{Z}_2)
\]

is zero. If \( \text{Sq}^{2k} \) were not zero then the formulae of Wu (see Wu [1],
Kervaire [2]) would imply that $W$ had a non-trivial Stiefel-Whitney class in dimension $\leq 2k$.

Since $\partial W$ is a homology sphere it follows by Poincare duality that the matrix of intersection numbers has determinant $\pm 1$. But a quadratic form with determinant $\pm 1$ which takes on only even values must have index divisible by 8. (Compare Milnor [4].) This completes the proof.

**Definition.** The residue class of $\frac{1}{8} I(W)$ modulo $\frac{1}{8} I_k$ will be denoted by $\lambda^*(M)$.

3.3 **Lemma.** The properties of being (1) a homotopy $n$-sphere, and (2) the boundary of a $\pi$-manifold, are invariant under $J$-equivalence; and are preserved by the sum operation $\#$.

Hence the manifolds which have these properties give rise to a subgroup of $\Theta^R$.

**Definition.** This subgroup will be denoted by $\Theta^R(\partial \pi)$.

3.4 **Lemma.** The invariant $\lambda^*(M)$ depends only on the $J$-equivalence class of $M$. Furthermore

$$\lambda^*(M_1 \# M_2) = \lambda^*(M_1) + \lambda^*(M_2).$$

The proofs of 3.3 and 3.4 are straightforward. Hence $\lambda^*$ gives rise to a homomorphism

$$\lambda^*: \Theta^{4k-1}(\partial \pi) \rightarrow \mathbb{Z} \frac{1}{8} I_k$$

It will be proved in Sections 4, 5 that $\lambda^*$ is an isomorphism, at least for $k > 1$.

The principal difficulty with the invariant $\lambda^*$ is that it is extremely difficult to compute. For example it would be very interesting to evaluate $\lambda^*$ for the topological spheres which are constructed in Milnor [1, 5] and Shimada [1]. The invariant $\lambda$ which is defined in these papers is somewhat weaker, but much easier to compute.
The numbers \( \frac{1}{6k} I_k \) can be described as follows. Let \( B_k \) denote the \( k \)-th Bernoulli number:

\[
B_1 = \frac{1}{6}, \quad B_2 = \frac{1}{30}, \quad \ldots, \quad B_6 = 691/2730, \quad \ldots
\]

Define \( j_k \) as the order of the image

\[
J_{4k-1}(S^q) \subseteq \pi_{q+4k-1} (S^q) \quad \text{for large } q.
\]

Define \( a_k \) to be 2 if \( k \) is odd and 1 if \( k \) is even. Then according to Kervaire and Milnor [1]:

3.5 **Lemma.** \( I_k \) is equal to

\[
2^{2k-1} (2^{2k-1} - 1) B_k j_k a_k / k.
\]

The only unknown quantity here is the integer \( j_k \).

3.6 **Lemma.** \( j_k \) is a multiple of the denominator of \( B_k / 4k \).

**Proof.** For \( k \) even, this is proved in Kervaire and Milnor [1].

For \( k \) odd this follows from the arguments of that paper, together with the following:

**Theorem of Hirzebruch (Not yet published.)** If the unbounded manifold \( M^{4k} \) has Stiefel-Whitney class \( w_2 \) equal to zero, and if \( k \) is odd, then the \( \hat{A} \)-genus \( \hat{A} (M^{4k}) \) is an even integer.

On the other hand an upper bound for \( j_k \) is given by the order of the largest cyclic subgroup of \( \pi_{q+4k-1} (S^q) \). The p-primary component of \( \pi_{q+4k-1} (S^q) \) is known for \( k < p^{2(p-1)}/2 \) for any prime \( p \). (See Toda [1,2].) The full group is known (to me) only for \( k = 1, 2, 3 \).

It turns out that the upper bound for the p-primary factor of \( j_k \) is exactly equal to the lower bound in each known case.
Combining the preceding information, we have:

3.7 Lemma. The number \( \frac{1}{8} \mathcal{I}_k \) is equal to \( a_k 2^{2k-2(2^{2k-1}-1)} \) (numerator \( a_k/k \)), multiplied by an integer whose prime factors \( p \) satisfy \( p^2(p-1) \leq 2k \). In particular

\[
\frac{1}{8} \mathcal{I}_1 = 2, \quad \frac{1}{8} \mathcal{I}_2 = 28, \quad \frac{1}{8} \mathcal{I}_3 = 992, \quad \frac{1}{8} \mathcal{I}_4 \text{ equals 8128 times a power of 2.}
\]

§4. Construction of \((4k-1)\)-manifolds

The following is perhaps the simplest example of a symmetric matrix with determinant \( \pm 1 \), with only even elements on the diagonal, and with index different from zero. (Compare Milnor [4].)

\[
\begin{vmatrix}
2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 1 & 0 & -1 & 0 & 0 & 0 \\
0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\
\end{vmatrix}
\]

We will construct a manifold \( W^{4k} \) which has the above intersection matrix.

Let \( T \) be a tubular neighborhood of the diagonal in \( S^{2k} \times S^{2k} \); say the set of all pairs \( (x,y) \) with distance \( d(x,y) \leq \varepsilon \). Thus \( T \) is a \( 4k \)-manifold having the homotopy type of \( S^{2k} \). The intersection number of the fundamental \( 2k \)-cycle with itself is \( +2 \).
Let $\alpha : S^{2k} \to S^{2k}$ be the "twelve hour rotation" which leaves the north pole $p$ fixed, and satisfies $\alpha(x) = -x$ for $x$ on the equator. Let $T' = (1 \times \alpha) T$ be the set of pairs $(x,y)$ with $d(x,\alpha y) \leq \varepsilon$. Then $T$ and $T'$ intersect only in a small neighborhood of the pair $(p,p)$, and a small neighborhood of the pair $(-p,-p)$.

The universal covering space of $T \cap T'$ consists of infinitely many disjoint copies of $T$ and infinitely many disjoint copies of $T'$. Numbering these copies $T_i$ and $T'_i$, we may assume that each $T'_i$ intersects only $T_i$ and $T_{i+1}$.

Define $W_1$ as the subset

$$T_1 \cup T'_1 \cup T_2 \cup T'_2 \cup T_3 \cup T'_3 \cup T_4 \cup T'_4$$

of this universal covering space. Thus $W_1$ is a topological $4k$-manifold, having the homotopy type of a union of eight $2k$-spheres with a single point in common. Choosing an appropriate basis for $H_{2k}(W_1)$, the intersection matrix is as follows:

$$
\begin{pmatrix}
2 & 1 \\
1 & 2 & 1 \\
& 1 & 2 & 1 \\
& & 1 & 2 & 1 \\
& & & 1 & 2 & 1 \\
& & & & 1 & 2 \\
\end{pmatrix}
$$

To correct this intersection matrix it is necessary to introduce an intersection between $T'_1$ and $T'_3$, so as to obtain an intersection number $-1$. Choose a rotation of $S^{2k} \times S^{2k}$ which carries a region of $T'$ near the "equator" onto a region of $T$ near the "equator", so as to obtain an intersection number of
-1. Matching the corresponding regions of $T'_1$ and $T'_3$, we obtain a topological manifold $W_2$, with the required intersection matrix.

