NONCOMMUTATIVE LOCALIZATION
Andrew Ranicki (Edinburgh and MPIM, Bonn)

http://www.maths.ed.ac.uk/~aar

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Introduction

- The talk will describe:
  - the localization $\Sigma^{-1}A$ of a ring $A$ inverting a set $\Sigma$ of $A$-module morphisms,
  - the exact sequences relating the algebraic $K$- and $L$-groups of $A$ and $\Sigma^{-1}A$,
  - the applications to manifolds with fundamental group a generalized free product or an $HNN$ extension, and to submanifolds of codimension 1 and 2.
- These topics have been studied for nearly 50 years by many authors – notably Pierre Vogel.
Absolute and relative $K$- and $L$-groups

- The **absolute** algebraic $K$- and $L$-groups $K_*(A)$, $L_*(A)$ of a ring $A$ are defined using the subcategory

  $\text{Proj}(A) = \{\text{f.g. projective } A\text{-modules}\} \subset \text{Mod}(A) = \{A\text{-modules}\}$.

  Need an involution on $A$ for $L$-theory.

- For a ring morphism $f : A \to B$ use the $(B, A)$-bimodule structure on $B$

  $$B \times B \times A \to B ; (b, x, a) \mapsto b.x.f(a).$$

  to define the **change of rings** functor

  $$B \otimes_A - : \text{Mod}(A) \to \text{Mod}(B) ; M \mapsto B \otimes_A M$$

- Can use $B \otimes_A - : \text{Proj}(A) \to \text{Proj}(B)$ to define the **relative** $K$- and $L$-groups $K_*(f)$, $L_*(f)$ with long exact sequences

  $$\cdots \to K_n(A) \to K_n(B) \to K_n(f) \to K_{n-1}(A) \to \cdots,$$

  $$\cdots \to L_n(A) \to L_n(B) \to L_n(f) \to L_{n-1}(A) \to \cdots.$$
The universal localization I.

- $A$ = ring, $\Sigma = \text{a set of morphisms } s : P \rightarrow Q \text{ in } \text{Proj}(A)$.
- A ring morphism $A \rightarrow B$ is $\Sigma$-inverting if the induced morphisms in $\text{Proj}(B)$
  
  $$1 \otimes s : B \otimes_A P \rightarrow B \otimes_A Q \ (s \in \Sigma)$$

  are isomorphisms.
- A universal localization of $A$ is a $\Sigma$-inverting morphism $A \rightarrow \Sigma^{-1}A$ with the universal property: for any $\Sigma$-inverting morphism $A \rightarrow B$ there is a unique factorization

  $$\begin{array}{ccc}
  A & \rightarrow & B \\
  & \downarrow & \downarrow \\
  & \Sigma^{-1}A & \rightarrow \\
  \end{array}$$

  If $A \rightarrow \Sigma^{-1}A$ exists, it is unique up to isomorphism.
- In general $\text{Proj}(A) \rightarrow \text{Proj}(\Sigma^{-1}A)$ is not a localization of categories in the sense of Verdier, Zisman etc.
Localization in algebraic $K$- and $L$-theory

$A \to \Sigma^{-1}A$ induces the change of rings functor

$$\Sigma^{-1} : \text{Mod}(A) \to \text{Mod}(\Sigma^{-1}A) ; \ M \mapsto \Sigma^{-1}M = \Sigma^{-1}A \otimes_M A .$$

(Milnor, Bass, Quillen, Karoubi, Pardon, R., Vogel, Schofield, Neeman-R., \ldots, 1960’s – now) For certain $A \to \Sigma^{-1}A$

$$K_\ast(A \to \Sigma^{-1}A) = K_{\ast -1}(T(A, \Sigma)) , \ L_\ast(A \to \Sigma^{-1}A) = L_\ast(T(A, \Sigma))$$

with $T(A, \Sigma)$ the torsion exact category of homological dimension 1 $A$-modules $M$ with

$$\Sigma^{-1}M = 0 .$$

Such expressions of relative $K$- and $L$-groups as absolute $K$- and $L$-groups are always interesting!

Pierre Vogel (1980’s) pioneered the use of noncommutative localization in study of knots and links. Motivated by the Wall surgery obstruction $L$-theory, the Cappell-Shaneson homology surgery $\Gamma$-theory and the algebraic theory of surgery (R.).
Some applications of the torsion $K$- and $L$-groups to topology I.

- $A = \mathbb{Z} \rightarrow \Sigma^{-1}A = \mathbb{Q}$, $T(A, \Sigma) = \{\text{finite abelian groups}\}$. 
  $\mathbb{Q}/\mathbb{Z}$-valued linking forms in $T(A, \Sigma)$ for arbitrary manifolds $M$, on $T_*(M) = \text{torsion}(H_*(M))$ (Seifert, deRham 1930’s).
- $A = \mathbb{Z}[t, t^{-1}] \rightarrow \Sigma^{-1}A = \text{quotient field} :$ the Reidemeister torsion of knots $S^n \subset S^{n+2}$ (Milnor), and the Blanchfield linking form for knot complement $S^{n+2}\setminus S^n$ (1950-1960’s).
- A map $h : M^m \rightarrow X^m$ of $m$-dimensional manifolds can be made transverse at an $n$-dimensional submanifold $Y^n \subset X$, with $N^n = h^{-1}(Y) \subset M$ also an $n$-dimensional submanifold. If $h$ is a homotopy equivalence the restriction $h| : N \rightarrow Y$ will not in general be a homotopy equivalence.
- Surgery splitting obstruction theory for $m - n = 1$ or 2 is closely related to the $K$- and $L$-groups of appropriate universal localizations $A = \mathbb{Z}[\pi_1(X)] \rightarrow \Sigma^{-1}A$. (1970 – . . . , still work in progress).
Some applications of the torsion $K$- and $L$-groups to topology II.

