AN EXPLICIT PROJECTION
by Andrew Ranicki

A module $P$ over a ring $A$ is f.g. (finitely generated) projective if it is isomorphic to the image $\text{im}(p : A^n \longrightarrow A^n)$ of a projection $p = p^2 : A^n \longrightarrow A^n$ of a f.g. free module $A^n$. Projective modules and the projective class groups $K_0(A)$, $\widetilde{K}_0(A)$ entered topology via the work of Swan [8] on finite group actions on homotopy spheres, and more generally via the finiteness obstruction theory of Wall [10]. In various papers (Munkholm and Ranicki [5], Ranicki [7], Lück [1], Pedersen and Weibel [6], Lück and Ranicki [2]) it has actually been found more convenient to work with the projections rather than the modules.

In this note an explicit projection is obtained for the f.g. projective $A$-module constructed by the standard Mayer-Vietoris procedure (Milnor [4,§2]) from an automorphism of a f.g. free $A'$-module, with $A$ and $A'$ related by a cartesian square of rings

\[
\begin{array}{ccc}
A & \longrightarrow & B \\
\uparrow f' & & \downarrow g \\
B' & \longrightarrow & A'
\end{array}
\]

with $f' : A \longrightarrow B'$ and $g : B \longrightarrow A'$ onto. In view of the theorem of Swan that the map $\widetilde{K}_0(\mathbb{Z}[G]) \longrightarrow \widetilde{K}_0(\mathbb{Q}[G])$ is trivial this is the generic construction of f.g. projective $\mathbb{Z}[G]$-modules for finite groups $G$. By way of an example an explicit projection is constructed for a generator of $\widetilde{K}_0(\mathbb{Z}[\mathbb{Q}(8)]) = \mathbb{Z}_2$, with $\mathbb{Q}(8)$ the quaternion group $Q(8)$ of order 8. This is the simplest example of a group $G$ with non-trivial reduced projective class group $\widetilde{K}_0(\mathbb{Z}[G])$.

A commutative square of rings (as above) is cartesian if the sequence of additive groups

\[
0 \longrightarrow A \xrightarrow{(f \ f')} B \oplus B' \xrightarrow{(g \ -g')} A' \longrightarrow 0
\]

is exact.

Given an automorphism $\alpha' : A^m \longrightarrow A^m$ of a f.g. free $A'$-module define the pullback f.g. projective $A$-module

\[
P(\alpha') = \{(x, x') \in B^n \oplus B'^n \mid \alpha'(g(x)) = g'(x') \in A^m\}
\]

which fits into an exact sequence of additive groups

\[
0 \longrightarrow P(\alpha') \longrightarrow B^n \oplus B'^m \xrightarrow{(\alpha'g \ -g')} A^m \longrightarrow 0
\]
with \( A \) acting by

\[
A \times P(\alpha') \longrightarrow P(\alpha'); \ (a, (x, x')) \longmapsto (f(a)x, f'(a)x')
\]

The construction is used to define the connecting map \( \partial \) in the Mayer-Vietoris exact sequence (Milnor [4, §4]) of algebraic \( K \)-groups

\[
\begin{array}{c}
K_1(A) \xrightarrow{\left( \begin{array}{c} f \\ f' \end{array} \right)} K_1(B) \oplus K_1(B') \xrightarrow{(g \ -g')} K_1(A') \xrightarrow{\partial} \\
K_0(A) \xrightarrow{\left( \begin{array}{c} f \\ f' \end{array} \right)} K_0(B) \oplus K_0(B') \xrightarrow{(g \ -g')} K_0(A')
\end{array}
\]

with

\[
\partial : K_1(A') \longrightarrow K_0(A); \ \tau(\alpha' : A'^n \longrightarrow A'^n) \longmapsto [P(\alpha')] - [A^n].
\]

Given \( A' \)-module automorphisms \( \alpha' : A'^n \longrightarrow A'^n, \alpha'' : A'^m \longrightarrow A'^m \), and also a \( B \)-module morphism \( \beta : B^n \longrightarrow B^m \) and a \( B' \)-module morphism \( \beta' : B'^m \longrightarrow B'^m \) such that the square

\[
\begin{array}{ccc}
A'^n & \xrightarrow{g(\beta)} & A'^m \\
\alpha' \downarrow & & \downarrow \alpha'' \\
A'^m & \xrightarrow{g'(\beta')} & A'^m
\end{array}
\]

commutes let

\[
(\beta, \beta') : P(\alpha') \longrightarrow P(\alpha'') ; \ (x, x') \longmapsto (\beta(x), \beta'(x'))
\]

be the pullback \( A \)-module morphism.