This manifold $W_2$ is differentiable except along eight "angles" which have been introduced in the boundary. Let $W_3$ be a differentiable manifold obtained by straightening these angles. (See the appendix.)

Unfortunately the transition from $W_1$ to $W_2$ changed the homotopy type. In fact the fundamental group $\pi_1(W_2) = \pi_1(W_3)$ is infinite cyclic. Next we will kill this fundamental group. A generator can be represented by a simple closed differentiable curve $C$ lying on the boundary of $W_3$.

Choose an imbedding $h: S^1 \times D^{4k-2} \to \partial W_3$ which carries $S^1 \times 0$ onto the given curve $C$. Let $W_4$ be the space obtained from the disjoint union $W_3 \cup D^2 \times D^{4k-2}$ by identifying $S^1 \times D^{4k-2}$ with its image under $h$. Then $W_4$ is simply connected. In fact $W_4$ has the same homotopy type as $W_1$; but the same intersection matrix as $W_2$ or $W_3$.

This space $W_4$ is a differentiable manifold, except along the "angle" corresponding to $S^1 \times S^{4k-3}$. Let $W_5$ be a differentiable manifold obtained by "straightening" this angle.

4.1 Theorem. $W_5$ is a parallelizable $4k$-manifold with boundary $M_0$ which is a homology $(4k-1)$-sphere. In fact for $k > 1$, $M_0$ is a homotopy sphere. The index $I(W_5)$ equals $-8$.

Thus the invariant $\Lambda'(M_0)$ is defined and equal to $+1$.

4.2 Corollary. The homomorphism $\Lambda'$ from $\theta^{4k-1}(\partial \nu)$ to the cyclic group of order $\frac{1}{8} I_k$ is onto, providing that $k > 1$. 
4.3 Corollary. The group $\pi_{4k-1}^h$ is non-trivial, providing that $k > 1$.

Proof that $W_0$ is parallelizable. The only obstruction to parallelizability lies in the group

$$H^{2k}(W_0; \pi_{2k-1}(SO_{4k})).$$

But $H_{2k}(W_0)$ is generated by eight cycles, each of which is contained in a sub-manifold diffeomorphic to $T \subset S^{2k} \times S^{2k}$. Since $S^{2k}$ is a $\pi$-manifold, it follows that $T$ is a $\pi$-manifold, hence parallelizable. Therefore $W_0$ is parallelizable.

Computation of $H_*(M_0^c)$. Since the groups $H_1(W_0)$ have no torsion, it follows by Poincaré-Lefschetz duality that $H_1(W_0, M_0^c) \simeq H_{4k-1}(W_0)$. The natural homomorphism

$$H_1(W_0) \to H_1(W_0, M_0^c) \simeq \text{Hom}(H_{4k-1}(W_0), \mathbb{Z})$$

is determined by the matrix of intersection numbers.

Now recall that $H_0(W_0) = \mathbb{Z}$; that $H_i(W_0) = 0$ for $i \neq 0, 2k$; and that the matrix of intersection numbers in dimension $2k$ has determinant $+1$. Plugging this information into the exact sequence of the pair $(W_0, M_0)$, it follows that $M_0^c$ has the homology of a $(4k-1)$-sphere.

Proof that $M_0^c$ is simply connected, providing that $k > 1$. (For $k$ equal to 1 the group $\pi_1(M_0^c)$ depends on the choice of the curve $C$. If $C$ could be chosen so that $\pi_1(M_0^c) = 0$, then $M_0^c$ would provide a counter-example to the Poincaré hypothesis.)

Let $K \subset W_3$ denote the union of 8 copies of $S^{2k}$, one in the center of each $T_i$ and $T_i'$. Since the codimension $2k$ of $K$ is greater
than 2, it follows that
\[ \pi_2(W_3, W_3 - K) = 0. \]

But it is clear that \( \partial W_3 \) is a deformation retract of \( W_3 - K \). Hence \( \pi_2(W_3, \partial W_3) = 0 \). From the exact sequence of this pair it follows that \( \pi_1(\partial W_3) \) is infinite cyclic, generated by the closed curve \( C \).

The manifold \( \partial W_4 \) can be obtained from \( \partial W_3 \) in two steps, as follows. (Compare Lemma 5.3.)

1) Remove a tubular neighborhood of \( C \subset \partial W_3 \). Since the codimension \( 4k - 2 \) of \( C \) in \( \partial W_3 \) is greater than 2, it follows that \( \pi_1(\partial W_3 - C) \) is also infinite cyclic.

2) Fill in the resulting hole with a copy of \( D^2 \times S^{4k - 3} \).

The effect of this addition on the fundamental group is to kill the generator. Hence \( \pi_1(\partial W_4) = 0 \).

Since the differentiable manifold \( M_0 = \partial W_0 \) is homeomorphic to \( \partial W_4 \), this completes the proof that \( M_0 \) is a homotopy sphere. Since the index \( I(W_0) \) is easily shown to be \( +8 \), this proves Theorem 4.1.

§5. Simplifying manifolds by surgery

This section will describe an operation, suggested to the author by Thom, which can be used to kill off the lower homotopy groups of a manifold. To illustrate the method, the following will first be proved.

5.1 Theorem. Let \( M \) be an unbounded \( 4k \)-manifold which is almost parallelizable. (That is there exists a finite subset \( F \), so that \( M - F \) is parallelizable.) Then there exists an unbounded \( 4k \)-manifold \( M' \) which satisfies:

1) the index \( I(M') \) equals \( I(M) \),
2) \( M' \) is also almost parallelizable, and
3) \( M' \) is \( (2k-1) \)-connected.
Remark 1. It is not possible to kill off any further homotopy groups: If \( M' \) were 2k-connected, then the index \( I(M') \) would have to be zero.

Remark 2. The hypothesis that \( M \) is almost parallelizable is essential here. As an example, for \( n=12 \), the complex projective space \( \mathbb{P}^6(C) \) has index 1. But for any 5-connected 12-manifold, the index must be divisible by \( I_3 = 7936 \). (This follows since the only obstruction to almost parallelizability lies in \( H^6(M; \pi_5(SO_{12})) = 0 \).)