$m - n = 2$. The computation of the cobordism groups $C_n$ of knots $S^n \subset S^{n+2}$ in dimensions high ($n \geq 2$ Kervaire, Levine 1970) and low (Cochran-Orr-Teichner 2001, $n = 1$).

Homology surgery theory (Cappell-Shaneson, 1970's)

The computation of the high-dimensional boundary link cobordism groups (Duval 1984, Sheiham 2003).

$m - n = 1$. If $Y^n \subset X^{n+1}$ is 2-sided $\pi_1(X)$ has the structure of a generalized free product or an $HNN$ extension

$$\pi_1(Y) \longrightarrow \pi_1(X \setminus Y).$$

If $\pi_1(Y) \to \pi_1(X)$ is injective Waldhausen and Cappell (1970's) have decomposed $K_*(\mathbb{Z}[\pi_1(X)])$ and $L_*(\mathbb{Z}[\pi_1(X)])$, with splitting obstructions in Nil- and UNil-groups for $n \geq 5$.

Can interpret decompositions in terms of the $K$- and $L$-groups of a certain universal localization $A \to \Sigma^{-1}A$, with the Nil- and UNil-obstructions living in the torsion $K$- and $L$-groups.
Commutative localization of rings

The localization of a ring $A$ inverting a multiplicatively closed subset $S \subset A$ of central non-zero divisors with $1 \in S$ is the ring $S^{-1}A$ of fractions $a/s \ (a \in A, s \in S)$, where

$$\frac{a}{s} = \frac{b}{t}$$

if and only if $at = bs$.

Usual addition and multiplication

$$\frac{a}{s} + \frac{b}{t} = (at + bs)/(st) \ , \ (\frac{a}{s})(\frac{b}{t}) = (as)/(bt) .$$

The canonical morphism $A \to S^{-1}A; a \mapsto a/1$ is injective.

Localization is a direct limit, with an isomorphism of rings

$$\lim_{\longleftarrow} \left( \frac{A}{s} \to A \to \cdots \right) \overset{\cong}{\longrightarrow} S^{-1}A \ ; \ [a] \mapsto a/s .$$

If $A$ is commutative and $\mathfrak{p} \in \text{spec}(A)$ is a prime ideal the localization $(A\setminus \mathfrak{p})^{-1}A = A_{\mathfrak{p}}$ is a local ring, corresponding to the "localization" of an algebraic variety at the point $\mathfrak{p}$.
The classical ring of noncommutative fractions

Let $A$ be a ring. A multiplicatively closed subset $\Sigma \subset A$ of non-zero divisors satisfies the **Ore condition** if for all $a \in A$, $s \in \Sigma$ there exist $b \in A$, $t \in \Sigma$ with $ta = bs \in A$.

The **classical ring of fractions** or **Ore localization** $\Sigma^{-1}A$ is the ring of noncommutative fractions

$$\Sigma^{-1}A = (\Sigma \times A)/\sim$$

with $(s, a) \sim (t, b)$ iff there exist $u, v \in A$ that

$$us = vt \in \Sigma, \ ua = vb \in A.$$  

$\Sigma^{-1}A$ is the universal localization of $A$ inverting $\Sigma$. Injective canonical ring morphism

$$A \to \Sigma^{-1}A; \ a \mapsto (1, a).$$

Ore localization can be used to construct quotient skewfield $S^{-1}A$ of certain noncommutative “integral domain” $A$. 

The universal localization II.

- **Theorem** (P.M. Cohn, 1971, G.M. Bergman 1974)
  A universal localization $A \to \Sigma^{-1}A$ exists for any ring $A$ and any set $\Sigma$ of morphisms in $\text{Proj}(A)$.

- **Proof** By generators and relations.

- **Example** Let $\Sigma = \{s : A^n \to A^n\}$, with

  $$s = (s_{ij})_{1 \leq i,j \leq n}, s_{ij} \in A.$$

  The universal localization $\Sigma^{-1}A$ is defined by adding to $A$ the $n^2$ entries $s'_{ij}$ of a formal inverse $s' = s^{-1}$, and setting the relations given by

  $$ss' = s's = I_n.$$

- In general, $A \to \Sigma^{-1}A$ is not injective, and it is even possible that $\Sigma^{-1}A = 0$, e.g. if $0 \in \Sigma$. 
The normal form I.

