NONCOMMUTATIVE LOCALIZATION

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Introduction

- The talk will describe:
  - the localization $\Sigma^{-1}A$ of a ring A inverting a set Σ of A-module morphisms,
  - the exact sequences relating the algebraic K- and L-groups of A and $\Sigma^{-1}A$,
  - the applications to manifolds with fundamental group a generalized free product or an HNN extension, and to submanifolds of codimension 1 and 2.
- These topics have been studied for nearly 50 years by many authors – notably Pierre Vogel.

Absolute and relative K- and L-groups

- The **absolute** algebraic K- and L-groups $K_*(A)$, $L_*(A)$ of a ring A are defined using the subcategory

  $$\mathrm{Proj}(A) = \{\text{f.g. projective } A\text{-modules}\} \subset \mathrm{Mod}(A) = \{A\text{-modules}\} .$$

  Need an involution on A for L-theory.
- For a ring morphism $f : A \to B$ use the (B, A)-bimodule structure on B

  $$B \times B \times A \to B \; ; \; (b, x, a) \mapsto b.x.f(a) \; .$$

  to define the **change of rings** functor

  $$B \otimes_A - \; : \; \mathrm{Mod}(A) \to \mathrm{Mod}(B) \; ; \; M \mapsto B \otimes_A M$$

- Can use $B \otimes_A - : \mathrm{Proj}(A) \to \mathrm{Proj}(B)$ to define the **relative** K- and L-groups $K_*(f)$, $L_*(f)$ with long exact sequences

  $$\cdots \longrightarrow K_n(A) \longrightarrow K_n(B) \longrightarrow K_n(f) \longrightarrow K_{n-1}(A) \longrightarrow \cdots ,$$

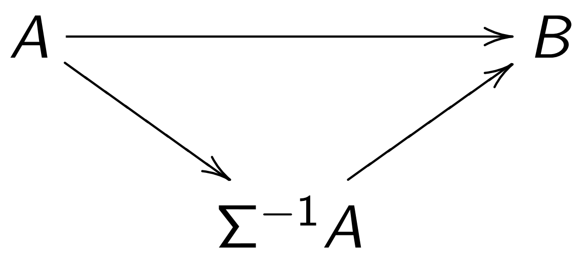
  $$\cdots \longrightarrow L_n(A) \longrightarrow L_n(B) \longrightarrow L_n(f) \longrightarrow L_{n-1}(A) \longrightarrow \cdots .$$

The universal localization I.

- A = ring, Σ = a set of morphisms $s : P \to Q$ in Proj(A).
- A ring morphism $A \to B$ is Σ-**inverting** if the induced morphisms in Proj(B)

$$1 \otimes s \ : \ B \otimes_A P \to B \otimes_A Q \ (s \in \Sigma)$$

  are isomorphisms.
- A **universal localization** of A is a Σ-inverting morphism $A \to \Sigma^{-1}A$ with the universal property : for any Σ-inverting morphism $A \to B$ there is a unique factorization



  If $A \to \Sigma^{-1}A$ exists, it is unique up to isomorphism.
- In general Proj(A) $\to$ Proj($\Sigma^{-1}A$) is not a localization of categories in the sense of Verdier, Zisman etc.

Localization in algebraic K- and L-theory

- $A \to \Sigma^{-1}A$ induces the change of rings functor

  $\Sigma^{-1}\ :\ \mathrm{Mod}(A) \to \mathrm{Mod}(\Sigma^{-1}A)\ ;\ M \mapsto \Sigma^{-1}M = \Sigma^{-1}A \otimes_M A\ .$

- (Milnor, Bass, Quillen, Karoubi, Pardon, R., Vogel, Schofield, Neeman-R., ..., 1960's – now) For certain $A \to \Sigma^{-1}A$

  $K_*(A \to \Sigma^{-1}A) = K_{*-1}(T(A,\Sigma))\ ,\ L_*(A \to \Sigma^{-1}A) = L_*(T(A,\Sigma))$

  with $T(A,\Sigma)$ the **torsion** exact category of homological dimension 1 A-modules M with

  $$\Sigma^{-1}M\ =\ 0\ .$$

  Such expressions of relative K- and L-groups as absolute K- and L-groups are always interesting!

- Pierre Vogel (1980's) pioneered the use of noncommutative localization in study of knots and links. Motivated by the Wall surgery obstruction L-theory, the Cappell-Shaneson homology surgery Γ-theory and the algebraic theory of surgery (R.).

Some applications of the torsion K- and L-groups to topology I.

- $A = \mathbb{Z} \to \Sigma^{-1}A = \mathbb{Q}$, $T(A, \Sigma) = \{$finite abelian groups$\}$. $\mathbb{Q}/\mathbb{Z}$-valued linking forms in $T(A, \Sigma)$ for arbitrary manifolds M, on $T_*(M) = \text{torsion}(H_*(M))$ (Seifert, deRham 1930's).
- $A = \mathbb{Z}[t, t^{-1}] \to \Sigma^{-1}A =$ quotient field : the Reidemeister torsion of knots $S^n \subset S^{n+2}$ (Milnor), and the Blanchfield linking form for knot complement $S^{n+2} \backslash S^n$ (1950-1960's).
- A map $h : M^m \to X^m$ of m-dimensional manifolds can be made transverse at an n-dimensional submanifold $Y^n \subset X$, with $N^n = h^{-1}(Y) \subset M$ also an n-dimensional submanifold. If h is a homotopy equivalence the restriction $h| : N \to Y$ will not in general be a homotopy equivalence.
- Surgery splitting obstruction theory for $m - n = 1$ or 2 is closely related to the K- and L-groups of appropriate universal localizations $A = \mathbb{Z}[\pi_1(X)] \to \Sigma^{-1}A$. (1970 – ..., still work in progress).