Proof of 5.1. If \( M \) has several components, let \( M_1 \) denote the connected sum of these components. It is not hard to show that \( M_1 \) is almost parallelizable, and that \( I(M_1) = I(M) \).

Suppose by induction that \( M \) is \((q-1)\)-connected, where \( 0 < q < 2k \). Any given element \( \alpha \in \pi_q(M) \) can be represented by an imbedding \( f : S^q \to M \).

(This presents no difficulty since the dimension \( 4k \) is greater than \( 2q \). Compare Whitney [1].)

5.2 Lemma. Let \( f : S^q \to M^n \) be an imbedding, with \( q < \frac{1}{2}n \); and suppose that the bundle \( f^*(\tau^n) \) induced from the tangent bundle of \( M^n \) is trivial. Then the normal bundle \( \nu^{n-q} \) is trivial.

Proof. Let \( o_1^k \) denote the trivial \( SO_k \)-bundle over \( S^q \), and let \( \tau^q \) denote the tangent bundle. It is well known that the Whitney sum \( \tau^q \oplus o_1 \) is trivial. We are assuming that the bundle \( \nu^{n-q} \oplus \tau^q \) \( \sim f^*(\tau^n) \) is trivial. Therefore

\[
\nu^{n-q} \oplus o^{q+1} \sim \nu^{n-q} \oplus \tau^q \oplus o_1^1 \sim \nu^n \oplus o^1 \sim o^{n+1}
\]

is trivial. That is: the inclusion \( SO_{n-q} \to SO_{n+1} \) carries the
$\mathbb{S}^{n-q}$ bundle $\mathcal{V}^{n-q}$ into the trivial bundle. But the homomorphism

$$\pi_{q-1}(\mathbb{S}^{n-q}) \to \pi_{q-1}(\mathbb{S}^{n+1})$$

is an isomorphism in the stable range $n-q > q$. This completes the proof.

Let $T$ be a tubular neighborhood of $f(S^q)$. Then $T$ can be identified with the total space of the $D^{4k-q}$ bundle which is associated with $\mathcal{V}^{4k-q}$. Choosing a specific product structure for $\mathcal{V}^{4k-q}$, it follows that $T$ is homeomorphic to $S^q \times D^{4k-q}$. Let $M_1$ denote a differentiable manifold obtained from $M$ by

1) removing the interior of $T$, and
2) pasting a copy of $D^{q+1} \times S^{4k-q-1}$ in its place, matching the common boundary $S^q \times S^{4k-q-1}$.

5.3 Lemma. The manifold $M_1$ is also $(q-1)$-connected. Furthermore $\pi_q(M_1)$ is isomorphic to $\pi_q(M)/(\alpha)$, where $(\alpha)$ denotes the normal subgroup generated by $\alpha$.

Proof. Since $f(S^q)$ has codimension $4k-q$ in $M$, it follows that $\pi_i(M - f(S^q))$ is isomorphic to $\pi_i(M)$ for $i < 4k-q-1$. In particular

$$\pi_i(M - f(S^q)) = 0 \text{ for } i < q,$$

and

$$\pi_q(M - f(S^q)) = \pi_q(M).$$

The manifold $M - f(S^q)$ can be obtained from $M_1$ by removing the sphere $0 \times S^{4k-q-1}$ of codimension $q+1$.

Therefore

$$\pi_i(M_1) = \pi_i(M - f(S^q)) = 0 \text{ for } i < q.$$  

Case $q = 1$. Then $\pi_1(M_1)$ can be computed as follows. The manifold $M_1$ can be obtained from $M - T$ by first adjoining a 2-cell.
$D^2 \times \text{(constant)}$ and then adjoining a $4k$-cell. The first operation introduces the relation $\alpha = 0$ into the fundamental group; while the second operation leaves the group unchanged.

**Case 2.** $q > 1$. Then the group $\pi_{q+1}(M_1, M-f(S^q))$ is isomorphic to $H_{q+1}(M_1, M-f(S^q)) \cong \mathbb{Z}$. In the exact sequence

$$Z \longrightarrow \pi_q(M-f(S^q)) \longrightarrow \pi_q(M_1) \longrightarrow 0,$$

it is clear that $\partial(1) = \alpha$, so that $\pi_q(M_1) \cong \pi_q(M)/\langle \alpha \rangle$, as required. This completes the proof of 5.3.

**5.4 Lemma.** If the product structure for the normal bundle $\nu^{4k-q}$ is correctly chosen, then the manifold $M_1$ will also be almost parallelizable.

Before giving the proof, here is a description of some vector fields on $S^q \subset D^{q+1}$. Let $\varepsilon_1, \ldots, \varepsilon_{q+1}$ be the standard basis for the tangent vector space of $D^{q+1}$. The outward normal vector at a point $(t_1, \ldots, t_{q+1}) \in S^q$ is $\zeta = t_1 \varepsilon_1 + \ldots + t_{q+1} \varepsilon_{q+1}$. Let $\varepsilon'_1$ denote the projection of $\varepsilon_1$ into the tangent bundle of $S^q$. Thus $\varepsilon'_1 = \varepsilon_1 - t_1 \zeta$, so that $\varepsilon_1 = \varepsilon'_1 + t_1 \zeta$.

**Proof of 5.4.** Choose some field $\varphi_1$ of vectors normal to $f(S^q)$. The "endpoints" of the vectors $\varphi_1$ sweep out a subset of $\partial T$ which will be denoted by $S^q \times (1, 0, \ldots, 0)$. The outward normal vector to $\partial T$ at a point of $S^q \times (1, 0, \ldots, 0)$ will also be denoted by $\varphi_1$ (since it can be considered as a translate of the vector $\varphi_1$ at the corresponding point of $f(S^q)$). Now consider the vector fields

$$\varepsilon'_1 + t_1 \varphi_1, \quad \varepsilon'_2 + t_2 \varphi_1, \ldots, \quad \varepsilon'_{q+1} + t_{q+1} \varphi_1,$$

along $S^q \times (1, 0, \ldots, 0)$. These are orthogonal unit vectors in
the tangent bundle of \( M \).

Since \( M \) is almost parallelizable, there exists a field 
\((\psi_1, \ldots, \psi_{4k})\) of \(4k\)-frames which is defined over \( M - F \). Here \( F \) 
denotes some finite set which we may assume is disjoint from \( T \).