(Gerasimov, Malcolmson, 1981) Assume $\Sigma$ consists of all the morphisms $s : P \to Q$ in $\text{Proj}(A)$ such that
$1 \otimes s : \Sigma^{-1}P \to \Sigma^{-1}Q$ is an isomorphism in $\text{Proj}(\Sigma^{-1}A)$.
(Can enlarge any $\Sigma$ to have this). Then every element $x \in \Sigma^{-1}A$ is (non-uniquely) of the form $x = fs^{-1}g$ for some

$$((s : P \to Q) \in \Sigma, f : P \to A, g : A \to Q).$$

For f.g. projective $A$-modules $M, N$ every $\Sigma^{-1}A$-module morphism $x : \Sigma^{-1}M \to \Sigma^{-1}N$ is of the form $x = fs^{-1}g$ for some

$$((s : P \to Q) \in \Sigma, f : P \to N, g : M \to Q).$$

Diagram:

```
M  \downarrow g  \downarrow s  \downarrow f  \downarrow N
  \downarrow        \downarrow        \downarrow        \downarrow
Q  \downarrow        \downarrow        \downarrow        \downarrow
  \downarrow        \downarrow        \downarrow        \downarrow
P
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Addition by $fs^{-1}g + f's'^{-1}g' = (f \oplus f')(s \oplus s')^{-1}(g \oplus g')$. 
The normal form II.

For f.g. projective $M, N$, a $\Sigma^{-1}A$-module morphism $fs^{-1}g : \Sigma^{-1}M \to \Sigma^{-1}N$ is such that $fs^{-1}g = 0$ if and only if there is a commutative diagram of $A$-module morphisms

\[
\begin{pmatrix}
  s & 0 & 0 & g \\
  0 & s_1 & 0 & 0 \\
  0 & 0 & s_2 & g_2 \\
  f & f_1 & 0 & 0
\end{pmatrix}
\]

with $s, s_1, s_2, (p, p_1, p_2), (q, q_1, q_2)^T \in \Sigma$. (Exercise: check that diagram $\implies fs^{-1}g = 0$).
Noncommutative localization via chain complexes

For any $A$-module chain complexes $C, D$ let $[C, D]_A$ be the group of chain homotopy classes of chain maps $C \to D$.

(Vogel 1982, Neeman+R 2001) For any $A, \Sigma$ and finite chain complex $C$ in $\text{Proj}(A)$ define the $A$-module chain complex

$$E(C) = \lim B$$

with $C \to B$ chain maps in $\text{Proj}(A)$ such that $B$ is finite and

$$H_*(\Sigma^{-1}C) \cong H_*(\Sigma^{-1}B).$$

There is a canonical chain homotopy class of $A$-module chain maps $E(C) \to \Sigma^{-1}C$. In general, the $A$-module morphisms $H_*(E(C)) = \lim H_*(B) \to H_*(\Sigma^{-1}C)$ are not isomorphisms.

Universal coefficient spectral sequence

$$E^2_{i,j} = \text{Tor}^A_i(\Sigma^{-1}A, H_j(E(A))) \Longrightarrow$$

$$H_k(\Sigma^{-1}E(A)) = \begin{cases} H_0(E(A)) = \Sigma^{-1}A & \text{if } k = 0 \\ 0 & \text{if } k \geq 1. \end{cases}$$
Chain complex interpretation of the normal form

- \( fs^{-1}g \in \Sigma^{-1}A \) is a chain homotopy class of chain maps \((f, s, g) : A \to E(A)\).

- \( fs^{-1}g = 0 \in \Sigma^{-1}A \) if and only if there exists chain homotopy \( m : (f, s, g) \simeq 0 : A \to E(A) \). Take \( P_1 = P_2 = Q_1 = Q_2 = 0 \) for simplicity:
Homology and cohomology with coefficients

Given a connected CW complex $X$ let $A = \mathbb{Z}[\pi_1(X)]$, and let $C(\tilde{X})$ be the free $A$-module cellular chain complex of the universal cover $\tilde{X}$.

Given a ring morphism $f : A = \mathbb{Z}[\pi_1(X)] \to B$ define the $B$-coefficient homology and cohomology of $X$ to be the $B$-modules

\[
H_*(X; B) = H_*(B \otimes_A C(\tilde{X})) ,
\]

\[
H^*(X; B) = H_*(\text{Hom}_B(B \otimes_A C(\tilde{X}), B)) .
\]

If $i : Y \to X$ is a map of connected CW complexes which induces an isomorphism of $B$-coefficient homology

\[
i_* : H_*(Y; B) \cong H_*(X; B)
\]

the relative $A$-module chain complex

\[
C = \text{algebraic mapping cone}(i : C(\tilde{Y}) \to C(\tilde{X}))
\]

is $B$-contractible, with $\tilde{Y} = i^*\tilde{X}$ the pullback cover of $Y$. 
The universal localization of a ring morphism

Example (Vogel, 1982) Given a ring morphism $f : A \to B$ let $\Sigma$ be the set of morphisms $s : P \to Q$ in $\text{Proj}(A)$ such that

