AN INTRODUCTION TO EXOTIC SPHERES AND SINGULARITIES

Andrew Ranicki (Edinburgh)
http://www.maths.ed.ac.uk/~aar

Edinburgh, 4 May, 2012

The original papers

- J. Milnor, **On manifolds homeomorphic to the 7-sphere**, Annals of Maths. 64, 399-405 (1956)
- M. Kervaire and J. Milnor, **Groups of homotopy spheres I.**, Annals of Maths. 77, 504–537 (1963)
- F. Pham, **Formules de Picard-Lefschetz généralisées et ramification des intégrales**, Bull. Soc. Math. France 93, 333-367 (1965)
- E. Brieskorn, **Beispiele zur Differentialtopologie von Singularitäten**, Inventiones math. 2, 1–14 (1966)
- F. Hirzebruch, **Singularities and exotic spheres**, Seminaire Bourbaki 314, 1966/67
- J. Milnor, **Singular points of complex hypersurfaces**, Annals of Maths. Study 61 (1968)

Homotopy spheres

- A **homotopy** m**-sphere** Σ^m is a differentiable oriented m-dimensional manifold which is homotopy equivalent to S^m.
- For $m \geqslant 5$ Σ^m is homeomorphic to S^m.
- Σ^m is **standard** if it is diffeomorphic to S^m.
- Σ^m is **exotic** if it is not diffeomorphic to S^m.
- In this lecture will describe the construction and main properties of the **Brieskorn spheres**, which arise as the links of the isolated singularities of complex hypersurfaces.

The original exotic spheres

- The original exotic 7-spheres Σ^7 of Milnor (1956) were constructed as boundaries $\Sigma^7 = \partial F$ of the (D^4, S^3)-bundles over S^4
$$(D^4, S^3) \to (F, \partial F) \to S^4$$
of the 4-plane vector bundles over S^4 classified by particular elements in
$$\pi_4(BSO(4)) \; = \; \mathbb{Z} \oplus \mathbb{Z} \; .$$
- The exotic nature of Σ^7 detected by the defect
$$\text{signature}(F) - \langle \mathcal{L}(F), [F] \rangle \in \mathbb{Q}$$
of the Hirzebruch signature theorem for an 8-dimensional manifold F with $\partial F = \Sigma^7$.
- Kervaire and Milnor (1963) showed that there are 28 differentiable structures on S^7.

Bounding exotic spheres

- A homotopy m-sphere Σ^m **bounds** if $\Sigma^m = \partial F$ for a framed $(m+1)$-dimensional manifold F.
- Pairs $(F, \partial F)$, $(F', \partial F')$ are **cobordant** if there exists an orientation-preserving diffeomorphism $\partial F \cong \partial F'$ such that $F \cup_\partial -F'$ is a framed boundary. The cobordism classes constitute a group bP_{m+1} under connected sum.
- Kervaire-Milnor (1963) computed bP_{m+1} to be a quotient of the simply-connected surgery obstruction group

$$P_{m+1} \ = \ L_{m+1}(\mathbb{Z}) \ .$$

- No obstruction to simply-connected odd-dimensional surgery, $P_{2n-1} = L_{2n-1}(\mathbb{Z}) = 0$, so that $bP_{2n-1} = 0$: every bounding homotopy $(2n-2)$-sphere Σ^{2n-2} is standard.

The bounding odd-dimensional homotopy spheres I.

- Every bounding homotopy $(2n-3)$-sphere is the boundary $\Sigma^{2n-3} = \partial F$ of an $(n-2)$-connected framed $(2n-2)$-dimensional manifold F^{2n-2} constructed by plumbing together μ copies of $\tau_{S^{n-1}}$ using a nonsingular $(-1)^{n-1}$-quadratic form $(H_{n-1}(F) = \mathbb{Z}^\mu, b, q)$ over $\mathbb{Z}$.
- The rel ∂ surgery obstruction of $(F, \partial F) \to (D^{2n-2}, S^{2n-3})$ is

$$\sigma(F) = \begin{cases} \text{signature}(F)/8 \\ \text{Kervaire}(F) \end{cases}$$
$$\in P_{2n-2} = L_{2n-2}(\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n \text{ is odd} \\ \mathbb{Z}_2 & \text{if } n \text{ is even} \end{cases}.$$

- $\text{Kervaire}(F) = \text{Arf}(H_{n-1}(F;\mathbb{Z}_2), q)$ is the Arf invariant of the quadratic form q determined by the framing.
- The surjection $b : P_{2n-2} \to bP_{2n-2}; \sigma(F) \mapsto \partial F$ is a precursor of the Wall realization of surgery obstructions.
- The groups bP_{2n-2} are cyclic finite.