**Proposition** Given an \( A' \)-module automorphism \( \alpha' : A'^n \longrightarrow A'^n \) and any lifts of \( \alpha', \alpha'^{-1} \) to \( B \)-module endomorphisms \( \beta, \gamma : B^n \longrightarrow B^n \), there is defined an \( A \)-module projection

\[
p(\alpha') = \begin{pmatrix}
((2 - \beta \gamma)\beta \gamma, 1) & ((2 - \beta \gamma)(1 - \beta \gamma)\beta, 0) \\
(\gamma(1 - \beta \gamma), 0) & ((1 - \gamma \beta)^2, 0)
\end{pmatrix} : A^n \oplus A^n \longrightarrow A^n \oplus A^n
\]

such that up to isomorphism

\[
P(\alpha') = \text{im}(p(\alpha')).
\]
Proof: Lift the Whitehead lemma identity of $A'$-module automorphisms
\[
\begin{pmatrix} \alpha' & 0 \\ 0 & \alpha'^{-1} \end{pmatrix} = \begin{pmatrix} 1 & \alpha' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha'^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & \alpha' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\]

\[: A^n \oplus A^n \longrightarrow A^n \oplus A^n\]
to define a $B$-module automorphism
\[
\phi = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\gamma & 1 \end{pmatrix} \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} (2 - \beta \gamma) \beta & \beta \gamma - 1 \\ 1 - \gamma \beta & \gamma \end{pmatrix}
\]

\[: B^n \oplus B^n \longrightarrow B^n \oplus B^n\]
with inverse
\[
\phi^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -\beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \begin{pmatrix} 1 & -\beta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \gamma & 1 - \gamma \beta \\ \beta \gamma - 1 & (2 - \beta \gamma) \beta \end{pmatrix}
\]

\[: B^n \oplus B^n \longrightarrow B^n \oplus B^n\].

Identifying $A^n \oplus A^n = A^{2n}$ define an $A$-module isomorphism
\[
h = (\phi, 1) : P(\alpha') \oplus P(\alpha'^{-1}) \longrightarrow P(1 : A^{2n} \longrightarrow A^{2n}) = A^{2n}
\]
with inverse
\[
h^{-1} = (\phi^{-1}, 1) : P(1 : A^{2n} \longrightarrow A^{2n}) = A^{2n} \longrightarrow P(\alpha') \oplus P(\alpha'^{-1}) .
\]

It is now immediate from the identity
\[
p(\alpha') = h(1 \oplus 0)h^{-1} : A^{2n} \overset{h^{-1}}{\longrightarrow} P(\alpha') \oplus P(\alpha'^{-1}) \overset{1 \oplus 0}{\longrightarrow} P(\alpha') \oplus P(\alpha'^{-1}) \overset{h}{\longrightarrow} A^{2n}
\]
that $p(\alpha') : A^{2n} \longrightarrow A^{2n}$ is a projection with image isomorphic to $P(\alpha')$. Explicitly, the restriction of $h$ defines an $A$-module isomorphism
\[
P(\alpha') \longrightarrow \text{im}(p(\alpha')) ; (x, x') \longmapsto ((2 - \beta \gamma)\beta(x), x') \oplus ((1 - \gamma \beta)(x), 0) .
\]

\[\square\]