**Assertion.** It is possible to deform this field \((\psi_1, \ldots, \psi_{4k})\) 
so as to obtain a field \((\psi_1', \ldots, \psi_{4k}')\) of \(4k\)-frames such that 
along \( S^q \times (1, 0, \ldots, 0) \):

\[
\psi_1' = \varepsilon_1' + t_1 \varphi_1', \ldots, \psi_{q+1}' = \varepsilon_{q+1}' + t_{q+1} \varphi_1' .
\]

This is proved as follows. Define a matrix \( a_{ij}(t_1, \ldots, t_{q+1}) \) 
by the formulae

\[
\varepsilon_i' + t_i \varphi_1 = \sum_{j=1}^{4k} a_{ij} \psi_j' , 
\quad i = 1, 2, \ldots, q + 1 .
\]

This defines a map from \( S^q \) to the Stiefel manifold \( V_{4k, q+1} \).
This map is null-homotopic since \( \pi_q(V_{4k, q+1}) = 0 \). Hence it can 
be lifted to a null-homotopic map of \( S^q \) into \( V_{4k, 4k} \)

\[
(t_1, \ldots, t_{q+1}) \longrightarrow \| a_{ij} \| , \quad i = 1, \ldots, 4k .
\]

Let \( \psi_i' = \psi_i \) outside of a neighborhood of \( S^q \times (1, 0, \ldots, 0) \) but 
let

\[
\psi_i' = \sum a_{ij} \psi_j
\]

for points in \( S^q \times (1, 0, \ldots, 0) \), and for all \( i \). The null-
homotopy can now be used to define \( \psi_i' \) throughout the neighborhood.

We may assume that the vectors \( \psi_i' \) along \( f(S^q) \) are translates 
of those along \( S^q \times (1, 0, \ldots, 0) \).

Now choose the product structure for the normal bundle of \( f(S^q) \) 
which is determined by the field.
of normal \((4k-q)\)-frames. In terms of this product structure, construct the manifold

\[ M_1 = M-(\text{interior } T) \cup D^{q+1} \times S^{4k-q-1}. \]

The \(4k\)-frame \(\psi_1', \ldots, \psi_{4k}'\) in \(M-(\text{interior } T)\) can be extended throughout \(D^{q+1} \times (1,0,\ldots,0)\) as follows. Note that the vector \(\phi_{1}\) along \(S^q \times (1,0,\ldots,0)\) can be identified with the normal vector to \(S^q\) in \(D^{q+1}\). Hence the vectors \(\psi_i' = \phi_i + t_i \theta_i\) \((1 \leq i \leq q + 1)\) along \(S^q \times (1,0,\ldots,0)\) can be identified with the standard basis for the tangent bundle of \(D^{q+1} \times (1,0,\ldots,0)\). Hence these vectors \(\psi_1', \ldots, \psi_{q+1}'\) can be extended.

The remaining vectors \(\psi_{q+2}', \ldots, \psi_{4k}'\) are normal to \(D^{q+1}\). The projection \(S^q \times S^{4k-q-1} \rightarrow S^{4k-q-1}\) carries these vectors onto a fixed \((4k-q-1)\)-frame at the point \((1,0,\ldots,0)\in S^{4k-q-1}\). Hence it is certainly possible to extend \((\psi_{q+2}', \ldots, \psi_{4k}')\) over \(D^{q+1} \times (1,0,\ldots,0)\) as a field of normal \((4k-q-1)\)-frames.

Thus a field of \(4k\)-frames has been defined over the subset

\[(M-(\text{interior } T)-F) \cup (D^{q+1} \times (1,0,\ldots,0))\]

of \(M_1 - F\). The complement of this set in \(M_1 - F\) consists of a single \(4k\)-cell: \((\text{interior } D^{q+1}) \times (S^{4k-q-1} - \text{point})\). Let \(F'\) consist of \(F\) together with a single point in this cell. Then it is clearly possible to extend the \(4k\)-frame field throughout \(M_1 - F'\). This completes the proof that \(M_1\) is almost parallelizable.

Remark. It cannot be proved that \(M_1\) is parallelizable, even assuming that \(M\) is parallelizable. As an example take \(M = S^1 \times S^3, M_1 = S^4\).
5.5 **Lemma** (Thom). The manifold $M_1$ belongs to the same cobordism class as $M$.

**Proof.** Let $W$ be the space obtained from the disjoint union of $M \times [0,1]$ and $D^{q+1} \times D^{4k-q}$ by pasting together $T \times [1]$ and $S^q \times D^{4k-q}$, using the product structure for $T$ constructed above. A differentiable manifold $W_\perp$ is obtained from $W$ by "straightening" the angle $\partial T \times [1] = S^q \times S^{4k-q-1}$. It is clear that $\partial W_\perp$ is the disjoint union of $M_\perp$ and $-M$, which completes the proof.

**Proof of 5.1.** Suppose that $M$ is $(q,l)$-connected, almost parallelizable, and that $\pi_q(M)$ has $r$ generators. The above construction yields a manifold $M_\perp$ which is $(q,l)$-connected, almost parallelizable, and such that $\pi_q(M_\perp)$ has $r$ generators. Iterating the construction $r$ times, this yields a manifold $M_r$ which is $q$-connected. Continue by induction on $q$ until we obtain a manifold $M'$ which is $(2k-1)$-connected.

According to 5.5 the manifold $M'$ has the same cobordism class as $M$. Therefore the index $I(M')$ is equal to $I(M)$. (Compare Thom [1].) This completes the proof of 5.1.

5.6 **Theorem.** Let $M$ be a homology sphere of dimension $4k-1$, $k > 1$, which bounds an $\Lambda$-manifold. If $\chi(M) = 0$ then $M$ bounds a contractible manifold.

5.7 **Corollary.** If $M$ is a homotopy $(4k-1)$-sphere with $\chi(M) = 0$, then $M$ is $J$-equivalent to $S^{4k-1}$; providing that $k > 1$.

5.8 **Corollary.** For $k > 1$ the group $\Theta^{4k-1}(\partial \Pi)$ is cyclic of order $\frac{1}{8} I_k$.

The proof of 5.6 is similar to that of 5.1, but also uses the following three results.
5.9. **Theorem.** Let \( W \) be a simply connected manifold of dimension \( 2n, n > 2 \). Then every element of \( \pi_n(W) \) is represented by an imbedding \( f: S^n \to W \).

The proof is a modification of Whitney's proof that every \( n \)-manifold can be imbedded in \( 2n \)-space. (See Whitney [2].) Details will not be given. I do not know whether this theorem is true for \( n = 2 \).

5.10. **Theorem.** Suppose that a quadratic form over the integers has determinant \( \pm 1 \), index 0, and takes on only even values. Then it is equivalent to a quadratic form with matrix \( \text{diag}(U, U, \ldots, U) \), where \( U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \).

