MANIFOLDS AND DUALITY

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- Classification of manifolds
- Uniqueness Problem
- Existence Problem
- Quadratic algebra
- Applications
Manifolds

- An \( n \)-dimensional manifold \( M^n \) is a topological space which is locally homeomorphic to \( \mathbb{R}^n \).
  - compact, oriented, connected.

- Classification of manifolds up to homeomorphism.
  - For \( n = 1 \): circle
  - For \( n = 2 \):
    - sphere, torus, \ldots, handlebody.
  - For \( n \geq 3 \): in general impossible.
The Uniqueness Problem

- Is every homotopy equivalence of $n$-dimensional manifolds $f : M^n \rightarrow N^n$ homotopic to a homeomorphism?
  
  - For $n = 1, 2$: Yes.
  
  - For $n \geq 3$: in general No.
The Poincaré conjecture

- Every homotopy equivalence \( f : M^3 \to S^3 \) is homotopic to a homeomorphism.

  - Stated in 1904 and still unsolved!

- Theorem
  \((n \geq 5: \text{Smale, 1960, } n = 4: \text{Freedman, 1983})\)

Every homotopy equivalence \( f : M^n \to S^n \) is homotopic to a homeomorphism.
Old solution of the Uniqueness Problem

- Surgery theory works best for \( n \geq 5 \).
  - From now on let \( n \geq 5 \).

- **Theorem**
  (Browder, Novikov, Sullivan, Wall, 1970)
  A homotopy equivalence \( f : M^n \to N^n \) is homotopic to a homeomorphism if and only if two obstructions vanish.

- The 2 obstructions of surgery theory:
  1. In the **topological** \( K \)-theory of vector bundles over \( N \).
  2. In the **algebraic** \( L \)-theory of quadratic forms over the fundamental group ring \( \mathbb{Z}[\pi_1(N)] \).
Traditional surgery theory

- **Advantage:**
  - Suitable for computations.

- **Disadvantages:**
  - Inaccessible.
  - A complicated mix of topology and algebra.
  - Passage from a homotopy equivalence to the obstructions is indirect.
  - Obstructions are not independent.
Wall’s programme

• “The theory of quadratic structures on chain complexes should provide a simple and satisfactory algebraic version of the whole setup.”


• Such a theory is now available.

Siebenmann’s theorem

- The kernel groups of a map $f : M \to N$ are the relative homology groups
  \[ K_r(x) = H_{r+1}(f^{-1}(x) \to \{x\}) \quad (x \in N). \]

- Exact sequence
  \[
  \cdots \to K_r(x) \to H_r(f^{-1}(x)) \to H_r(\{x\}) \to K_{r-1}(x) \to \cdots.
  \]

- $K_\ast(x) = 0$ for a homeomorphism $f$.

- **Theorem** (Siebenmann, 1972)
  A homotopy equivalence $f : M^n \to N^n$ with
  \[ K_\ast(x) = 0 \quad (x \in N) \]
  is homotopic to a homeomorphism.
New solution of the Uniqueness Problem

- The total surgery obstruction $s(f)$ of a homotopy equivalence $f : M^n \to N^n$ is the cobordism class of
  
  - the sheaf of $\mathbb{Z}$-module chain complexes
  
  - with $n$-dimensional Poincaré duality
  
  - over $N$

  - with stalk homology $K_*(x)$ ($x \in N$).

- Cobordism and Poincaré duality are algebraic.

- **Theorem** A homotopy equivalence $f$ is homotopic to a homeomorphism if and only if $s(f) = 0$. 
Poincaré duality

• **Theorem** (Poincaré, 1895)
  The homology and cohomology of a compact oriented $n$-dimensional manifold $M$ are isomorphic:

  $$H^{n-r}(M) \cong H_r(M) \quad (r = 0, 1, 2, \ldots) .$$

• **Definition** (Browder, 1962)
  An $n$-dimensional duality space $X$ is a space with isomorphisms:

  $$H^{n-r}(X) \cong H_r(X) \quad (r = 0, 1, 2, \ldots) .$$
The Existence Problem

- Is an \( n \)-dimensional duality space \( X \) homotopy equivalent to an \( n \)-dimensional manifold?

  - For \( n = 1, 2 \): Yes.
  
  - For \( n \geq 3 \): in general No.
Old solution of the Existence Problem

- **Theorem**
  (Browder, Novikov, Sullivan, Wall, 1970)
  An $n$-dimensional duality space $X$ is homotopy equivalent to an $n$-dimensional manifold if and only if 2 obstructions vanish.

- The 2 obstructions (as for Uniqueness):
  
  1. In the **topological** $K$-theory of vector bundles over $X$.

  2. In the **algebraic** $L$-theory of quadratic forms over the fundamental group ring $\mathbb{Z}[\pi_1(X)]$.

- Same (dis)advantages as for the old solution of the Uniqueness Problem.
The Theorem of Galewski and Stern

- The kernel groups $K_r(x)$ of an $n$-dimensional duality space $X$ fit into the exact sequence

$$
\cdots \rightarrow K_r(x) \rightarrow H^{n-r}(\{x\}) \rightarrow H_r(X,X\setminus\{x\}) \rightarrow K_{r-1}(x) \rightarrow \cdots.
$$

- $K_*(x) = 0$ for a manifold.