Some applications of the torsion K- and L-groups to topology II.

- $m-n=2$. The computation of the cobordism groups C_n of knots $S^n \subset S^{n+2}$ in dimensions high ($n \geqslant 2$ Kervaire, Levine 1970) and low (Cochran-Orr-Teichner 2001, $n=1$).
- Homology surgery theory (Cappell-Shaneson, 1970's)
- The computation of the high-dimensional boundary link cobordism groups (Duval 1984, Sheiham 2003).
- $m-n=1$. If $Y^n \subset X^{n+1}$ is 2-sided $\pi_1(X)$ has the structure of a generalized free product or an HNN extension

  $$\pi_1(Y) \rightrightarrows \pi_1(X\backslash Y)\ .$$

  If $\pi_1(Y) \to \pi_1(X)$ is injective Waldhausen and Cappell (1970's) have decomposed $K_*(\mathbb{Z}[\pi_1(X)])$ and $L_*(\mathbb{Z}[\pi_1(X)])$, with splitting obstructions in Nil- and UNil-groups for $n \geqslant 5$.
- Can interpret decompositions in terms of the K- and L-groups of a certain universal localization $A \to \Sigma^{-1}A$, with the Nil- and UNil-obstructions living in the torsion K- and L-groups.

Commutative localization of rings

- The **localization** of a ring A inverting a multiplicatively closed subset $S \subset A$ of central non-zero divisors with $1 \in S$ is the ring $S^{-1}A$ of fractions a/s ($a \in A, s \in S$), where

$$a/s \ = \ b/t \text{ if and only if } at \ = \ bs \ .$$

  Usual addition and multiplication

$$a/s + b/t \ = \ (at + bs)/(st) \ , \ (a/s)(b/t) \ = \ (as)/(bt) \ .$$

- The canonical morphism $A \to S^{-1}A; a \mapsto a/1$ is injective.
- Localization is a direct limit, with an isomorphism of rings

$$\varinjlim_{s \in S} (A \xrightarrow{s} A \longrightarrow \cdots) \xrightarrow{\cong} S^{-1}A \ ; \ [a] \mapsto a/s \ .$$

- If A is commutative and $\mathfrak{P} \in \text{spec}(A)$ is a prime ideal the localization $(A \backslash \mathfrak{P})^{-1}A = A_{\mathfrak{P}}$ is a local ring, corresponding to the "localization" of an algebraic variety at the point $\mathfrak{P}$.

The classical ring of noncommutative fractions

- Let A be a ring. A multiplicatively closed subset $\Sigma \subset A$ of non-zero divisors satisfies the **Ore condition** if for all $a \in A$, $s \in \Sigma$ there exist $b \in A$, $t \in \Sigma$ with $ta = bs \in A$.
- The **classical ring of fractions** or **Ore localization** $\Sigma^{-1}A$ is the ring of noncommutative fractions

$$\Sigma^{-1}A \;=\; (\Sigma \times A)/\sim$$

with $(s,a) \sim (t,b)$ iff there exist $u, v \in A$ that

$$us \;=\; vt \in \Sigma \;,\; ua \;=\; vb \in A \;.$$

- $\Sigma^{-1}A$ is the universal localization of A inverting Σ. Injective canonical ring morphism

$$A \to \Sigma^{-1}A \;;\; a \mapsto (1,a) \;.$$

- Ore localization can be used to construct quotient skewfield $S^{-1}A$ of certain noncommutative "integral domain" A.

The universal localization II.

- **Theorem** (P.M. Cohn, 1971, G.M. Bergman 1974)
  A universal localization $A \to \Sigma^{-1}A$ exists for any ring A and any set Σ of morphisms in $\mathrm{Proj}(A)$.
- **Proof** By generators and relations.
- **Example** Let $\Sigma = \{s : A^n \to A^n\}$, with

  $$s = (s_{ij})_{1 \leqslant i,j \leqslant n} \, , s_{ij} \in A \, .$$

  The universal localization $\Sigma^{-1}A$ is defined by adding to A the n^2 entries s'_{ij} of a formal inverse $s' = s^{-1}$, and setting the relations given by

  $$ss' = s's = I_n \, .$$

- In general, $A \to \Sigma^{-1}A$ is not injective, and it is even possible that $\Sigma^{-1}A = 0$, e.g. if $0 \in \Sigma$.

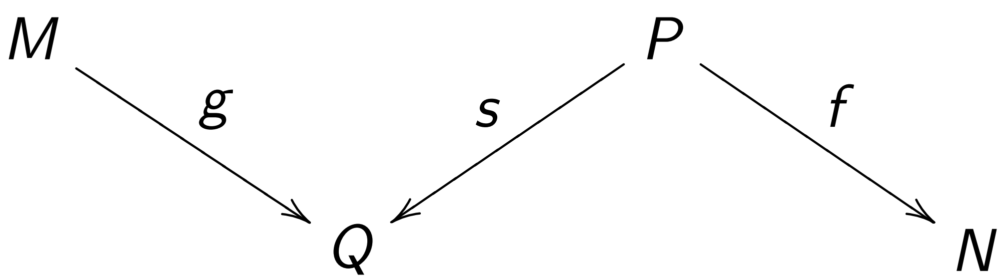
The normal form I.