**Proof.** This follows from theorems 1, 2 of Milnor [4] (making use of theorems of Eichler, etc).

(The following remark is due to H. Sah. In order to prove 5.10 it is sufficient to prove that there exists an isotropic vector: that is an \( \alpha \neq 0 \) such that the value \( \langle \alpha, \alpha \rangle \) of the quadratic form is zero. The existence of an isotropic vector is not difficult to prove; using the Hasse-Minkowski theorem that such a vector exists if and only if (1) the form is indefinite and (2) for each prime \( p \) the corresponding form over the \( p \)-adic numbers has an isotropic vector. Given such an \( \alpha \), let \( \alpha' = \alpha/\sqrt{r} \) be indivisible. Since the determinant is \( \pm 1 \), there exists \( \beta \) with \( \langle \alpha', \beta \rangle = 1 \). Define

\[
\beta_1 = \beta - \frac{1}{2} < \beta, \beta > \alpha_1.
\]

Then

\[
\langle \alpha_1, \alpha_1 \rangle = \langle \beta_1, \beta_1 \rangle = 0, \langle \alpha_1, \beta_1 \rangle = 1.
\]
Now consider the set of all $\gamma$ which satisfy $<\alpha_1, \gamma> = <\beta_1, \gamma> = 0$. By induction on the rank we can choose a basis for this set so that the matrix has the required form.

5.11. Lemma. Let $f: S^{2k} \to W^{4k}$ be an imbedding, and suppose that

1) the homology class $\beta$ of $f(S^{2k})$ has self intersection number $<\beta, \beta> = 0$ and
2) the induced bundle $f^*(\tau^{4k})$ over $S^{2k}$ is trivial.

Then the normal bundle $v^{2k}$ is trivial.

Proof. Just as in 5.2 it is seen that $v^{2k}$ corresponds to an element $\alpha \in \pi_{2k-1}(SO_{2k})$ which is annihilated by the homomorphism

$$\pi_{2k-1}(SO_{2k}) \to \pi_{2k-1}(SO_{2k+1}).$$

Since the group $\pi_{2k-1}(SO_{2k+1})$ is already stable, it follows from the exact sequence

$$\pi_{2k}(S^{2k}) \approx Z \xrightarrow{\partial} \pi_{2k-1}(SO_{2k}) \to \pi_{2k-1}(SO_{2k+1})$$

that $\alpha = \partial n$ for some $n \in Z$.

The element $\partial 1 \in \pi_{2k-1}(SO_{2k})$ corresponds to the tangent bundle of $S^{2k}$, with Euler class equal to twice the generator of $H^{2k}(S^{2k})$. Therefore the Euler class of $v^{2k}$ is equal to $2n$ times a generator of $H^{2k}(S^{2k})$. But this Euler class can be interpreted as the selfintersection number $<\beta, \beta>$ times a generator. Therefore $2n = 0$, hence $\alpha = 0$. This completes the proof.
Proof of 5.6. Let $M$ be a $(4k-1)$-manifold with $\lambda^*(M) = 0$. An argument similar to the proof of 5.1 shows that $M$ bounds a manifold $W$ which is almost parallelizable (hence parallelizable) and $(2k-1)$-connected. The index $I(W)$ is congruent to zero modulo $I_k$. Hence there exists an almost parallelizable $4k$-manifold $N$, without boundary, which satisfies $I(N) = -I(W)$. By 5.1 we may assume that $N$ is also $(2k-1)$-connected.

Now consider the sum $W_1 = W \# N$. This is a parallelizable $4k$-manifold with index zero, and with boundary $M$. The self-intersection matrix of $W_1$ has determinant $\pm 1$ by the Poincare duality theorem, and has only even elements on the diagonal. (Compare the proof of 3.2.) Therefore, according to 5.10, it is possible to choose a basis $\{\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_r\}$ for $H_{2k}(W_1)$ so that the intersection matrix is given by:

$$<\alpha_i, \alpha_j> = 0, <\beta_i, \beta_j> = 0, <\alpha_i, \beta_j> = \delta_{i,j}.$$ 

(Here $\delta_{i,j}$ is a Kronecker delta.)

According to 5.9 there exists an imbedding $f: S^{2k} \rightarrow W_1$ which represents the homology class $\alpha_i$. According to 5.11 the normal bundle of $f(S^{2k})$ is trivial. Hence we can remove a tubular neighborhood and replace it by $D^{2k+1} \times S^{2k-1}$, yielding a new manifold $W_2$.

From the pair $(W_1, W_1 - f(S^{2k}))$ we obtain an exact sequence

$$\cdots \rightarrow 0 \rightarrow H_{2k}(W_1 - f(S^{2k})) \rightarrow H_{2k}(W_1) \rightarrow \cdots$$
where $j(\gamma)$ is the intersection number of $\gamma$ with the homology class 
$\alpha_1$ of $f(S^{2k})$. Therefore $H_{2k}(W_1 - f(S^{2k}))$ is the subgroup of $H_{2k}(W_1)$
generated by $\{\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_r\}$.

From the pair $(W_2, W_1 - f(S^{2k}))$ we obtain an exact sequence

$$
\rightarrow \mathbb{Z} \xrightarrow{\partial} H_{2k}(W_1 - f(S^{2k})) \rightarrow H_{2k}(W_2) \xrightarrow{j} 0 \rightarrow \cdots
$$

where $\partial 1$ is the class $\alpha_1$. Hence $H_{2k}(W_2)$ is freely generated by
the classes $\{\alpha_2, \ldots, \alpha_r, \beta_2, \ldots, \beta_r\}$. Note that the intersection num-
bers of these classes in $W_2$ is the same as that in $W_1$. In fact
any $2k$-cycle in $W_2$ can be deformed so that it does not intersect
the submanifold $\emptyset \times S^{2k-1}$ which has codimension $2k + 1$.

Now choose an imbedding $f_2: S^{2k} \rightarrow W_2$ which represents the
class $\alpha_2$. We may assume that $f_2(S^{2k})$ is contained in the
parallelizable manifold

$$W_2 - (\emptyset \times S^{2k-1}) = W_1 - f(S^{2k}),$$

hence the normal bundle is trivial. Iterating this procedure $r$ times, we obtain a manifold $W_{r+1}$ which is $2k$-connected, and therefore
contractible. This completes the proof of 5.6.

This argument can be modified slightly to prove the following.

5.12. Theorem. The groups $\Theta_5(\partial \tau)$ and $\Theta_{13}(\partial \tau)$ are zero.

Proof. Let $M^5 = \emptyset W^6$ where $W^6$ is parallelizable. Just as
above, we may assume that $W^6$ is $2$-connected. The self intersection
matrix of $H_3(W^6)$ is skew symmetric with determinant $\pm 1$. Hence it
is equivalent to a matrix of the form diag($U'^1, U'^1, \ldots, U'^1$) where
$U' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. (See for example Veblen [1] pg. 183.) The normal bundle of any 3-sphere in $W^6$ is trivial since $\pi_2(SO_3) = 0$. Hence the argument above shows that we can kill $H_3(W^6)$.