Example: Given a finite group $G$ consider the Rim cartesian square of rings
\[
\begin{array}{ccc}
Z[G] & \longrightarrow & Z[G]/N \\
\downarrow \epsilon & & \downarrow \epsilon \\
Z & \longrightarrow & Z/|G|
\end{array}
\]
in which all the morphisms are onto, with

\[ N = \sum_{g \in G} g \in \mathbb{Z}[G], \quad \epsilon : \mathbb{Z}[G] \longrightarrow \mathbb{Z}; \quad g \mapsto 1. \]

The canonical isomorphism of rings \( \mathbb{Z}[G] \longrightarrow (\mathbb{Z}[G]/N, 1, \mathbb{Z}) \) has inverse

\[ (\mathbb{Z}[G]/N, 1, \mathbb{Z}) \longrightarrow \mathbb{Z}[G]; \quad (b, b') \mapsto a + (b' - \epsilon(a))(N/|G|) \]

with \( a \in \mathbb{Z}[G] \) any lift of \( b \in \mathbb{Z}[G]/N \) (so that \( \epsilon(a) \equiv b' \pmod{|G|} \)). In this case the boundary map in the Mayer-Vietoris sequence is given by

\[ \partial : K_1(\mathbb{Z}/|G|) = (\mathbb{Z}/|G|)^\times \longrightarrow K_0(\mathbb{Z}[G]); \]

\[ \tau(\alpha') \longmapsto [\text{im}(p(\alpha'))] - [\mathbb{Z}[G]^2] \]

for any unit \( \alpha' \in (\mathbb{Z}/|G|)^\times \), with \( \beta, \gamma \in \mathbb{Z} \) such that \([\beta] = \alpha', \quad [\gamma] = \alpha'^{-1} \in \mathbb{Z}/|G|\), and \( p(\alpha') \) the \( \mathbb{Z}[G] \)-module projection

\[ p(\alpha') = \begin{pmatrix}
1 - (1 - \beta \gamma)^2(N/|G|) & (2 - \beta \gamma)(1 - \beta \gamma)\beta(N/|G|) \\
\gamma(1 - \beta \gamma)(N/|G|) & (1 - \gamma \beta)^2(N/|G|)
\end{pmatrix} 
: \mathbb{Z}[G] \oplus \mathbb{Z}[G] \longrightarrow \mathbb{Z}[G] \oplus \mathbb{Z}[G]. \]

\[ \square \]

**Example** For the quaternion group of order 8

\[ G = Q(8) = \{ \pm 1, \pm i, \pm j, \pm k \} \]

and the unit \( \alpha' = 3 \in (\mathbb{Z}/8)^\times \) take \( \beta = \gamma = 3 \in \mathbb{Z} \) in the previous Example. By the Proposition the corresponding projection

\[ p(\alpha') = \begin{pmatrix}
21N \\
-3N
\end{pmatrix} : \mathbb{Z}[Q(8)] \oplus \mathbb{Z}[Q(8)] \longrightarrow \mathbb{Z}[Q(8)] \oplus \mathbb{Z}[Q(8)] \]

is such that \( P(\alpha') \cong \text{im}(p(\alpha')) \) is a f.g. projective \( \mathbb{Z}[Q(8)] \)-module isomorphic to the two-sided ideal

\[ \langle 3, N \rangle = \text{im}(\langle 3, N \rangle : \mathbb{Z}[Q(8)] \oplus \mathbb{Z}[Q(8)] \longrightarrow \mathbb{Z}[Q(8)]) \subset \mathbb{Z}[Q(8)] \]

of the type considered by Swan [8,§6], with an isomorphism

\[ \langle 3, N \rangle \longrightarrow P(\alpha') ; \quad 3x + Ny \longmapsto x(1, 3) + y(0, 8) \quad (x, y \in \mathbb{Z}[Q(8)]). \]
The reduced projective class

\[ \partial \tau(3) = [P(\alpha')] \in \tilde{K}_0(\mathbb{Z}[Q(8)]) = \mathbb{Z}/2 \]

represents the generator (Martinet [3]). As noted in [3] \(P(\alpha')\) is isomorphic to
the f.g. projective \(\mathbb{Z}[Q(8)]\)-module \(P_3\) defined by the f.g. free \(\mathbb{Z}\)-module \(\mathbb{Z}^8\) on 8 generators \(\{e_0\} \cup \{e_s|s \in Q(8), s \neq 1\}\), with \(Q(8)\) acting by

\[ se_0 = e_0, \quad se_{s-1} = 3e_0 - \sum_{t \neq 1} e_t \ (s \in Q(8)), \]

\[ se_t = e_{st} \ (t \neq 1, s^{-1}). \]