The bounding odd-dimensional homotopy spheres II.

- bP_{4m} is cyclic of order $\sigma_m/8$ with

$$\sigma_m = \epsilon_m 2^{2m-2}(2^{2m-1}-1)\text{numerator}(B_m/4m)$$

where B_m is the mth Bernoulli number, and $\epsilon_m = 2$ or 1, according as to whether m is odd or even.

- $bP_8 = \mathbb{Z}_{28}$, generated by one of the Milnor 1956 examples.
-

$$bP_{4m+2} = \begin{cases} 0 & \text{if there exists a framed } (4m+2)\text{-dimensional manifold} \\ \mathbb{Z}_2 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 0 & \text{for } m = 0,1,3,7,15 \\ \mathbb{Z}_2 & \text{for } m \neq 0,1,3,7,15,31\ . \end{cases}$$

- $bP_{126} = \mathbb{Z}_2$ or 0.

The Brieskorn-Hirzebruch-Pham-Milnor construction

- For any $a = (a_1, a_2, \ldots, a_n)$ with $a_1, a_2, \ldots, a_n \geqslant 2$ the map

  $$P_a \ : \ \mathbb{C}^n \to \mathbb{C} \ ; \ (z_1, z_2, \ldots, z_n) \mapsto z_1^{a_1} + z_2^{a_2} + \cdots + z_n^{a_n}$$

  has an isolated singularity at

  $$(0, 0, \ldots, 0) \in P_a^{-1}(0) \ = \ \text{complex hypersurface} \subset \mathbb{C}^n \ .$$

- The '(star,link)'-pair of the singularity is a framed $(2n-2)$-dimensional manifold with boundary $(F, \partial F) \subset \mathbb{C}^n$ constructed near the singular point. The complexity of the singularity is measured by the differential topology of $(F, \partial F)$.
- A **Brieskorn sphere** is a link $\partial F = \Sigma^{2n-3}$ which happens to be a homotopy $(2n-3)$-sphere, necessarily bounding.
- Σ^{2n-3} can be exotic.

The hypersurface $\Xi_a(t)$

- Terminology of Brieskorn (1966)
- For $t \in \mathbb{C}$ define the hypersurface

$$\begin{aligned} \Xi_a(t) &= P_a^{-1}(t) \\ &= \{(z_1, z_2, \dots, z_n) \in \mathbb{C}^n \mid z_1^{a_1} + z_1^{a_1} + \cdots + z_n^{a_n} = t\} \subset \mathbb{C}^n \end{aligned}$$

- $\Xi_a(t)$ is non-compact if $n \geqslant 2$.
- For $t \neq 0$ $\Xi_a(t)$ is nonsingular, an open $(2n-2)$-dimensional manifold, with a diffeomorphism

$$\Xi_a(t) \cong \Xi_a(1) .$$

- Write $\Xi_a(1) = \Xi_a$.

The star F_a and link Σ_a of the singular point $(0,0,\dots,0) \in \Xi_a(0)$

- $\Xi_a(0)$ has an isolated singularity at $(0,0,\dots,0)$, with $\Xi_a(0)\backslash\{(0,0,\dots,0)\}$ an open $(2n-2)$-dimensional manifold
- For $t \neq 0$ the **star of the singularity** is the compact framed $(2n-2)$-dimensional manifold

$$F_a(t) \; = \; \Xi_a(t) \cap D^{2n} \subset D^{2n} \; .$$

  ($F_a(t)$ denoted $M_a(t)$ by Brieskorn).
- The **link of the singularity** is

$$\Sigma_a(t) \; = \; \partial F_a(t) \; = \; \Xi_a(t) \cap S^{2n-1} \subset S^{2n-1}$$

- For $t \neq 0$ with $|t|$ sufficiently small the (star, link) pair is independent of t, and written

$$(F_a(t), \Sigma_a(t)) \; = \; (F_a, \Sigma_a) \; ,$$

  with a diffeomorphism $F_a \backslash \partial F_a \cong \Xi_a$.