The argument in dimension 13 is similar, using the fact that $\pi_6(SO_7) = 0$. This completes the proof of 5.12.

Remark. The following assertion will be proved in a later paper. For any $n$ of the form $4k + 1$, the group $\Theta^n(\partial W)$ is either zero or cyclic of order two. The proof will make use of the Arf invariant of a certain quadratic form over the field $\mathbb{Z}_2$.

5.13. Theorem. The groups $\Theta^6(\partial W)$ and $\Theta^{14}(\partial W)$ are zero.

Outline of proof. Let $M^{2k} = \partial W^{2k+1}$, where $W^{2k+1}$ is parallelizable. Just as above we may assume that $W^{2k+1}$ is $(k-1)$-connected. Furthermore the group $H_k(W^{2k+1}; \mathbb{Q})$ with rational coefficients is not difficult to kill. Thus we may assume that $H_k(W^{2k+1})$ is a finite group. Any element of this group is represented by an imbedded $k$-sphere with trivial normal bundle. Hence one can form $W^{2k+1}$ as before. However the homology group $H_k(W^{2k+1})$ depends on the particular product structure which is chosen for the normal bundle. The following question arises: Given an arbitrary normal vector field $\varphi_1$, does there exist a field of normal $(k+1)$-frames $(\varphi_1, \ldots, \varphi_{k+1})$? For $k$ equal to 1, 3 or 7 this is possible, since the homomorphism

$$\pi_k(SO_{k+1}) \rightarrow \pi_k(S^k)$$

is onto. Hence it is possible to choose $(\varphi_1, \ldots, \varphi_{k+1})$ so that
$H_k(W^2_{k+1})$ is smaller than $H_k(W^2_{k+1})$. However for other values of $k$ this homomorphism is not onto, so that the proof does not go through.

§6. The group $\mathfrak{g}^n/\mathfrak{g}^n(\partial \pi)$

The main result of this section will be that the factor group $\mathfrak{g}^n/\mathfrak{g}^n(\partial \pi)$ is always finite. [This is the group of all homotopy $n$-spheres modulo those which bound $\pi$-manifolds.] Upper bounds for this group are given, but no lower bounds. It is possible that every homotopy sphere is the boundary of a $\pi$-manifold.

Let $M \subset \mathbb{R}^{n+q}$ be a homology sphere, with $q > n$. The only obstruction to triviality of the normal bundle is an element

$$\mathfrak{c} \in H^n(M^n; \pi_{n-1}(SO_q)) \approx \pi_{n-1}(SO_q).$$

This coefficient group has been computed as follows by R. Bott [1]:

$$\begin{array}{c|cccccc}
\frac{n \text{ modulo } 8}{\pi_{n-1}(SO_q)} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
\mathbb{Z} & \mathbb{Z}_2 & \mathbb{Z}_2 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}$$

(This table is valid for $q > n \geq 2$.)

If $n$ is congruent to 3, 5, 6 or 7 modulo 8, this clearly implies that $\mathfrak{c}$ is zero.

If $n$ is equal to $4k$ then the obstruction class $\mathfrak{c}$ can be identified with a certain fraction of the Pontrjagin class $p_k(M^n)$. (See Kervaire [3] or Kervaire and Milnor [1].) But Hirzebruch's index formula (Hirzebruch [1] p. 85) implies that the Pontrjagin class of a homology sphere is zero. Again it follows that $\mathfrak{c} = 0$.

Finally suppose that $n$ is congruent to 1 or 2 modulo 8, so that $\pi_{n-1}(SO_q) \approx \mathbb{Z}_2$. A theorem of Rohlin asserts that the obstruction class $\mathfrak{c}$
is annihilated by the homomorphism

$$J_{n-1}: \pi_{n-1}(SO_q) \longrightarrow \pi_{n+q-1}(S^q) .$$

(See Rohlin [1] or Kervaire and Milnor [1].) If $J_{n-1}$ is non-trivial, it follows that $\alpha' = 0$. This proves:

6.1 Theorem. Every homology $n$-sphere is a $\pi$-manifold, unless

1) $n = 1$ or 2 modulo 8, and

2) the homomorphism $J_{n-1}$ is zero.

For $n = 2$ it is well known that $J_1$ is an isomorphism. For $n = 9, 10$ we have:

6.2 Lemma of Kervaire [5]. The homomorphisms $J_8$ and $J_9$ are non-trivial.

Therefore:

6.3 Corollary. For $n < 17$ every homology $n$-sphere is a $\pi$-manifold.

I do not know whether conditions (1) and (2) of 6.1 are ever satisfied. However in any case the following is true.

6.4 Lemma. For any $n$ the homotopy $n$-spheres which are $\pi$-manifolds form a subgroup $\Theta^n(\pi) \subset \Theta^n$ which has index at most 2.

Proof. We may assume that $\pi_{n-1}(SO_q) \cong \mathbb{Z}_2$. The obstruction correspondence $M^n \longrightarrow \sigma(M^n) \in \pi_{n-1}(SO_q)$ is easily seen to be additive, and invariant under $J$-equivalence. This completes the proof.

Now let $M^n$ be any $\pi$-manifold without boundary, and consider the question: Is $M^n$ the boundary of a $\pi$-manifold? The theory of Thom [2] can be used to give an answer as follows.

Choose an imbedding of $M^n$ in the interior of a cube $[0, 1] \times \ldots \times [0, 1] = I^{n+q}$, and choose a field $\phi$ of normal $q$-frames. Then the
Thom construction yields a map
\[(\nu^{n+q}, \partial \nu^{n+q}) \rightarrow (s^q, \text{base point}),\]
and hence a homotopy class
\[t(\varphi) \in \pi_{n+q}(s^q).\]

proof of Lemma 1.) This class is zero if and only if there exists a
\(\nu\)-manifold \(W \subseteq \nu^{n+q+1}\) such that

1) \(\partial W = M^n \times [0]\), and
2) the field \(\varphi\) of normal \(q\)-frames can be extended throughout \(W\).