- **Theorem** (Galewski and Stern, 1977)
  A polyhedral duality space $X$ with

  $$
  K_*(x) = 0 \ (x \in X) \ (a \ homology\ manifold)
  $$

  is homotopy equivalent to a manifold.
New solution of the Existence Problem

• The total surgery obstruction \( s(X) \) of an \( n \)-dimensional duality space \( X \) is the cobordism class of
  
  – the sheaf of \( \mathbb{Z} \)-module chain complexes

  – with \((n - 1)\)-dimensional Poincaré duality

  – over \( X \)

  – with stalk homology \( K_\ast(x) \) (\( x \in X \)).

• Cobordism and Poincaré duality are algebraic.

• Theorem A duality space \( X \) is homotopy equivalent to a manifold if and only if \( s(X) = 0 \).
Quadratic algebra

- Chain complexes with the homological properties of manifolds and duality spaces.

- An $n$-dimensional duality complex is a chain complex

\[ C_n \xrightarrow{d} C_{n-1} \xrightarrow{d} C_{n-2} \rightarrow \ldots \rightarrow C_0 \ (d^2 = 0) \]

with isomorphisms

\[ H^{n-r}(C) \cong H_r(C) \ (r = 0, 1, 2, \ldots) \ . \]

- generalized quadratic forms

- cobordism of duality complexes
Local and global duality complexes

$X = \text{connected space}$

- The global surgery group $L_n(\mathbb{Z}[\pi_1(X)])$ of Wall is the cobordism group of $n$-dimensional duality complexes of $\mathbb{Z}[\pi_1(X)]$-modules.
  - Generalized Witt groups.

- The local surgery group $H_n(X; \mathbb{L}(\mathbb{Z}))$ is the cobordism group of $n$-dimensional duality complexes of $\mathbb{Z}$-module sheaves over $X$.
  - Generalized homology with coefficients $L_\ast(\mathbb{Z})$. 
The surgery exact sequence

- **Theorem** The local and global surgery groups are related by the exact sequence

\[
\ldots \rightarrow H_n(X; \mathbb{L}(\mathbb{Z})) \xrightarrow{A} L_n(\mathbb{Z}[\pi_1(X)]) \rightarrow S_n(X) \rightarrow H_{n-1}(X; \mathbb{L}(\mathbb{Z})) \rightarrow \ldots .
\]

- The **assembly map** \( A \) is the passage from local to global duality.

- The **structure group** \( S_n(X) \) is the cobordism group of \((n-1)\)-dimensional local duality complexes over \( X \) which are globally underlinetrivial.
The total surgery obstructions

- **Uniqueness**: the total surgery obstruction of a homotopy equivalence \( f : M^n \to N^n \)

  \[ s(f) \in S_{n+1}(N). \]

  - \( s(f) \) is the cobordism class of the \( n \)-dimensional globally trivial local duality complex with stalk homology the kernels \( K_*(x) \) \( (x \in N) \).

- **Existence**: the total surgery obstruction of an \( n \)-dimensional duality space \( X \)

  \[ s(X) \in S_n(X). \]

  - \( s(X) \) is the cobordism class of the \( (n - 1) \)-dimensional globally trivial local duality complex with stalk homology the kernels \( K_*(x) \) \( (x \in X) \).
Topology and homotopy theory

- The difference between the topology of manifolds and the homotopy theory of duality spaces = the difference between the cobordism theories of the local and global duality complexes.

\[
\begin{array}{ccc}
\text{manifolds} & \to & \text{local duality} \\
\downarrow & & \downarrow A \\
\text{duality spaces} & \to & \text{global duality}
\end{array}
\]

- Converse of Poincaré duality:
  A duality space with sufficient local duality is homotopy equivalent to a manifold.
The Novikov and Borel conjectures

- The Novikov conjecture on the homotopy invariance of the higher signatures is algebraic:
  
  \[ A : H_*(B\pi; \mathbb{L}(\mathbb{Z})) \to L_*(\mathbb{Z}[\pi]) \] is rationally injective, for every group \( \pi \).

- The Borel conjecture on the existence and uniqueness of aspherical manifolds is algebraic:
  
  \[ A : H_*(B\pi; \mathbb{L}(\mathbb{Z})) \to L_*(\mathbb{Z}[\pi]) \] is an isomorphism if \( B\pi \) is a duality space.

- The various solution methods can now be turned into algebra:

  - topology, geometry, analysis (\( C^* \)-algebra), index theorems, \ldots
Applications

- algebraic computations of the $L$-groups
  - number theory
- singular spaces
  - algebraic varieties
- differential geometry
  - hyperbolic geometry
- non-compact manifolds
  - controlled topology
- 3- and 4-dimensional manifolds