$$1 \otimes s : B \otimes_A P \to B \otimes_A Q$$

is an isomorphism in $\text{Proj}(B)$. Then $f$ factorizes as

$$f : A \to \Sigma^{-1}A \to B.$$ 

In favourable circumstances a finite chain complex $C$ in $\text{Proj}(A)$ has $H_*(B \otimes_A C) = 0$ if and only if $H_*(\Sigma^{-1}C) = 0$.

Proposition (Dicks-Sontag 1978, Farber-Vogel 1992, Ara-Dicks 2007) For $\mu \geq 1$ let $f : A = \mathbb{Z}[F_\mu] \to B = \mathbb{Z}$ be the augmentation $F_\mu \mapsto 1$, and let

$$\Sigma = \{ s : A^k \to A^k \mid 1 \otimes s : B^k \to B^k \text{ invertible} \}.$$ 

Then $\Sigma^{-1}\mathbb{Z}[F_\mu]$ is such that a finite chain complex $C$ in $\text{Proj}(A)$ has $H_*(B \otimes_A C) = 0$ if and only if $H_*(\Sigma^{-1}C) = 0$,
Noncommutative localization and codimension 2 knotting

Let $i : N^n \subset M^{n+2}$ be a codimension 2 embedding with exterior $P = M \setminus N$. Assume a factorization

$$A = \mathbb{Z}[\pi_1(P)] \to \Sigma^{-1}A \to B = \mathbb{Z}[\pi_1(M)]$$

such that a finite chain complex $C$ in Proj$(A)$ has $H_*(B \otimes_A C) = 0$ if and only if $H_*(\Sigma^{-1}C) = 0$.

The Alexander duality isomorphisms

$$H_*(P; B) \cong H^{n+2-*}(M, N; B)$$

show that $H_*(P; B)$ depends only on the homotopy class of $i$, and does not detect knotting.

The fundamental group $\pi_1(P)$ detects unknotting using group theory. The homology $H_*(P; \Sigma^{-1}A)$ detects unknotting using homological algebra.

Example For boundary links $N^n = \bigcup S^n \subset M^{n+2} = S^{n+2}$ use

$$A = \mathbb{Z}[F_{\mu}] \to \Sigma^{-1}A \to B = \mathbb{Z}. $$
Modules over a universal localization

- **Proposition** A $\Sigma^{-1}A$-module $M$ is an $A$-module such that the $A$-module morphism

$$M \to \Sigma^{-1}M ; \ x \mapsto 1 \otimes x$$

is an isomorphism.

- **Proof** The $A$-module morphism

$$\Sigma^{-1}A \to \Sigma^{-1}A \otimes_A \Sigma^{-1}A ; \ x \mapsto 1 \otimes x$$

is an isomorphism.

- **Definition** A $\Sigma^{-1}A$-module $N$ is **induced** if $N = \Sigma^{-1}M$ for an $A$-module $M$.

In favourable cases it is possible to express the algebraic $K$- and $L$-theory of $\Sigma^{-1}A$ in terms of $A$-modules.
An Ore localization $A \to \Sigma^{-1}A$ is flat

- **Proposition** (i) For any induced $\Sigma^{-1}A$-module chain complex $D$ there exists an $A$-module chain complex $C$ with $D = \Sigma^{-1}C$.
(ii) $D$ is chain contractible if and only if there exist $A$-module morphisms $\Gamma : C_r \to C_{r+1}$ with the $\Sigma^{-1}A$-module morphisms 
\[
1 \otimes (d\Gamma + \Gamma d) : \Sigma^{-1}C_r \to \Sigma^{-1}C_r
\]
isomorphisms.

- **Corollary 1** $\Sigma^{-1}A$ is a flat $A$-module: the functor $\Sigma^{-1} : \text{Mod}(A) \to \text{Mod}(\Sigma^{-1}A)$ is exact. In fact, an $A$-module sequence $M \to M' \to M''$ is exact if and only if the $\Sigma^{-1}A$-module sequence $\Sigma^{-1}M \to \Sigma^{-1}M' \to \Sigma^{-1}M''$ is exact.

- **Corollary 2** For any $A$-module $M$
\[
\text{Tor}^A_i(\Sigma^{-1}A, M) = 0 \ (i \geq 1).
\]

- **Corollary 3** For any $A$-module chain complex $C$
\[
H_*(\Sigma^{-1}C) = \Sigma^{-1}H_*(C).
\]
A universal localization $A \to \Sigma^{-1}A$ need not be flat

- In general, if $M$ is an $A$-module and $C$ is an $A$-module chain complex

  $$\text{Tor}_A^*(\Sigma^{-1}A, M) \neq 0 \quad \text{and} \quad H_*(\Sigma^{-1}C) \neq \Sigma^{-1}H_*(C).$$

- **Example** The universal cover of the complement $S^1 \vee S^1$ of the trivial link $S^1 \cup S^1 \subset S^3$. Let $x_1, x_2$ be noncommuting indeterminates over $\mathbb{Z}$. The universal localization $\Sigma^{-1}A$ of $A = \mathbb{Z}\langle x_1, x_2 \rangle$ inverting $\Sigma = \{x_1\}$ is not flat. The 1-dimensional f.g. free $A$-module chain complex