Now let \(\varphi\) range over all possible fields of normal \(q\)-frames. The
set of all homotopy classes \(t(\varphi)\) will be denoted by
\[t'(M^n) \subseteq \pi_{n+q}(s^q).\]
Evidently \(M^n\) bounds a \(\nu\)-manifold if and only if
\[0 \in t'(M^n).\]

6.5 Lemma. If \(M_1\) and \(M_2\) are \(\nu\)-manifolds, then
\[t'(M_1 \# M_2) \supseteq t'(M_1) + t'(M_2).\]

(I do not know whether equality holds.) Proof. Let \(W\) be a manifold
formed from the disjoint union of \(M_1 \times [0, 1]\), \(M_2 \times [0, 1]\) and \(D^n \times D^1\)
by matching \(D^n \times [-1]\) with a cell in \(M_1\); matching \(D^n \times [1]\) with
a cell in \(M_2\); and then straightening corners. If the orientations
are correct, then \(\partial W\) will be the disjoint union of \(M_1 \# M_2\), \(-M_1\) and
\(-M_2\). Furthermore \(W\) has the homotopy type of the union of \(M_1\) and \(M_2\)
with a single point in common.

Choose an imbedding of \(W\) in \(R^{n+q} \times [0, 1]\) so that \(-M_1\) and \(-M_2\)
go into \(R^{n+q} \times [0]\), while \(M_1 \# M_2\) goes into \(R^{n+q} \times [1]\). Now given
fields \(\varphi_1, \varphi_2\) of normal \(q\)-frames on \(M_1\) and \(M_2\) respectively, there
exists an extension \(\psi\) which is defined throughout \(W\).
If $\varphi$ denotes the restriction of $\psi$ to $M_1 \neq M_2$, then it is clear that $t(\varphi) = t(\varphi_1) + t(\varphi_2)$. This completes the proof of 6.5.

Now consider the special case $M = S^n$. Every field $\varphi$ of normal $\mathfrak{q}$-frames determines an element

$$\alpha \in \pi_n(S^0_q) .$$

Kervaire has shown that $t(\varphi)$ is equal to $\pm J_n(\alpha)$. (See Kervaire [4].) Since any $\alpha$ may occur this proves:

6.6 Lemma. The set $t'(S^n)$ is equal to $\text{Image } J_n \subset \pi_{n+q}(S^q)$.

Applying 6.5 to the identity

$$M^n \# S^n = M^n$$

this shows that $t'(M^n) \supset t'(M^n) + (\text{image } J_n)$. In other words $t'(M^n)$ is a union of cosets of $(\text{image } J_n)$. This suggests that we define $t(M^n)$ as the subset of

$$\text{cokernel } J_n = \pi_{n+q}(S^q)/(\text{image } J_n)$$

which corresponds to $t'(M^n)$.

6.7 Theorem. The Thom construction yields a correspondence

$$M^n \longrightarrow t(M^n) \subset (\text{cokernel } J_n)$$

with the following properties:

a) $t(M^n)$ is defined and non-vacuous for every unbounded $\pi$-manifold.

b) $t(M^n)$ contains 0 if and only if $M^n$ bounds a $\pi$-manifold.

c) $t(M_1 \# M_2) \supset t(M_1) + t(M_2)$.

d) $t(S^n) = \{0\}$.

e) If $M_1$ is $J$-equivalent to $M_2$ then $t(M_1) = t(M_2)$.

f) If $M^n$ is a homotopy sphere, then $t(M^n)$ consists of a single element.

Proof. Properties (a) through (d) follow from the discussion above. Property (e) follows immediately from the definition. To prove (f) recall
that $M^n \# (-M^n)$ is $J$-equivalent to $S^n$. Therefore $(0) \supset t(M^n) + t(-M^n)$. But this would be impossible if $t(M^n)$ contained more than one element.

6.8 **Corollary.** The factor group $\theta^N(\pi)/\theta^N(3\pi)$ is naturally isomorphic to a subgroup of $(\text{cokernel } J_n)$.

6.9 **Corollary.** This factor group is finite for every $n$. Hence the subgroup $\theta^N(3\pi) \subset \theta^N$ has finite index.

To conclude this section, here is a summary of what is known about the group $(\text{cokernel } J_n)$. Toda has computed the $p$-primary component of the stable group $\pi_{n+q}(S^q)$ for the range $n < 2p^2(p-1) - 3$. (See Toda [2].) Combining this information with §3.6 the $p$-primary component of $(\text{cokernel } J_n)$ is determined for the same range. As an example (compare Milnor [3]):

**Assertion.** The $p$-primary component of $(\text{cokernel } J_n)$ is zero for $n < 2p(p-1)-2$, and is $Z_p$ for $n = 2p(p-1)-2$.

The 2-primary component can be determined for $n \leq 13$, making use of Toda [1], together with §6.2 and §3.6. The following is a tabulation of the first thirteen groups.

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14 \to 19</th>
</tr>
</thead>
<tbody>
<tr>
<td>coker $J_n$</td>
<td>0</td>
<td>$Z_2$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$Z_2$</td>
<td>0</td>
<td>$Z_2$</td>
<td>$Z_2^+$</td>
<td>$Z_2$</td>
<td>0</td>
<td>0</td>
<td>$Z_3$</td>
<td>2-group</td>
</tr>
</tbody>
</table>

Since $\theta^2$ is known to be zero, the first unsolved case occurs for $n = 6$. Is the group $\theta^6/\theta^6(3\pi)$ non-trivial?
§7. Discussion

Combining the results of the preceding sections, we have the following estimate of $\Theta^n$ for small values of $n$.

$\Theta^1 = \Theta^2 = \Theta^5 = 0,$
$\Theta^6$ is either 0 or $Z_2$,
$\Theta^7$ is cyclic of order 28,
$\Theta^9$ has order at most 8,
$\Theta^{11}$ is cyclic of order 992,
$\Theta^{13}$ is either 0 or $Z_3$,
$\Theta^{14}$ is a 2-group,
$\Theta^{15}$ has order 127 times a power of 2. This group contains an element of order 8126.

Evidently the biggest hiatus in the results is the following.

**Problem 1.** Are the groups $\Theta^{2k}(\Theta^n)$ finite for $k \neq 1, 3, 7$? A solution would probably be based on a detailed study of $(2k+1)$-manifolds which are $(k-1)$-connected. (Compare §5.13.)

Another outstanding problem is the decision as to whether every homotopy sphere bounds a $\pi$-manifold. (See §6.)

**Problem 2.** Is there any theory which related the invariant $t(M^n) \subset (\text{cokernel } J_n)$ with the topology of $M^n$? In particular does this invariant vanish for a homotopy sphere?

Another question would be the relationship between this paper and the Poincare hypothesis.
Problem 3. Does there exist a homotopy 3-sphere $M$ such that $\lambda^i(M) \neq 0$?

Such a manifold could not be homeomorphic to $S^3$. In fact J. Munkres, S. Smale and J. H. C. Whitehead have proved that the differentiable structure of a topological 3-manifold is unique up to diffeomorphism.