- (Gerasimov, Malcolmson, 1981) Assume Σ consists of all the morphisms $s : P \to Q$ in $\mathrm{Proj}(A)$ such that $1 \otimes s : \Sigma^{-1}P \to \Sigma^{-1}Q$ is an isomorphism in $\mathrm{Proj}(\Sigma^{-1}A)$. (Can enlarge any Σ to have this). Then every element $x \in \Sigma^{-1}A$ is (non-uniquely) of the form $x = fs^{-1}g$ for some

$$((s : P \to Q) \in \Sigma\,,\ f : P \to A\,,\ g : A \to Q\,)\ .$$

- For f.g. projective A-modules M, N every $\Sigma^{-1}A$-module morphism $x : \Sigma^{-1}M \to \Sigma^{-1}N$ is of the form $x = fs^{-1}g$ for some

$$((s : P \to Q) \in \Sigma\,,\ f : P \to N\,,\ g : M \to Q\,)\ .$$



Addition by $fs^{-1}g + f's'^{-1}g' = (f \oplus f')(s \oplus s')^{-1}(g \oplus g')$.

The normal form II.

- For f.g. projective M, N, a $\Sigma^{-1}A$-module morphism $fs^{-1}g : \Sigma^{-1}M \to \Sigma^{-1}N$ is such that $fs^{-1}g = 0$ if and only if there is a commutative diagram of A-module morphisms

$$\begin{pmatrix} s & 0 & 0 & g \\ 0 & s_1 & 0 & 0 \\ 0 & 0 & s_2 & g_2 \\ f & f_1 & 0 & 0 \end{pmatrix}$$

$$P \oplus P_1 \oplus P_2 \oplus M \longrightarrow Q \oplus Q_1 \oplus Q_2 \oplus N$$

$$\begin{pmatrix} p & p_1 & p_2 & m \end{pmatrix} : P \oplus P_1 \oplus P_2 \oplus M \to L, \qquad \begin{pmatrix} q & q_1 & q_2 & n \end{pmatrix}^T : L \to Q \oplus Q_1 \oplus Q_2 \oplus N$$

with $s, s_1, s_2, \begin{pmatrix} p & p_1 & p_2 \end{pmatrix}, \begin{pmatrix} q & q_1 & q_2 \end{pmatrix}^T \in \Sigma$.

(Exercise: check that diagram $\Longrightarrow fs^{-1}g = 0$).

Noncommutative localization via chain complexes

- For any A-module chain complexes C, D let $[C, D]_A$ be the group of chain homotopy classes of chain maps $C \to D$.
- (Vogel 1982, Neeman+R 2001) For any A, Σ and finite chain complex C in $\mathrm{Proj}(A)$ define the A-module chain complex

$$E(C) \;=\; \varinjlim B$$

with $C \to B$ chain maps in $\mathrm{Proj}(A)$ such that B is finite and

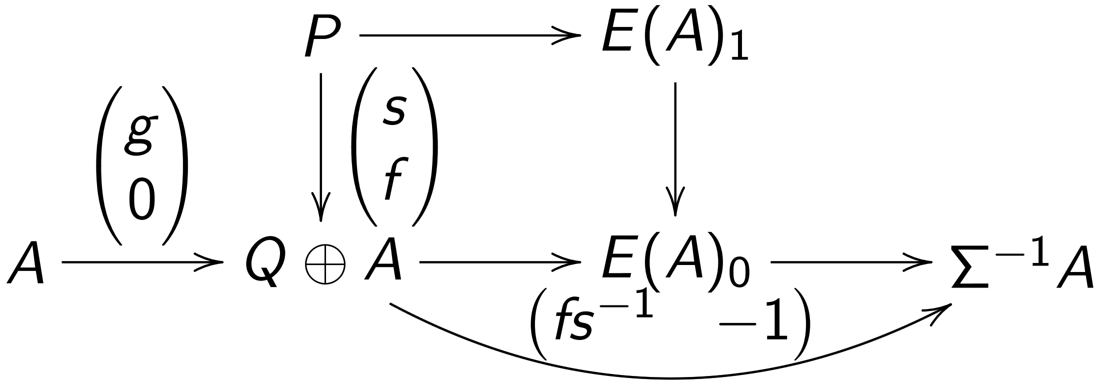
$$H_*(\Sigma^{-1}C) \;\cong\; H_*(\Sigma^{-1}B) \;.$$

- There is a canonical chain homotopy class of A-module chain maps $E(C) \to \Sigma^{-1}C$. In general, the A-module morphisms $H_*(E(C)) = \varinjlim H_*(B) \to H_*(\Sigma^{-1}C)$ are not isomorphisms.
- Universal coefficient spectral sequence

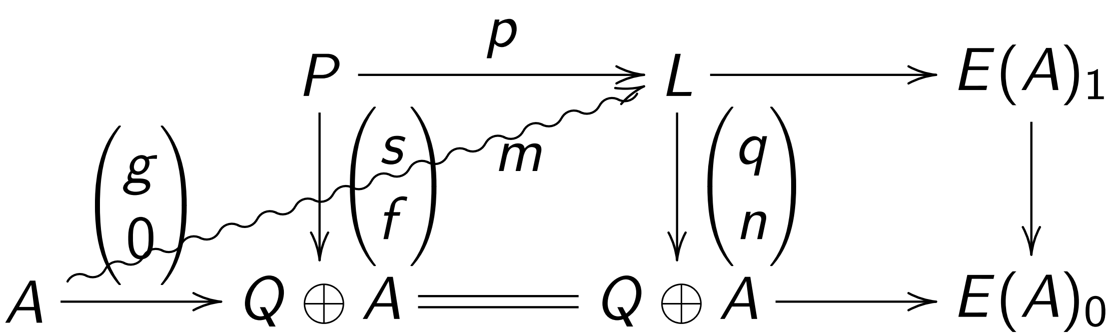
$$E^2_{i,j} \;=\; \mathrm{Tor}^A_i(\Sigma^{-1}A, H_j(E(A))) \Longrightarrow$$
$$H_k(\Sigma^{-1}E(A)) \;=\; \begin{cases} H_0(E(A)) = \Sigma^{-1}A & \text{if } k = 0 \\ 0 & \text{if } k \geqslant 1 \;. \end{cases}$$

Chain complex interpretation of the normal form

- $fs^{-1}g \in \Sigma^{-1}A$ is a chain homotopy class of chain maps $(f, s, g) : A \to E(A)$.