  $$d_C = (x_1 \ x_2) : C_1 = A \oplus A \to C_0 = A$$

  is a resolution of $H_0(C) = \mathbb{Z}$, with $H_1(C) = 0$ and $H_1(\Sigma^{-1}C) = \text{Tor}_1^A(\Sigma^{-1}A, H_0(C)) = \Sigma^{-1}A \neq \Sigma^{-1}H_1(C) = 0$.

- **Proposition** $\Sigma^{-1}A$ is a flat $A$-module if and only if $\Sigma^{-1}A$ is an Ore localization (Beachy, Teichner, 2003).
Chain complex lifting

- A **lift** of a f.g. free $\Sigma^{-1}A$-module chain complex $D$ is a f.g. projective $A$-module chain complex $C$ with a chain equivalence $\Sigma^{-1}C \simeq D$.
- For an Ore localization $\Sigma^{-1}A$ one can lift every $n$-dimensional f.g. free $\Sigma^{-1}A$-module chain complex $D$, for any $n \geq 0$.
- For a universal localization $\Sigma^{-1}A$ one can only lift for $n \leq 2$ in general.
- **Proposition** (Neeman+R., 2001) For $n \geq 3$ there are lifting obstructions in $\text{Tor}^A_i(\Sigma^{-1}A, \Sigma^{-1}A)$ for $i \geq 2$.
- $\text{Tor}^A_1(\Sigma^{-1}A, \Sigma^{-1}A) = 0$ always.
- (Krause, 2005) General result characterizing the localizations such that chain complex lifting $=$ localization of triangulated categories.
Stable flatness

Definition A universal localization $\Sigma^{-1}A$ is stably flat if
\[ \text{Tor}_i^A(\Sigma^{-1}A, \Sigma^{-1}A) = 0 \quad (i \geq 2). \]

$\Sigma^{-1}A$ is stably flat if and only if $H_i(E(A)) = 0$ for all $i > 0$, if and only if $E(C) \rightarrow \Sigma^{-1}C$ is a homology equivalence for every finite chain complex $C$ in Proj($A$).

For stably flat $\Sigma^{-1}A$ have stable exactness:
\[ H_*(\Sigma^{-1}C) = H_*(E(C)) = \lim_{\rightarrow B} \Sigma^{-1}H_*(B). \]

with the limit taken over all the chain maps $C \rightarrow B$ in Proj($A$) such that $B$ is finite and $H_*(\Sigma^{-1}C) \cong H_*(\Sigma^{-1}B)$.

Flat $\iff$ stably flat. If $\Sigma^{-1}A$ is flat (i.e. an Ore localization)
\[ \text{Tor}_i^A(\Sigma^{-1}A, M) = 0 \quad (i \geq 1) \]

for every $A$-module $M$. The special case $M = \Sigma^{-1}A$ gives that $\Sigma^{-1}A$ is stably flat.
A universal localization which is not stably flat

- Given a ring extension $R \subset S$ and an $S$-module $M$ let
  
  $$K(M) = \ker(S \otimes_R M \to M).$$

- **Theorem** (Neeman, R. and Schofield, 2005)
  
  (i) The universal localization of the ring

  $$A = \begin{pmatrix} R & S & S \\ 0 & R & S \\ 0 & 0 & R \end{pmatrix} = P_1 \oplus P_2 \oplus P_3 \text{ (columns)}$$

  inverting $\Sigma = \{P_1 \subset P_2, P_2 \subset P_3\}$ is $\Sigma^{-1}A = M_3(S)$.

  (ii) If $S$ is a flat $R$-module then

  $$\text{Tor}_{n-1}^A(\Sigma^{-1}A, \Sigma^{-1}A) = M_n(K^n(S)) \ (n \geq 3).$$

  (iii) If $R$ is a field and $\dim_R(S) = d$ then

  $$K^n(S) = K(K(\ldots K(S)\ldots)) = R^{(d-1)^n d}.$$ 

  If $d \geq 2$, e.g. $S = R[x]/(x^d)$, then $\Sigma^{-1}A$ is not stably flat.
Change of rings in algebraic $K$-theory

- $K_\ast(A) = K_\ast(\text{Proj}(A))$ (Bass, Quillen).
- A finite chain complex $C$ in $\text{Proj}(A)$ has a **projective class**
  \[
  [C] = \sum_{r=0}^{\infty} (-1)^r [C_r] \in K_0(A) = \{\text{projective class group}\}.
  \]
- For a contractible finite chain complex $C$ in $\text{Proj}(A)$ a choice of bases determines the **Whitehead torsion** using any chain contraction $\Gamma : 0 \simeq 1 : C \to C$
  \[
  \tau(C) = \tau(d + \Gamma : C_{\text{odd}} \to C_{\text{even}}) \in K_1(A).
  \]
- For $f : A \to B$ a $B$-contractible finite chain complex $C$ in $\text{Proj}(A)$ with $[C] = 0 \in K_0(A)$ has a **Reidemeister torsion**
  \[
  \tau(B \otimes_A C) \in \text{im}(K_1(B) \to K_1(f)) = \text{coker}(f_* : K_1(A) \to K_1(B))
  \] using any choice of bases for $C$.
- (Milnor 1966) Algebraic $K$-theory interpretation of the Reidemeister torsion of a knot using $A = \mathbb{Z}[t, t^{-1}] \to B = \mathbb{Q}^\times$. 