Problem 4. Are the homotopy spheres $M_0^{4k-1}$ homeomorphic to $S^{4k-1}$? (See §4. Note that $k$ must be $\geq 2$.)

The following seems to be a very deep question.

Problem 5. Are $J$-equivalent manifolds necessarily diffeomorphic?

An affirmative answer would imply the generalized Poincaré hypothesis for differentiable manifolds. For if $M$ is a homotopy $n$-sphere then $M \# (-M)$ is $J$-equivalent to $S^n$. But if $M \# M_1$ is diffeomorphic to $S^n$ then an argument due to Mazur [1] implies that $M$ itself is homeomorphic to $S^n$.

Most known invariants of differentiable manifolds depend only on the $J$-equivalence class. For example:

Assertion. If $M_1$ is $J$-equivalent to $M_2$ then some homotopy equivalence $M_1 \rightarrow M_2$ is covered by a bundle map $T_1^n \rightarrow T_2^n$ between the tangent bundles.

Proof. Suppose that the boundaries $M_1$ and $-M_2$ are deformation retracts of $W$. Choose a non-singular vector field on $W$ which points out of $W$ along $M_1$ and into $W$ along $M_2$. The orthogonal complement of this vector field in $T^{n+1}$ yields an $SO_n$-bundle $\xi^n$.
over $W$. Now the bundle maps $\tau_1^n \to \mathcal{F}^n \leftarrow \tau_2^n$ can be used to construct the required bundle map.

**Problem 6.** Is the "simple homotopy type" of $M$ invariant under $J$-equivalence? (See J. H. C. Whitehead [1], [4].)

**Appendix: Pasting and straightening**

Let $R_+$ denote the set of real numbers $t$ with $0 \leq t < \infty$.

**Assertion.** If $W$ is a differentiable manifold with boundary, then there exists a neighborhood $U$ of $\partial W$, and a diffeomorphism $h: \partial W \times R_+ \to U$ which satisfies the identity $h(x,0) = x$.

(A proof of this assertion is given in Milnor [2]. Alternatively this may be taken as part of the definition of "manifold with boundary").

Given two manifolds $W_1, W_2$ and an orientation reversing diffeomorphism $f: \partial W_1 \to \partial W_2$, let $M$ denote the space obtained from the disjoint union of $W_1$ and $W_2$ by identifying each $x \in \partial W_1$ with $f(x)$. 
8.1. Lemma. The topological manifold $M$ can be given a differentiable structure which is compatible with that of $W_1$ and $W_2$.

8.2. Lemma. If two such differentiable structures are given, then the resulting differentiable manifolds are diffeomorphic.

Proof of 8.1. Choose neighborhoods $U_1$ of $\partial W_1$ in $W_1$, and diffeomorphisms

$$h_1 : \partial W_1 \times \mathbb{R}^+ \rightarrow U_1$$

as above. A homeomorphism

$$h : \partial W_1 \times \mathbb{R} \rightarrow U_1 \cup U_2 \subseteq M$$

is defined by the formula

$$h(x,t) = \begin{cases} h_1(x,t) & \text{for } t \geq 0 \\ h_2(fx,-t) & \text{for } t \leq 0. \end{cases}$$

Taking $h$ to be a diffeomorphism, this defines the required differentiable structure.

Proof of 8.2. Let $M$ and $M'$ be the two differentiable manifolds. Choose a contravariant vector field $\varphi_1$ along the boundary of $W_1$ which points out of $W_1$. Considering $W_1$ and $W_2$ as submanifolds of $M$, this yields a vector field $\varphi_2$ along the boundary of $W_2$ which points into $W_2$. On the other hand, considering $W_1$ and $W_2$ as submanifolds of $M'$, the field $\varphi_1'$ corresponds to some other field $\varphi_2'$ along the boundary of $W_2$.

Choose a diffeomorphism $g_2 : W_2 \rightarrow W_2$ which leaves $\partial W_2$ point-wise fixed, and carries the vector field $\varphi_2$ into $\varphi_2'$. (The construction is not difficult.) Then a homeomorphism $g : M \rightarrow M'$ is
obtained by combining $g_2$ with the identity map of $W_1$. It is easily verified that $g$ and $g^{-1}$ are differentiable of class $C^1$.

Approximate $g$ by a $C^\infty$-differentiable map $g'$; where the approximation must be close enough so that the Jacobian of $g'$ has rank $n$ everywhere. (See Whitney [1].) Then $g': M \to M'$ is the required diffeomorphism.

Several times in this paper it has been necessary to consider $n$-manifolds with boundary which are differentiable except along some $(n-2)$-dimensional submanifold of the boundary. The simplest example of such an object is the quadrant $R_+ \times R_+ \subset R^2$. This example can be "straightened" by introducing new coordinates as follows. Map $R_+ \times R_+$ onto the half-plane $R \times R_+$ by the correspondence

$$(r \cos \theta, r \sin \theta) \mapsto (r \cos 2\theta, r \sin 2\theta)$$

for $0 \leq r$, $0 \leq \theta \leq \frac{\pi}{2}$. Thus $f$ is a diffeomorphism, except at the singular point. Another example is provided by the three-quarter-plane $R_+ \times R \cup R \times R_+$. This can be straightened by the transformation

$$(r \cos \theta, r \sin \theta) \mapsto (r \cos ((2\theta+\pi)/3), r \sin ((2\theta+\pi)/3)),$$

for $0 \leq r$, $-\frac{\pi}{2} \leq \theta \leq \pi$.

A higher dimensional example is given as follows. Let $W_1$ and $W_2$ be differentiable manifolds with boundary. Then $W_1 \times W_2$ is differentiable except along $\partial W_1 \times \partial W_2$. Some neighborhood $U_1 \times U_2$ of this singular set is "diffeomorphic" to

$$(\partial W_1 \times \partial W_2) \times (R_+ \times R_+).$$
Form a new differentiable manifold $W$ as follows. Take the disjoint union of $W_1 \times W_2 = \partial W_1 \times \partial W_2$ and $\partial W_1 \times \partial W_2 \times \mathbb{R} \times \mathbb{R}_+$, and identify

$$h_1(x_1, r \cos \theta) \times h_2(x_2, r \sin \theta) \in U_1 \times U_2$$

with

$$(x_1, x_2, r \cos 2\theta, r \sin 2\theta)$$

for each $x_1 \in \partial W_1, x_2 \in \partial W_2, 0 < r, 0 \leq \theta \leq \frac{\pi}{2}$. This construction will be referred to as "straightening the angle". Note that the differentiable structure of $\partial W_1 \times W_2$, and of $W_1 \times \partial W_2$, is left fixed, so that Lemma 8.2 applies to their union $\partial W$.

A similar construction works for each of the examples considered in this paper.

References


