THE TOTAL SURGERY OBSTRUCTION

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The homotopy types of manifolds

- **Manifold** = compact oriented topological manifold.
- An n-**dimensional manifold** M is defined by the property that every $x \in M$ has an open neighbourhood $U \subset M$ homeomorphic to $\mathbb{R}^n$, so

$$(U, U\backslash\{x\}) \;\cong\; (\mathbb{R}^n, \mathbb{R}^n\backslash\{0\}) \;.$$

- A manifold M is an n-**dimensional homology manifold**

$$H_*(M, M\backslash\{x\}) \;\cong\; H_*(\mathbb{R}^n, \mathbb{R}^n\backslash\{0\}) = \begin{cases} \mathbb{Z} & \text{for } * = n \\ 0 & \text{for } * \neq n \;. \end{cases}$$

- A homology manifold M has **Poincaré duality**

$$H^{n-*}(M) \;\cong\; H_*(M) \;.$$

- The **total surgery obstruction** $s(X)$ is a homotopy invariant of a space X with n-dimensional Poincaré duality which measures the failure of X to have the homotopy type of a manifold. It is a complete invariant for $n > 4$.

The local-to-global assembly in homology

- The **local homology groups** of a space X at $x \in X$ are
$$H_*(X)_x \;=\; H_*(X, X\backslash\{x\}) \;.$$
- For any homology class $[X] \in H_n(X)$ the images
$$[X]_x \in \mathrm{im}(H_n(X) \to H_n(X, X\backslash\{x\}))$$
can be viewed as $\mathbb{Z}$-module morphisms
$$[X]_x \;:\; H^{n-*}(\{x\}) = H_*(\mathbb{R}^n, \mathbb{R}^n\backslash\{0\}) \to H_*(X, X\backslash\{x\}) \; (x \in X) \;.$$
- The diagonal map
$$\Delta \;:\; X \to X \times X \;;\; x \mapsto (x, x)$$
sends $[X] \in H_n(X)$ to the chain homotopy class
$$\Delta[X] \;=\; [X] \cap - \in H_n(X \times X) \;=\; H_0(\mathrm{Hom}_{\mathbb{Z}}(C(X)^{n-*}, C(X)))$$
of the cap product $\mathbb{Z}$-module chain map $[X] \cap - : C(X)^{n-*} \to C(X)$, **assembling** $[X]_x$ $(x \in X)$ to $\Delta[X] = [X] \cap - : H^{n-*}(X) \to H_*(X)$.

The duality theorems

- Let X be a connected space with universal cover $\widetilde{X}$ and fundamental group $\pi_1(X)=\pi$, homology and compactly supported cohomology

$$\begin{aligned} H_*(\widetilde{X}) &= H_*(C(\widetilde{X}))\ , \\ H^*(\widetilde{X}) &= H_{-*}(\mathrm{Hom}_{\mathbb{Z}[\pi]}(C(\widetilde{X}),\mathbb{Z}[\pi]))\ . \end{aligned}$$

- **Poincaré duality** If X is an n-dimensional manifold with fundamental class $[X]\in H_n(X)$ then the local $\mathbb{Z}$-module Poincaré duality isomorphisms

$$[X]_x\cap - \ :\ H^{n-*}(\{x\}) \cong H_*(X,X\backslash\{x\})\ (x\in X)$$

  assemble to the global $\mathbb{Z}[\pi]$-module Poincaré duality isomorphisms

$$[X]\cap - \ :\ H^{n-*}(\widetilde{X}) \cong H_*(\widetilde{X})\ .$$

- **Poincaré-Lefschetz duality** An n-dimensional manifold with boundary $(X,\partial X)$ has a fundamental class $[X]\in H_n(X,\partial X)$ and $\mathbb{Z}[\pi]$-module isomorphisms $[X]\cap - : H^{n-*}(\widetilde{X},\widetilde{\partial X}) \cong H_*(\widetilde{X})$.