\[\text{K}\ast\text{theory}\]
The algebraic $K$-theory localization exact sequence I.

- Assume each $(s : P \to Q) \in \Sigma$ is injective and $A \to \Sigma^{-1}A$ is injective. The torsion exact category $T(A, \Sigma)$ has objects $A$-modules $T$ with $\Sigma^{-1}T = 0$, hom. dim. $(T) = 1$.

- **Example** $T = \text{coker}(s)$ for $s \in \Sigma$.

- **Theorem** (Bass, 1968 for central, Schofield, 1985 for universal $\Sigma^{-1}A$). Exact sequence

$$K_1(A) \to K_1(\Sigma^{-1}A) \xrightarrow{\partial} K_0(T(A, \Sigma)) \to K_0(A) \to K_0(\Sigma^{-1}A) ,$$

$$\partial(\tau(fs^{-1}g : \Sigma^{-1}M \to \Sigma^{-1}N)) \quad (M, N \text{ based f.g. free})$$

$$= \left[\text{coker}\left(\begin{pmatrix} f & 0 \\ s & g \end{pmatrix} : P \oplus M \to N \oplus Q\right)\right] - \left[\text{coker}(s : P \to Q)\right].$$

- **Example** If $A = \mathbb{Z}$, $\Sigma = \mathbb{Z}\setminus\{0\}$ then

$$\Sigma^{-1}A = \mathbb{Q}, \quad T(A, \Sigma) = \{\text{finite abelian groups}\},$$

$$\partial : \text{coker}(K_1(A) \to K_1(\Sigma^{-1}A)) = \mathbb{Q}^\bullet/\{\pm1\} \xrightarrow{\cong} K_0(T(A, \Sigma)) = \bigoplus_{p \text{ prime}} \mathbb{Z} ; \quad p^n \mapsto (0, \ldots, 0, n, 0, \ldots).$$
The algebraic $K$-theory localization exact sequence II.

- **Example** The boundary map in the Schofield exact sequence for an injective universal localization $A \to \Sigma^{-1}A$

  \[
  \partial : K_1(\Sigma^{-1}A) \to K_0(T(A, \Sigma)) ; \quad \tau(D) \mapsto [C]
  \]

  sends the Whitehead torsion $\tau(D)$ of a contractible based f.g. free $\Sigma^{-1}A$-module chain complex $D$ to the projective class $[C]$ of any f.g. projective $A$-module chain complex $C$ such that $\Sigma^{-1}C \simeq D$.

- **Theorem** (Quillen, 1972, Grayson, 1980) Higher $K$-theory localization exact sequence for Ore localization $\Sigma^{-1}A$, by flatness

  \[
  \cdots \to K_n(A) \to K_n(\Sigma^{-1}A) \to K_{n-1}(T(A, \Sigma)) \to K_{n-1}(A) \to \cdots
  \]
The algebraic $K$-theory localization exact sequence III.

- **Theorem** (Neeman + R., 2001)
  If $A \to \Sigma^{-1}A$ is injective and stably flat then:
  - there is a 'fibration sequence of exact categories'
    \[
    T(A, \Sigma) \to \text{Proj}(A) \to \text{Proj}(\Sigma^{-1}A)
    \]
    (actually need chain complexes)
  - every induced f.g. projective $\Sigma^{-1}A$-module chain complex can be lifted,
  - there is a localization exact sequence
    \[
    \cdots \to K_n(A) \to K_n(\Sigma^{-1}A) \to K_{n-1}(T(A, \Sigma)) \to K_{n-1}(A) \to \cdots
    \]
**Algebraic \( L \)-theory**

- Let \( A \) be an associative ring with 1, and with an involution \( A \to A; a \mapsto \bar{a} \) used to identify left \( A \)-modules = right \( A \)-modules.

- **Example** A group ring \( A = \mathbb{Z}[\pi] \) with \( \bar{g} = g^{-1} \) for \( g \in \pi \).

- The **quadratic \( L \)-group** \( L_n(A) \) is the abelian group of cobordism classes \((C, \psi)\) of \( n \)-dimensional f.g. free \( A \)-module chain complexes \( C \) with an \( n \)-dimensional quadratic Poincaré duality
  \[
  \psi : H^{n-*}(C) \cong H_{*}(C).
  \]

- \( L_*(A) = L_{*-4}(A) \) are the Wall (1970) surgery obstruction groups.

- \( L_{2i}(A) = \) Witt group of \((-)^i\)-hermitian forms on f.g. free \( A \)-modules.
The algebraic $L$-theory localization exact sequence

- **Theorem** (Karoubi, Pardon (1970’s) for commutative localization, R. (1980) for Ore localization, Vogel (1982) for universal localization)

  For any injective universal localization $A \to \Sigma^{-1}A$ of rings with involution $T(A, \Sigma) \to P(A) \to P(\Sigma^{-1}A)$ determines an exact localization sequences

  $$\cdots \to L_n(A) \to L_n(\Sigma^{-1}A) \to L_n(T(A, \Sigma)) \to L_{n-1}(A) \to \cdots$$