- $fs^{-1}g = 0 \in \Sigma^{-1}A$ if and only if there exists chain homotopy $m : (f, s, g) \simeq 0 : A \to E(A)$. Take $P_1 = P_2 = Q_1 = Q_2 = 0$ for simplicity:



Homology and cohomology with coefficients

- Given a connected CW complex X let $A = \mathbb{Z}[\pi_1(X)]$, and let $C(\widetilde{X})$ be the free A-module cellular chain complex of the universal cover $\widetilde{X}$.
- Given a ring morphism $f : A = \mathbb{Z}[\pi_1(X)] \to B$ define the B-**coefficient homology** and **cohomology** of X to be the B-modules

$$H_*(X;B) = H_*(B \otimes_A C(\widetilde{X})) ,$$
$$H^*(X;B) = H_*(\mathrm{Hom}_B(B \otimes_A C(\widetilde{X}), B)) .$$

- If $i : Y \to X$ is a map of connected CW complexes which induces an isomorphism of B-coefficient homology

$$i_* : H_*(Y;B) \cong H_*(X;B)$$

the relative A-module chain complex

$$C = \text{algebraic mapping cone}(i : C(\widetilde{Y}) \to C(\widetilde{X}))$$

is B-contractible, with $\widetilde{Y} = i^*\widetilde{X}$ the pullback cover of Y.

The universal localization of a ring morphism

- **Example** (Vogel, 1982) Given a ring morphism $f : A \to B$ let Σ be the set of morphisms $s : P \to Q$ in Proj(A) such that
$$1 \otimes s \ : \ B \otimes_A P \to B \otimes_A Q$$
is an isomorphism in Proj(B). Then f factorizes as
$$f \ : \ A \to \Sigma^{-1}A \to B \ .$$
- In favourable circumstances a finite chain complex C in Proj(A) has $H_*(B \otimes_A C) = 0$ if and only if $H_*(\Sigma^{-1}C) = 0$.
- **Proposition** (Dicks-Sontag 1978, Farber-Vogel 1992, Ara-Dicks 2007) For $\mu \geqslant 1$ let $f : A = \mathbb{Z}[F_\mu] \to B = \mathbb{Z}$ be the augmentation $F_\mu \mapsto 1$, and let
$$\Sigma \ = \ \{s : A^k \to A^k \,|\, 1 \otimes s : B^k \to B^k \text{ invertible}\} \ .$$
Then $\Sigma^{-1}\mathbb{Z}[F_\mu]$ is such that a finite chain complex C in Proj(A) has $H_*(B \otimes_A C) = 0$ if and only if $H_*(\Sigma^{-1}C) = 0$,

Noncommutative localization and codimension 2 knotting

- Let $i : N^n \subset M^{n+2}$ be a codimension 2 embedding with exterior $P = M \backslash N$. Assume a factorization
$$A \ = \ \mathbb{Z}[\pi_1(P)] \to \Sigma^{-1}A \to B \ = \ \mathbb{Z}[\pi_1(M)]$$
such that a finite chain complex C in Proj(A) has $H_*(B \otimes_A C) = 0$ if and only if $H_*(\Sigma^{-1}C) = 0$.
- The Alexander duality isomorphisms
$$H_*(P;B) \ \cong \ H^{n+2-*}(M,N;B)$$
show that $H_*(P;B)$ depends only on the homotopy class of i, and does not detect knotting.
- The fundamental group $\pi_1(P)$ detects unknotting using group theory. The homology $H_*(P;\Sigma^{-1}A)$ detects unknotting using homological algebra.
- **Example** For boundary links $N^n = \bigcup\limits_{\mu} S^n \subset M^{n+2} = S^{n+2}$ use
$$A \ = \ \mathbb{Z}[F_\mu] \to \Sigma^{-1}A \to B \ = \ \mathbb{Z} \ .$$

Modules over a universal localization

- **Proposition** A $\Sigma^{-1}A$-module M is an A-module such that the A-module morphism

$$M \to \Sigma^{-1}M \; ; \; x \mapsto 1 \otimes x$$

  is an isomorphism.

- **Proof** The A-module morphism

$$\Sigma^{-1}A \to \Sigma^{-1}A \otimes_A \Sigma^{-1}A \; ; \; x \mapsto 1 \otimes x$$

  is an isomorphism.

- **Definition** A $\Sigma^{-1}A$-module N is **induced** if $N = \Sigma^{-1}M$ for an A-module M.
- In favourable cases it is possible to express the algebraic K- and L-theory of $\Sigma^{-1}A$ in terms of A-modules.