The triangulation of manifolds

- A manifold M is **triangulable** if it is homeomorphic to a finite simplicial complex, in which case it is a finite CW complex.
- An n-dimensional PL manifold is automatically a finite simplicial complex, and so triangulable.
- Cairns (1940): every differentiable manifold has a canonical PL triangulation.
- Kirby+Siebenmann (1969): (i) every n-dimensional manifold M has the homotopy type of a finite CW complex, and (ii) for $n > 4$ there exist M without a PL triangulation.
- Edwards (1977): for $n > 4$ there exist n-dimensional manifolds with non-PL triangulations.
- Freedman (1982)+Casson(1990): there exist non-triangulable 4-dimensional manifolds, e.g. the E_8-manifold.
- It is still not known whether there exist non-triangulable n-dimensional manifolds for $n > 4$.

CW complexes and $\mathbb{Z}[\pi]$-module chain complexes

- For any group π use the involution on the group ring $\mathbb{Z}[\pi]$

$$\mathbb{Z}[\pi] \to \mathbb{Z}[\pi] ~;~ \sum_{g \in \pi} n_g g \mapsto \sum_{g \in \pi} n_g g^{-1} ~ (n_g \in \mathbb{Z}, g \in \pi)$$

to define the **dual** of a left $\mathbb{Z}[\pi]$-module K to be the left $\mathbb{Z}[\pi]$-module

$$K^* ~=~ \mathrm{Hom}_{\mathbb{Z}[\pi]}(K, \mathbb{Z}[\pi]) ~,~ (gf)(x) = f(x)g^{-1} ~ (f \in K^*, x \in K) ~.$$

- If K is f.g. free then so is K^*, with a natural isomorphism $K \cong K^{**}$.
- Let X be a CW complex, and let $\widetilde{X}$ be regular cover of X with group of covering translations π. The cellular free $\mathbb{Z}[\pi]$-module chain complex $C(\widetilde{X})$ and its dual $C(\widetilde{X})^{-*}$ are given by

$$C(\widetilde{X})_r ~=~ H_r(\widetilde{X}^{(r)}, \widetilde{X}^{(r-1)}) ~,~ C(\widetilde{X})^r ~=~ C(\widetilde{X})^*_r ~.$$

- If X is finite then $C(\widetilde{X})$ and $C(\widetilde{X})^{-*}$ are f.g. free. A homology class

$$\widetilde{\phi} \in H_n(\widetilde{X} \times_\pi \widetilde{X}) ~=~ H_0(\mathrm{Hom}_{\mathbb{Z}[\pi]}(C(\widetilde{X})^{n-*}, C(\widetilde{X})))$$

is a chain homotopy class of chain maps $\widetilde{\phi} : C(\widetilde{X})^{n-*} \to C(\widetilde{X})$.

Geometric Poincaré complexes

- An n-**dimensional geometric Poincaré complex** X is a finite CW complex with a fundamental class $[X] \in H_n(X)$ such that
$$\Delta[X] \in H_n(\widetilde{X} \times_\pi \widetilde{X}) = H_0(\mathrm{Hom}_{\mathbb{Z}[\pi]}(C(\widetilde{X})^{n-*}, C(\widetilde{X})))$$
is a chain homotopy class of $\mathbb{Z}[\pi]$-module chain equivalences
$$\Delta[X] \ = \ [X] \cap - \ : \ C(\widetilde{X})^{n-*} \to C(\widetilde{X}) \ ,$$
with $\widetilde{X}$ the universal cover of X, $\pi = \pi_1(X)$ and
$$\Delta \ : \ X \ = \ \widetilde{X}/\pi \to \widetilde{X} \times_\pi \widetilde{X} \ ; \ [\widetilde{x}] \mapsto [\widetilde{x}, \widetilde{x}] \ .$$
- Every n-dimensional manifold M is homotopy equivalent to an n-dimensional geometric Poincaré complex X.
- There is a corresponding notion of an n-**dimensional geometric Poincaré pair** $(X, \partial X)$ with a fundamental class $[X] \in H_n(X, \partial X)$ and the Poincaré-Lefschetz chain equivalence of an n-dimensional manifold with boundary
$$\Delta[X] \ = \ [X] \cap - \ : \ C(\widetilde{X})^{n-*} \to C(\widetilde{X}, \widetilde{\partial X}) \ .$$