- Suppose that $A \to \Sigma^{-1}A \to B$ is such that a finite chain complex $C$ in $\text{Proj}(A)$ has $H_*(B \otimes_A C) = 0$ if and only if $H_*(\Sigma^{-1}C) = 0$. Then $L_*(\Sigma^{-1}A) = \Gamma_*(A \to B)$ are the Cappell-Shaneson homology surgery obstruction groups.

- $L_{2i}(T(A, \Sigma)) = \text{Witt group of } \Sigma^{-1}A/A$-valued $(-)^i$-hermitian linking forms on modules in $T(A, \Sigma)$. 
Morita theory

- For any ring $R$ and $k \geq 1$ let $M_k(R)$ be the ring of $k \times k$ matrices in $R$.

- **Proposition** The functors

  \[
  \{R\text{-modules}\} \to \{M_k(R)\text{-modules}\} ; \ M \mapsto \begin{pmatrix} R \\ R \\ \vdots \\ R \end{pmatrix} \otimes_R M ,
  \]

  \[
  \{M_k(R)\text{-modules}\} \to \{R\text{-modules}\} ; \ N \mapsto (R \ R \ \ldots \ R) \otimes_{M_k(R)} N
  \]

  are inverse equivalences of categories.

- **Proposition** $K_\ast(M_k(R)) = K_\ast(R)$, and for a ring with involution $L_\ast(M_k(R)) = L_\ast(R)$. 
Triangular matrix rings

Given rings $A_1, A_2$ and an $(A_1, A_2)$-bimodule $B$ define the triangular matrix ring

$$ A = \begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix}. $$

Proposition 1 The $A$-module category $\text{Mod}(A)$ is equivalent to the category of triples $M = (M_1, M_2, \mu)$ with $M_1$ an $A_1$-module, $M_2$ an $A_2$-module and $\mu : B \otimes_{A_2} M_2 \rightarrow M_1$ an $A_1$-module morphism.

Proposition 2 The functor

$$ \text{Proj}(A) \rightarrow \text{Proj}(A_1) \times \text{Proj}(A_2) ; \ M = (M_1, M_2, \mu) \mapsto ((A_1 B) \otimes_A M, (0 A_2) \otimes_A M) = (\text{coker}(\mu), M_2) $$

induces isomorphisms

$$ K_*(A) \cong K_*(A_1) \oplus K_*(A_2). $$
The universal localizations of a triangular matrix ring I.

- The columns of \( A = \begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix} \) are f.g. projective \( A \)-modules

\[
P_1 = \begin{pmatrix} A_1 \\ 0 \end{pmatrix} = (A_1, 0, 0),
\]  
\[
P_2 = \begin{pmatrix} B \\ A_2 \end{pmatrix} = (B, A_2, 1)
\]

such that \( P_1 \oplus P_2 = A \).

- Proposition If \( A \to C \) is a ring morphism such that there is a \( C \)-module isomorphism \( C \otimes_A P_1 \cong C \otimes_A P_2 \) then \( C = M_2(R) \) is the 2 \( \times \) 2 matrix ring of \( R = \text{End}_C(C \otimes_A P_1) \). The change of rings \( A \to C = M_2(R) \) is the \textbf{assembly} functor

\[
\text{Mod}(A) \to \text{Mod}(C) \cong \text{Mod}(R) \ ; \ M \mapsto (R \ R) \otimes_A M
\]

\[
= \text{coker}(R \otimes_{A_2} B \otimes_{A_1} M_1 \to (R \otimes_{A_1} M_1) \oplus (R \otimes_{A_2} M_2)).
\]
The universal localizations of a triangular matrix ring II.

**Theorem** (Schofield, Bergman, R., Sheiham 1974–2005)

Let $A = \begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix}$, $s \in B$. The universal localization of $A$ inverting

$$\Sigma = \left\{ \begin{pmatrix} s \\ 0 \end{pmatrix} : P_1 = \begin{pmatrix} A_1 \\ 0 \end{pmatrix} \to P_2 = \begin{pmatrix} B \\ A_2 \end{pmatrix} \right\}$$

is

$$\Sigma^{-1}A = M_2(R)$$

with $R$ the ring with one generator $x_b$ for each $b \in B$, and relations

- $x_b + x_{b'} = x_{b+b'}$ for all $b, b' \in B$,
- $x_ax_b = x_{ab}$ for all $a \in A_1$, $b \in B$,
- $x_s = 1$. 
The stable flatness theorem

**Theorem** If $B, R$ are flat $A_1$-modules and $B$ is a flat right $A_2$-module then the universal localization

$$A = \begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix} \to \Sigma^{-1}A = M_2(R)$$

is stably flat.

**Proof** The $A$-module $M = \begin{pmatrix} R \\ R \end{pmatrix}$ has a 1-dimensional flat $A$-module resolution

$$0 \to \begin{pmatrix} B \\ 0 \end{pmatrix} \otimes_{A_2} R \to \begin{pmatrix} A_1 \\ 0 \end{pmatrix} \otimes_{A_1} R \oplus \begin{pmatrix} B \\ A_2 \end{pmatrix} \otimes_{A_2} R \to M \to 0$$