An Ore localization $A \to \Sigma^{-1}A$ is flat

- **Proposition** (i) For any induced $\Sigma^{-1}A$-module chain complex D there exists an A-module chain complex C with $D = \Sigma^{-1}C$.
(ii) D is chain contractible if and only if there exist A-module morphisms $\Gamma : C_r \to C_{r+1}$ with the $\Sigma^{-1}A$-module morphisms

$$1 \otimes (d\Gamma + \Gamma d) \;:\; \Sigma^{-1}C_r \to \Sigma^{-1}C_r$$

isomorphisms.
- **Corollary 1** $\Sigma^{-1}A$ is a flat A-module: the functor $\Sigma^{-1} : \mathrm{Mod}(A) \to \mathrm{Mod}(\Sigma^{-1}A)$ is exact. In fact, an A-module sequence $M \to M' \to M''$ is exact if and only if the $\Sigma^{-1}A$-module sequence $\Sigma^{-1}M \to \Sigma^{-1}M' \to \Sigma^{-1}M''$ is exact.
- **Corollary 2** For any A-module M

$$\mathrm{Tor}_i^A(\Sigma^{-1}A, M) \;=\; 0 \;(i \geqslant 1)\;.$$

- **Corollary 3** For any A-module chain complex C

$$H_*(\Sigma^{-1}C) \;=\; \Sigma^{-1}H_*(C)\;.$$

A universal localization $A \to \Sigma^{-1}A$ need not be flat

- In general, if M is an A-module and C is an A-module chain complex

$$\mathrm{Tor}_*^A(\Sigma^{-1}A, M) \neq 0 \ , \ \ H_*(\Sigma^{-1}C) \neq \Sigma^{-1}H_*(C) \ .$$

- **Example** The universal cover of the complement $S^1 \vee S^1$ of the trivial link $S^1 \cup S^1 \subset S^3$. Let x_1, x_2 be noncommuting indeterminates over $\mathbb{Z}$. The universal localization $\Sigma^{-1}A$ of $A = \mathbb{Z}\langle x_1, x_2\rangle$ inverting $\Sigma = \{x_1\}$ is not flat. The 1-dimensional f.g. free A-module chain complex

$$d_C \ = \ (x_1 \ x_2) \ : \ C_1 \ = \ A \oplus A \to C_0 \ = \ A$$

is a resolution of $H_0(C) = \mathbb{Z}$, with $H_1(C) = 0$ and

$$H_1(\Sigma^{-1}C) \ = \ \mathrm{Tor}_1^A(\Sigma^{-1}A, H_0(C)) \ = \ \Sigma^{-1}A \neq \Sigma^{-1}H_1(C) \ = \ 0 \ .$$

- **Proposition** $\Sigma^{-1}A$ is a flat A-module if and only if $\Sigma^{-1}A$ is an Ore localization (Beachy, Teichner, 2003).

Chain complex lifting

- A **lift** of a f.g. free $\Sigma^{-1}A$-module chain complex D is a f.g. projective A-module chain complex C with a chain equivalence $\Sigma^{-1}C \simeq D$.
- For an Ore localization $\Sigma^{-1}A$ one can lift every n-dimensional f.g. free $\Sigma^{-1}A$-module chain complex D, for any $n \geqslant 0$.
- For a universal localization $\Sigma^{-1}A$ one can only lift for $n \leqslant 2$ in general.
- **Proposition** (Neeman+R., 2001) For $n \geqslant 3$ there are lifting obstructions in $\mathrm{Tor}_i^A(\Sigma^{-1}A, \Sigma^{-1}A)$ for $i \geqslant 2$.
- $\mathrm{Tor}_1^A(\Sigma^{-1}A, \Sigma^{-1}A) = 0$ always.
- (Krause, 2005) General result characterizing the localizations such that
  chain complex lifting = localization of triangulated categories

Stable flatness

- **Definition** A universal localization $\Sigma^{-1}A$ is **stably flat** if
$$\mathrm{Tor}_i^A(\Sigma^{-1}A, \Sigma^{-1}A) \ = \ 0 \quad (i \geqslant 2) \ .$$
- $\Sigma^{-1}A$ is stably flat if and only if $H_i(E(A)) = 0$ for all $i > 0$, if and only if $E(C) \to \Sigma^{-1}C$ is a homology equivalence for every finite chain complex C in $\mathrm{Proj}(A)$.
- For stably flat $\Sigma^{-1}A$ have stable exactness:
$$H_*(\Sigma^{-1}C) \ = \ H_*(E(C)) \ = \ \varinjlim_B \ \Sigma^{-1}H_*(B) \ .$$
with the limit taken over all the chain maps $C \to B$ in $\mathrm{Proj}(A)$ such that B is finite and $H_*(\Sigma^{-1}C) \cong H_*(\Sigma^{-1}B)$.
- Flat $\Longrightarrow$ stably flat. If $\Sigma^{-1}A$ is flat (i.e. an Ore localization)
$$\mathrm{Tor}_i^A(\Sigma^{-1}A, M) \ = \ 0 \quad (i \geqslant 1)$$
for every A-module M. The special case $M = \Sigma^{-1}A$ gives that $\Sigma^{-1}A$ is stably flat.

A universal localization which is not stably flat

- Given a ring extension $R \subset S$ and an S-module M let

$$K(M) \;=\; \ker(S \otimes_R M \to M)\ .$$

- **Theorem** (Neeman, R. and Schofield, 2005)
  (i) The universal localization of the ring

$$A \;=\; \begin{pmatrix} R & S & S \\ 0 & R & S \\ 0 & 0 & R \end{pmatrix} \;=\; P_1 \oplus P_2 \oplus P_3 \text{ (columns)}$$

  inverting $\Sigma = \{P_1 \subset P_2, P_2 \subset P_3\}$ is $\Sigma^{-1}A = M_3(S)$.
  (ii) If S is a flat R-module then

$$\mathrm{Tor}^A_{n-1}(\Sigma^{-1}A, \Sigma^{-1}A) \;=\; M_n(K^n(S)) \ (n \geqslant 3).$$

  (iii) If R is a field and $\dim_R(S) = d$ then

$$K^n(S) \;=\; K(K(\dots K(S)\dots)) \;=\; R^{(d-1)^n d}\ .$$

  If $d \geqslant 2$, e.g. $S = R[x]/(x^d)$, then $\Sigma^{-1}A$ is not stably flat.