The element defined by

\[ e_1 = 3e_0 - \sum_{t \neq 1} e_t \in P_3 \]

is such that

\[ se_1 = e_s \in P_3 \ (s \neq 1). \]

Thus

\[ Ne_1 = e_1 + \sum_{t \neq 1} e_t = 3e_0 \in P_3, \]

and there is defined a \(\mathbb{Z}[Q(8)]\)-module isomorphism

\[ (3, N) \rightarrow P_3 : 3x + Ny \mapsto xe_1 + ye_0. \]

\[ \square \]

Given a ring \(A\) and a multiplicative subset \(S \subset A\) of central non-zero divisors there is defined a cartesian square of rings

\[
\begin{array}{ccc}
A & \rightarrow & S^{-1}A \\
\downarrow & & \downarrow \\
\hat{A} & \rightarrow & S^{-1}\hat{A}
\end{array}
\]

with \(S^{-1}A\) the localization of \(A\) inverting \(S\), and

\[ \hat{A} = \lim_{\leftarrow s \in S} A/sA \]

the \(S\)-adic completion of \(A\). The algebraic \(K\)-theory Mayer-Vietoris exact sequence determined by such a square

\[
\begin{align*}
K_1(A) & \rightarrow K_1(S^{-1}A) \oplus K_1(\hat{A}) \rightarrow K_1(S^{-1}\hat{A}) \overset{\partial}{\rightarrow} \\
K_0(A) & \rightarrow K_0(S^{-1}A) \oplus K_0(\hat{A}) \rightarrow K_0(S^{-1}\hat{A})
\end{align*}
\]
is widely used in the computations of the $K$-groups of the group rings $A = \mathbb{Z}[G]$ of finite groups $G$, with $S = \mathbb{Z}-\{0\}$, $S^{-1}A = \mathbb{Q}[G]$. Again, the connecting map $\partial$ is defined by the pullback construction: if $\alpha : S^{-1}\hat{A}^n \longrightarrow S^{-1}\hat{A}^n$ is an automorphism of a f.g. free $S^{-1}\hat{A}$-module then the pullback

$$P(\alpha) = \{(x, y) \in S^{-1}A^n \oplus \hat{A}^n \mid \alpha(x) = y \in S^{-1}\hat{A}^n\}$$

is a f.g. projective $A$-module, and

$$\partial : K_1(S^{-1}\hat{A}) \longrightarrow K_0(A); \quad \tau(\alpha : S^{-1}\hat{A}^n \longrightarrow S^{-1}\hat{A}^n) \longmapsto [P(\alpha)] - [A^n].$$

It is possible to obtain an explicit projection for $P(\alpha)$ from the material in Appendix A of Swan [9], but the actual formula is much more complicated than in the cartesian case with onto maps. (I am grateful to Jim Davis for the reference to [9]).

REFERENCES

[1] W.Lück

*The transfer maps induced in the algebraic $K_0$- and $K_1$- groups by a fibration I.*

Math. Scand. 59, 93–121 (1986)


*Chain homotopy projections*


*Modules sur l’algèbre de groupe quaternionien*


*Introduction to algebraic $K$-theory*

Annals of Mathematics Studies 72, Princeton (1971)


*The projective class group transfer induced by an $S^1$-bundle*


[6] E.Pedersen and C.Weibel

*A non-connective delooping of algebraic $K$-theory*


*The algebraic theory of finiteness obstruction*

[8] R.G. Swan

*Periodic resolutions for finite groups*
Annals of Maths. 72, 267–291 (1960)

[9]  Projective modules over binary polyhedral groups

[10] C.T.C. Wall

*Finiteness conditions for CW complexes*
Annals of Maths. 81, 56–69 (1965)

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