and hence so does $\Sigma^{-1}A = M \oplus M$.

**Remark** $\text{Tor}_1^A((A_1 0), M) = \ker(B \otimes_{A_2} R \to R)$, so in general $\Sigma^{-1}A$ is not flat.
**HNN extensions**

- The **HNN extension ring** of ring morphisms $i_1, i_2 : S \to R$ is

  $$R *_{i_1, i_2} \{t\} = R * \mathbb{Z}/\{i_1(x)t = ti_2(x) | x \in S\}.$$ 

  For $j = 1, 2$ let $R_j = R$ with $(R, S)$-bimodule structure

  $$R \times R_j \times S \to R_j ; (q, r, s) \mapsto qri_j(s).$$

- The universal localization of $A = \begin{pmatrix} R & R_1 \oplus R_2 \\ 0 & S \end{pmatrix}$ inverting

  $$\Sigma = \{s_1, s_2 : \begin{pmatrix} R \\ 0 \end{pmatrix} \to \begin{pmatrix} R_1 \oplus R_2 \\ S \end{pmatrix}\}$$

  is

  $$\Sigma^{-1} A = M_2(R *_{i_1, i_2} \{t\}).$$

- **Proposition** If $i_1, i_2 : S \to R$ are split injections and $R_1, R_2$ are flat $S$-modules then $A \to \Sigma^{-1} A$ is injective and stably flat. The algebraic $K$-theory localization exact sequence has

  $$K_n(A) = K_n(R) \oplus K_n(S), \quad K_n(\Sigma^{-1} A) = K_n(R *_{i_1, i_2} \{t\}),$$

  $$K_n(T(A, \Sigma)) = K_n(S) \oplus K_n(S) \oplus \text{Waldhausen-} \text{Nil}_n.$$
Amalgamated free products

The **amalgamated free product** \( R_1 *_S R_2 \) is defined for ring morphisms \( S \to R_1, S \to R_2 \).

The universal localization of \( A = \begin{pmatrix} R_1 & 0 & R_1 \\ 0 & R_2 & R_2 \\ 0 & 0 & S \end{pmatrix} \) inverting

\[
\Sigma = \{ s_1 : \begin{pmatrix} R_1 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} R_1 \\ R_2 \\ S \end{pmatrix}, \ s_2 : \begin{pmatrix} 0 \\ R_2 \\ 0 \end{pmatrix} \to \begin{pmatrix} R_1 \\ R_2 \\ S \end{pmatrix} \}
\]

is \( \Sigma^{-1}A = M_3(R_1 *_S R_2) \).

**Proposition** If \( S \to R_1, S \to R_2 \) are split injections with \( R_1, R_2 \) flat \( S \)-modules then \( A \to \Sigma^{-1}A \) is injective and stably flat. The algebraic \( K \)-theory localization exact sequence has

\[
K_n(A) = K_n(R_1) \oplus K_n(R_2) \oplus K_n(S),
\]
\[
K_n(\Sigma^{-1}A) = K_n(R_1 *_S R_2),
\]
\[
K_n(T(A, \Sigma)) = K_n(S) \oplus K_n(S) \oplus \text{Waldhausen}-\tilde{\text{Nil}}_n.
\]
The algebraic \( L \)-theory of a triangular ring

- If \( A_1, A_2, B \) have involutions then \( A = \begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix} \) may not have an involution.

- Involution on \( A_1, A_2 \) and a symmetric isomorphism \( \beta : B \to \text{Hom}_{A_1}(B, A_1) \) give a "chain duality" involution on the derived category of \( A \)-module chain complexes.

- The dual of an \( A \)-module \( M = (M_1, M_2, \mu) \) is the \( A \)-module chain complex
  \[
  d = (\beta^{-1} \mu^*, 0) : C_1 = (M_1^*, 0, 0) \to C_0 = (B \otimes_{A_2} M_2^*, M_2^*, 1).
  \]

- The quadratic \( L \)-groups of \( A \) are just the relative \( L \)-groups in the sequence
  \[
  \cdots \to L_n(A_1) \to \otimes(B, \beta) L_n(A_2) \to L_n(A) \to L_{n-1}(A_1) \to \cdots .
  \]
The algebraic $L$-theory of amalgamated free products and $HNN$ extensions

- **Theorem** Let $R = R_1 \ast_S R_2$ be the amalgamated free product of split injections $S \to R_1$, $S \to R_2$ of rings with involution, and let $A \to \Sigma^{-1}A = M_3(R)$ be the universal localization of triangular $A$, as before. If $R_1, R_2$ are flat $S$-modules then

$$L_n(\Sigma^{-1}A) = L_n(R) = L_n(A) \oplus L_n(T(A, \Sigma)),$$

$$L_n(T(A, \Sigma)) = \text{Cappell-UNil}_n(R; S_1, S_2).$$

- Similarly for the UNil-groups of an $HNN$ extension $R \ast_{i_1,i_2} \{t\}$ of split injective morphisms $i_1, i_2 : S \to R$ of rings with involution with $R_1, R_2$ flat $S$-modules, and universal localization $\Sigma^{-1}A = M_2(R \ast_{i_1,i_2} \{t\}).$