Change of rings in algebraic K-theory

- $K_*(A) = K_*(\mathrm{Proj}(A))$ (Bass, Quillen).
- A finite chain complex C in $\mathrm{Proj}(A)$ has a **projective class**

$$[C] \;=\; \sum_{r=0}^{\infty}(-)^r[C_r] \in K_0(A) \;=\; \{\text{projective class group}\}\ .$$

- For a contractible finite chain complex C in $\mathrm{Proj}(A)$ a choice of bases determines the **Whitehead torsion** using any chain contraction $\Gamma : 0 \simeq 1 : C \to C$

$$\tau(C) \;=\; \tau(d + \Gamma : C_{odd} \to C_{even}) \in K_1(A)\ .$$

- For $f : A \to B$ a B-contractible finite chain complex C in $\mathrm{Proj}(A)$ with $[C] = 0 \in K_0(A)$ has a **Reidemeister torsion**

$$\tau(B \otimes_A C) \in \mathrm{im}(K_1(B) \to K_1(f)) \;=\; \mathrm{coker}(f_* : K_1(A) \to K_1(B))$$

  using any choice of bases for C.
- (Milnor 1966) Algebraic K-theory interpretation of the Reidemeister torsion of a knot using $A = \mathbb{Z}[t, t^{-1}] \to B = \mathbb{Q}^\bullet$.

The algebraic K-theory localization exact sequence I.

- Assume each $(s : P \to Q) \in \Sigma$ is injective and $A \to \Sigma^{-1}A$ is injective. The **torsion** exact category $T(A,\Sigma)$ has objects A-modules T with $\Sigma^{-1}T = 0, \text{hom. dim.}\,(T) = 1$.
- **Example** $T = \text{coker}(s)$ for $s \in \Sigma$.
- **Theorem** (Bass, 1968 for central, Schofield, 1985 for universal $\Sigma^{-1}A$). Exact sequence

$$K_1(A) \to K_1(\Sigma^{-1}A) \xrightarrow{\partial} K_0(T(A,\Sigma)) \to K_0(A) \to K_0(\Sigma^{-1}A) \ ,$$

$$\partial\big(\tau(fs^{-1}g : \Sigma^{-1}M \to \Sigma^{-1}N)\big) \ (M, N \text{ based f.g. free})$$

$$= \left[\text{coker}\left(\begin{pmatrix} f & 0 \\ s & g \end{pmatrix} : P \oplus M \to N \oplus Q\right)\right] - \left[\text{coker}(s : P \to Q)\right] .$$

- **Example** If $A = \mathbb{Z}$, $\Sigma = \mathbb{Z}\backslash\{0\}$ then

$$\Sigma^{-1}A \ = \ \mathbb{Q} \ , \ T(A,\Sigma) \ = \ \{\text{finite abelian groups}\} \ ,$$

$$\partial \ : \ \text{coker}(K_1(A) \to K_1(\Sigma^{-1}A)) = \mathbb{Q}^\bullet/\{\pm 1\} \xrightarrow{\cong} K_0(T(A,\Sigma)) \ = \ \bigoplus_{p \text{ prime}} \mathbb{Z} \ ; \ p^n \mapsto (0,\dots,0,n,0,\dots) \ .$$

The algebraic K-theory localization exact sequence II.

- **Example** The boundary map in the Schofield exact sequence for an injective universal localization $A \to \Sigma^{-1}A$

$$\partial \ : \ K_1(\Sigma^{-1}A) \to K_0(T(A,\Sigma)) \ ; \ \tau(D) \mapsto [C]$$

sends the Whitehead torsion $\tau(D)$ of a contractible based f.g. free $\Sigma^{-1}A$-module chain complex D to the projective class $[C]$ of any f.g. projective A-module chain complex C such that $\Sigma^{-1}C \simeq D$.

- **Theorem** (Quillen, 1972, Grayson, 1980) Higher K-theory localization exact sequence for Ore localization $\Sigma^{-1}A$, by flatness

$$\dots \to K_n(A) \to K_n(\Sigma^{-1}A) \to K_{n-1}(T(A,\Sigma)) \to K_{n-1}(A) \to \dots$$

The algebraic K-theory localization exact sequence III.

- **Theorem** (Neeman + R., 2001)
  If $A \to \Sigma^{-1}A$ is injective and stably flat then :
  - there is a 'fibration sequence of exact categories'

    $$T(A,\Sigma) \to \text{Proj}(A) \to \text{Proj}(\Sigma^{-1}A)$$

    (actually need chain complexes)
  - every induced f.g. projective $\Sigma^{-1}A$-module chain complex can be lifted,
  - there is a localization exact sequence

    $$\cdots \to K_n(A) \to K_n(\Sigma^{-1}A) \to K_{n-1}(T(A,\Sigma)) \to K_{n-1}(A) \to \ldots$$

  - e-print RA.0109118, Geometry and Topology (2004)

Algebraic L-theory

- Let A be an associative ring with 1, and with an involution $A \to A; a \mapsto \bar{a}$ used to identify

$$\text{left } A\text{-modules} \;=\; \text{right } A\text{-modules} \,.$$

- **Example** A group ring $A = \mathbb{Z}[\pi]$ with $\bar{g} = g^{-1}$ for $g \in \pi$.
- The **quadratic L-group** $L_n(A)$ is the abelian group of cobordism classes (C, ψ) of n-dimensional f.g. free A-module chain complexes C with an n-dimensional quadratic Poincaré duality

$$\psi \;:\; H^{n-*}(C) \;\cong\; H_*(C) \,.$$

- $L_*(A) = L_{*+4}(A)$ are the Wall (1970) surgery obstruction groups.
- $L_{2i}(A) =$ Witt group of $(-)^i$-hermitian forms on f.g. free A-modules.