The Milnor fibration

- The codimension 2 submanifold $(F_a, \Sigma_a) \subset (D^{2n}, S^{2n-1})$ is framed, i.e. extends to an embedding

$$(F_a, \Sigma_a) \times D^2 \subset (D^{2n}, S^{2n-1}) \ .$$

- Define the $(2n-1)$-dimensional manifold with boundary

$$(E_a, \partial E_a) \ = \ (\text{cl.}(S^{2n-1} \backslash \Sigma_a \times D^2), \Sigma_a \times S^1) \ .$$

- The **Milnor fibration** map

$$p \ : \ E_a \to S^1 \ ; \ (z_1, z_2, \dots, z_n) \mapsto \frac{z_1^{a_1} + z_2^{a_2} + \dots + z_n^{a_n}}{\|z_1^{a_1} + z_2^{a_2} + \dots + z_n^{a_n}\|}$$

  is the projection of a fibre bundle with fibre $p^{-1}(1) = F_a$.
- The **monodromy** automorphism $h : F_a \to F_a$ is such that

$$E_a \ = \ F_a \times I / \{(x, 0) \sim (h(x), 1) \,|\, x \in F_a\}$$

  with $p : E_a \to S^1 ; [x, \theta] \mapsto e^{2\pi i \theta}$ and

  $h| \ = \ \text{id.} \ : \ \partial F_a \ = \ \Sigma_a \to \Sigma_a \ , \ p| \ = \ \text{proj.} \ : \ \partial E_a \ = \ \Sigma_a \times S^1 \to S^1 \ .$

The join

- The **join** of topological spaces A, B is the space

  $$A*B \ = \ (A\times I\times B)/\{(a_1,0,b)\sim(a_2,0,b),\ (a,0,b_1)\sim(a,0,b_2)\}$$

  for all $a, a_1, a_2 \in A,\ b, b_1, b_2 \in B$.
- If the reduced homology groups $\tilde{H}_*(A)$, $\tilde{H}_*(B)$ are without torsion then

  $$\tilde{H}_{r+1}(A*B) \ = \ \sum_{i+j=r} \tilde{H}_i(A)\otimes\tilde{H}_j(B)\ .$$

- If A is non-empty, and B is path-connected, then $A*B$ is simply-connected.
- The join is associative, with a homeomorphism

  $$(A*B)*C \ \cong A*(B*C)\ .$$

The algebraic and differential topology of (F_a, Σ_a) I.

- Pham, Brieskorn, Hirzebruch and Milnor determined the algebraic and differential topology of (F_a, Σ_a), in particular the conditions under which Σ_a is a homotopy sphere, and determined the differentiable structure.
- The subspace of Ξ_a

$$\Xi_a^{real} \;=\; \{(z_1, \ldots, z_n) \in \Xi_a \,|\, z_j^{a_j} \text{ is real for } j = 1, 2, \ldots, n\}$$

has the following properties.
- Ξ_a^{real} is a compact deformation retract of $\Xi_a = F_a \backslash \Sigma_a$.
- $\Xi_a^{real} = G_1 * G_2 * \cdots * G_n$ is the join of the cyclic groups $G_j = \mathbb{Z}_{a_j}$ of order a_j, regarded as discrete spaces with a_j elements.
- Ξ_a^{real} is $(n-2)$-connected, with homotopy equivalences

$$\Xi_a^{real} \;\simeq\; \Xi_a \;\simeq\; F_a \;\simeq\; S^{n-1} \vee S^{n-1} \vee \ldots \vee S^{n-1}$$

involving $\mu = (a_1 - 1)(a_2 - 1) \ldots (a_n - 1)$ copies of S^{n-1}. μ is called the **Milnor number**, with $H_{n-1}(F_a) = \mathbb{Z}^{\mu}$.

The algebraic and differential topology of (F_a, Σ_a) II.