The algebraic L-theory localization exact sequence

- **Theorem** (Karoubi, Pardon (1970's) for commutative localization, R. (1980) for Ore localization, Vogel (1982) for universal localization)
  For any injective universal localization $A \to \Sigma^{-1}A$ of rings with involution $T(A,\Sigma) \to P(A) \to P(\Sigma^{-1}A)$ determines an exact localization sequences

$$\cdots \to L_n(A) \to L_n(\Sigma^{-1}A) \to L_n(T(A,\Sigma)) \to L_{n-1}(A) \to \ldots$$

- Suppose that $A \to \Sigma^{-1}A \to B$ is such that a finite chain complex C in $\mathrm{Proj}(A)$ has $H_*(B \otimes_A C) = 0$ if and only if $H_*(\Sigma^{-1}C) = 0$. Then $L_*(\Sigma^{-1}A) = \Gamma_*(A \to B)$ are the Cappell-Shaneson homology surgery obstruction groups.
- $L_{2i}(T(A,\Sigma))$ = Witt group of $\Sigma^{-1}A/A$-valued $(-)^i$-hermitian linking forms on modules in $T(A,\Sigma)$.

Morita theory

- For any ring R and $k \geqslant 1$ let $M_k(R)$ be the ring of $k \times k$ matrices in R.
- **Proposition** The functors

$$\{R\text{-modules}\} \to \{M_k(R)\text{-modules}\}\ ;\ M \mapsto \begin{pmatrix} R \\ R \\ \vdots \\ R \end{pmatrix} \otimes_R M\ ,$$

$$\{M_k(R)\text{-modules}\} \to \{R\text{-modules}\}\ ;$$
$$N \mapsto (R\ R\ \dots\ R) \otimes_{M_k(R)} N$$

are inverse equivalences of categories.
- **Proposition** $K_*(M_k(R)) = K_*(R)$, and for a ring with involution $L_*(M_k(R)) = L_*(R)$.

Triangular matrix rings

- Given rings A_1, A_2 and an (A_1, A_2)-bimodule B define the **triangular matrix ring**

$$A = \begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix} .$$

- **Proposition 1** The A-module category $\mathrm{Mod}(A)$ is equivalent to the category of triples $M = (M_1, M_2, \mu)$ with M_1 an A_1-module, M_2 an A_2-module and $\mu : B \otimes_{A_2} M_2 \to M_1$ an A_1-module morphism.
- **Proposition 2** The functor

$$\mathrm{Proj}(A) \to \mathrm{Proj}(A_1) \times \mathrm{Proj}(A_2) \ ; \ M = (M_1, M_2, \mu) \mapsto$$
$$((A_1 \ B) \otimes_A M, (0 \ A_2) \otimes_A M) = (\mathrm{coker}(\mu), M_2)$$

induces isomorphisms

$$K_*(A) \cong K_*(A_1) \oplus K_*(A_2) .$$

The universal localizations of a triangular matrix ring I.

- The columns of $A = \begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix}$ are f.g. projective A-modules

$$\begin{aligned} P_1 &= \begin{pmatrix} A_1 \\ 0 \end{pmatrix} = (A_1, 0, 0) , \\ P_2 &= \begin{pmatrix} B \\ A_2 \end{pmatrix} = (B, A_2, 1) \end{aligned}$$

  such that $P_1 \oplus P_2 = A$.
- **Proposition** If $A \to C$ is a ring morphism such that there is a C-module isomorphism $C \otimes_A P_1 \cong C \otimes_A P_2$ then $C = M_2(R)$ is the 2×2 matrix ring of $R = \mathrm{End}_C(C \otimes_A P_1)$. The change of rings $A \to C = M_2(R)$ is the **assembly** functor

$$\begin{aligned} \mathrm{Mod}(A) \to \mathrm{Mod}(C) \approx \mathrm{Mod}(R) \ ;\ M \mapsto (R\ \ R) \otimes_A M \\ = \mathrm{coker}(R \otimes_{A_2} B \otimes_{A_1} M_1 \to (R \otimes_{A_1} M_1) \oplus (R \otimes_{A_2} M_2)) \ . \end{aligned}$$

The universal localizations of a triangular matrix ring II.

- **Theorem** (Schofield, Bergman, R., Sheiham 1974–2005)
  Let $A = \begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix}$, $s \in B$. The universal localization of A inverting

  $$\Sigma \;=\; \{\begin{pmatrix} s \\ 0 \end{pmatrix} : P_1 = \begin{pmatrix} A_1 \\ 0 \end{pmatrix} \to P_2 = \begin{pmatrix} B \\ A_2 \end{pmatrix}\}$$

  is

  $$\Sigma^{-1}A \;=\; M_2(R)$$

  with R the ring with one generator x_b for each $b \in B$, and relations
  - $x_b + x_{b'} = x_{b+b'}$ for all $b, b' \in B$,
  - $x_{as}x_b = x_{ab}$ for all $a \in A_1$, $b \in B$,
  - $x_s = 1$.

The stable flatness theorem

- **Theorem** If B, R are flat A_1-modules and B is a flat right A_2-module then the universal localization

$$A = \begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix} \to \Sigma^{-1}A = M_2(R)$$

is stably flat.

- **Proof** The A-module $M = \begin{pmatrix} R \\ R \end{pmatrix}$ has a 1-dimensional flat A-module resolution

$$0 \to \begin{pmatrix} B \\ 0 \end{pmatrix} \otimes_{A_2} R \to \begin{pmatrix} A_1 \\ 0 \end{pmatrix} \otimes_{A_1} R \oplus \begin{pmatrix} B \\ A_2 \end{pmatrix} \otimes_{A_2} R \to M \to 0$$

and hence so does $\Sigma^{-1}A = M \oplus M$.

- **Remark** $\mathrm{Tor}_1^A((A_1\ 0), M) = \ker(B \otimes_{A_2} R \to R)$, so in general $\Sigma^{-1}A$ is not flat.

HNN extensions

- The *HNN* **extension ring** of ring morphisms $i_1, i_2 : S \to R$ is
$$R *_{i_1,i_2} \{t\} \;=\; R * \mathbb{Z}/\{i_1(x)t = ti_2(x) \,|\, x \in S\} \;.$$
For $j = 1, 2$ let $R_j = R$ with (R, S)-bimodule structure
$$R \times R_j \times S \to R_j \;;\; (q, r, s) \mapsto qri_j(s) \;.$$
- The universal localization of $A = \begin{pmatrix} R & R_1 \oplus R_2 \\ 0 & S \end{pmatrix}$ inverting
$\Sigma = \{s_1, s_2 : \begin{pmatrix} R \\ 0 \end{pmatrix} \to \begin{pmatrix} R_1 \oplus R_2 \\ S \end{pmatrix}\}$ is
$$\Sigma^{-1}A \;=\; M_2(R *_{i_1,i_2} \{t\}) \;.$$
- **Proposition** If $i_1, i_2 : S \to R$ are split injections and R_1, R_2 are flat S-modules then $A \to \Sigma^{-1}A$ is injective and stably flat. The algebraic K-theory localization exact sequence has
$$K_n(A) \;=\; K_n(R) \oplus K_n(S) \;,\; K_n(\Sigma^{-1}A) \;=\; K_n(R *_{i_1,i_2} \{t\}) \;,$$
$$K_n(T(A, \Sigma)) \;=\; K_n(S) \oplus K_n(S) \oplus \text{Waldhausen-}\widetilde{\text{Nil}}_n \;.$$

Amalgamated free products

- The **amalgamated free product** $R_1 *_S R_2$ is defined for ring morphisms $S \to R_1$, $S \to R_2$.
- The universal localization of $A = \begin{pmatrix} R_1 & 0 & R_1 \\ 0 & R_2 & R_2 \\ 0 & 0 & S \end{pmatrix}$ inverting

$$\Sigma \ = \ \{s_1 : \begin{pmatrix} R_1 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} R_1 \\ R_2 \\ S \end{pmatrix} \ , \ s_2 : \begin{pmatrix} 0 \\ R_2 \\ 0 \end{pmatrix} \to \begin{pmatrix} R_1 \\ R_2 \\ S \end{pmatrix}\}$$

  is $\Sigma^{-1}A = M_3(R_1 *_S R_2)$.
- **Proposition** If $S \to R_1$, $S \to R_2$ are split injections with R_1, R_2 flat S-modules then $A \to \Sigma^{-1}A$ is injective and stably flat. The algebraic K-theory localization exact sequence has

$$\begin{aligned} K_n(A) \ &= \ K_n(R_1) \oplus K_n(R_2) \oplus K_n(S) \ , \\ K_n(\Sigma^{-1}A) \ &= \ K_n(R_1 *_S R_2) \ , \\ K_n(T(A,\Sigma)) \ &= \ K_n(S) \oplus K_n(S) \oplus \text{Waldhausen-}\widetilde{\text{Nil}}_n \ . \end{aligned}$$

The algebraic L-theory of a triangular ring

- If A_1, A_2, B have involutions then $A = \begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix}$ may not have an involution.
- Involutions on A_1, A_2 and a symmetric isomorphism $\beta : B \to \mathrm{Hom}_{A_1}(B, A_1)$ give a "chain duality" involution on the derived category of A-module chain complexes.
- The dual of an A-module $M = (M_1, M_2, \mu)$ is the A-module chain complex

$$d = (\beta^{-1}\mu^*, 0) \ : \ C_1 = (M_1^*, 0, 0) \to C_0 = (B \otimes_{A_2} M_2^*, M_2^*, 1) \ .$$

- The quadratic L-groups of A are just the relative L-groups in the sequence

$$\cdots \to L_n(A_1) \to^{\otimes(B,\beta)} L_n(A_2) \to L_n(A) \to L_{n-1}(A_1) \to \dots \ .$$

The algebraic L-theory of amalgamated free products and HNN extensions

- **Theorem** Let $R = R_1 *_S R_2$ be the amalgamated free product of split injections $S \to R_1$, $S \to R_2$ of rings with involution, and let $A \to \Sigma^{-1}A = M_3(R)$ be the universal localization of triangular A, as before. If R_1, R_2 are flat S-modules then

$$L_n(\Sigma^{-1}A) \ = \ L_n(R) \ = \ L_n(A) \oplus L_n(T(A,\Sigma)) \ ,$$
$$L_n(T(A,\Sigma)) \ = \ \text{Cappell-UNil}_n(R; S_1, S_2) \ .$$

- Similarly for the UNil-groups of an HNN extension $R *_{i_1,i_2} \{t\}$ of split injective morphisms $i_1, i_2 : S \to R$ of rings with involution with R_1, R_2 flat S-modules, and universal localization $\Sigma^{-1}A = M_2(R *_{i_1,i_2} \{t\})$.