- The characteristic polynomial of the monodromy automorphism $h_* : H_{n-1}(F_a) \to H_{n-1}(F_a)$ is

$$\begin{aligned} \Delta_a(z) &= \det(z - h_* : H_{n-1}(F_a)[z] \to H_{n-1}(F_a)[z]) \\ &= \prod_{k=1}^{n} \prod_{0<i_k<a_k} (z - \omega_1^{i_1}\omega_2^{i_2}\dots\omega_n^{i_n}) \in \mathbb{Z}[z] \end{aligned}$$

  with $\omega_j = e^{2\pi i/a_j} \in S^1$.
- For $n \geqslant 4$ Σ_a is $(n-3)$-connected, with exact sequence

$$0 \to H_{n-1}(\Sigma_a) \to H_{n-1}(F_a) \xrightarrow{1-h_*} H_{n-1}(F_a) \to H_{n-2}(\Sigma_a) \to 0\ .$$

  Thus Σ_a is a homotopy $(2n-3)$-sphere if and only if

$$\Delta_a(1) = 1 \in \mathbb{Z}\ .$$

The Kervaire invariants of Brieskorn $(4m+1)$-spheres

- J. Levine, **Polynomial invariants of codimension two**, Annals of Maths. 84, 537–554 (1966)
- For $m \geqslant 1$ let $a = (a_1, a_2, \ldots, a_{2m+2})$ be such that Σ_a is a homotopy $(4m+1)$-sphere. The Kervaire invariant of F_a in $L_{4m+2}(\mathbb{Z}) = \{0, 1\}$ is

$$\begin{aligned} \sigma(F_a) &= \mathrm{Arf}(H_{2m+1}(F_a; \mathbb{Z}_2), q) \\ &= \begin{cases} 0 & \text{if } \Delta_a(-1) \equiv \pm 1 \bmod 8 \\ 1 & \text{if } \Delta_a(-1) \equiv \pm 3 \bmod 8 \end{cases} \end{aligned}$$

Brieskorn $(4m+1)$-spheres with Kervaire invariant 1

- The Brieskorn $(4m+1)$-sphere Σ_a for $a = (2, 2, \ldots, 2, 3)$ has Kervaire invariant

$$\sigma(F_a) \ = \ 1 \in L_{4m+2}(\mathbb{Z}) \ = \ \mathbb{Z}_2 \ = \ \{0, 1\} \ .$$

- If $bP_{4m+2} = \mathbb{Z}_2$ then $\Sigma_a \in bP_{4m+2}$ is the generator.
- The exotic 9-sphere $\Sigma_{(2,2,2,2,2,3)}$ generates $bP_{10} = \mathbb{Z}_2$. Diffeomorphic to the exotic Kervaire 9-sphere, originally constructed by plumbing together 2 copies of τ_{S^5} using the quadratic form of Arf invariant 1.

The signatures of Brieskorn $(4m-1)$-spheres

- For $m \geqslant 1$ let $a = (a_1, a_2, \ldots, a_{2m+1})$ be such that Σ_a is a homotopy $(4m-1)$-sphere.
- Hirzebruch (1966) computed the signature of F_a to be

$$\sigma(F_a) = \sigma_a^+ - \sigma_a^- \in \mathbb{Z}$$

with σ_a^+ the number of $(2m+1)$-tuples $j = (j_1, j_2, \ldots, j_{2m+1})$ of integers with $0 < j_k < a_k$ such that

$$0 < \sum_{k=1}^{2m+1} \frac{j_k}{a_k} < 1 \mod 2 \ ,$$

and σ_a^- the number of $(2m+1)$-tuples j such that

$$-1 < \sum_{k=1}^{2m+1} \frac{j_k}{a_k} < 0 \mod 2 \ .$$

Brieskorn $(4m-1)$-spheres with non-zero signatures

- The signatures/8 of the Brieskorn $(4m-1)$-spheres Σ_a for $a=(2,\dots,2,3,6k-1)$ are given by

$$\sigma(F_a)/8 \;=\; (-1)^m k \in L_{4m}(\mathbb{Z}) \;=\; \mathbb{Z}\ .$$

- The Brieskorn spheres Σ_a for $k=1,2,\dots,\sigma_m/8$ represent the $\sigma_m/8$ bounded differentiable structures in $bP_{4m}=\mathbb{Z}_{\sigma_m/8}$.
- In particular, $\Sigma_{(2,2,2,3,5)}$ is one of the original 1956 exotic 7-spheres of Milnor, generating $bP_8=\mathbb{Z}_{28}$.