The fundamental questions of surgery theory

- The fundamental questions are:
  - (i) Is an n-dimensional geometric Poincaré complex X homotopy equivalent to a manifold? (manifold existence)
  - (ii) Is a homotopy equivalence of n-dimensional manifolds $f : M \to N$ homotopic to a homeomorphism? (rigidity)
- It has been known since the 1960's that in general the answers are no!
- For $n > 4$ the Browder-Novikov-Sullivan-Wall theory provides a 2-stage obstruction theory **working outside** X for both (i) and (ii): a **primary** obstruction in the topological K-theory of vector bundles and spherical fibrations, and a **secondary** obstruction in the algebraic L-theory of quadratic forms.
- The total surgery obstruction unites the 2 BNSW obstructions into a single **internal** obstruction, but still relies on them for proof.

The converse of the Poincaré duality theorem

- The $\mathcal{S}$-**groups** of a space X are the relative homotopy groups

$$\mathcal{S}_n(X) \ = \ \pi_n(A : H(X; \mathbf{L}_\bullet(\mathbb{Z})) \to \mathbf{L}_\bullet(\mathbb{Z}[\pi_1(X)]))$$

of the **assembly** map A of algebraic L-theory spectra, with

$$\pi_*(\mathbf{L}_\bullet(\mathbb{Z}[\pi_1(X)])) \ = \ L_*(\mathbb{Z}[\pi_1(X)])$$

the Wall surgery obstruction groups, and

$$H(X; \mathbf{L}_\bullet(\mathbb{Z})) \ = \ X_+ \wedge \mathbf{L}_\bullet(\mathbb{Z})$$

the generalized homology spectrum of X with $\mathbf{L}_\bullet(\mathbb{Z})$-coefficients.
- The **total surgery obstruction** of an n-dimensional geometric Poincaré complex X is a homotopy invariant $s(X) \in \mathcal{S}_n(X)$ measuring the failure of local Poincaré duality in X, given X has global Poincaré duality.
- **Key idea** Need to measure failure only up to algebraic Poincaré cobordism, in order to have a homotopy invariant.
- **Theorem** (R., 1978) For $n > 4$ $s(X) = 0$ if and only if X is homotopy equivalent to an n-dimensional manifold.

The rel ∂ total surgery obstruction

- The mapping cylinder of a homotopy equivalence $f : M \to N$ of n-dimensional manifolds

$$L = (M \times I \sqcup N)/\{(x,1) \sim f(x) \,|\, x \in M\}$$

is an $(n+1)$-dimensional geometric Poincaré cobordism $(L; M, N)$ with manifold boundary components.
- The **rel ∂ total surgery obstruction** $s_\partial(L) \in \mathcal{S}_{n+1}(L)$ is such that for $n > 4$ and $\tau(f) = 0 \in Wh(\pi_1(N))$ the following conditions are equivalent:
  (a) $s_\partial(L) = 0$,
  (b) f is homotopic to a homeomorphism,
  (c) the inverse images $f^{-1}(x) \subset M$ $(x \in N)$ are acyclic, $\widetilde{H}_*(f^{-1}(x)) = 0$, up to algebraic Poincaré cobordism.
- Since the rigidity question (ii) is a relative ∂ form of manifold existence (i), will only address (i).

Vector bundles and spherical fibrations

- The k-plane vector bundles over a finite CW complex X are classified by the homotopy classes of maps $X \to BO(k)$.
- An n-dimensional differentiable manifold $M \subset S^{n+k}$ has tangent and normal bundles

$$\tau_M : M \to BO(n) \ , \ \nu_M : M \to BO(k)$$

  with Whitney sum the trivial $(n+k)$-plane vector bundle

$$\tau_M \oplus \nu_M \ = \ \epsilon^{n+k} \ : \ M \to BO(n+k) \ .$$

- Similarly for topological bundles, with classifying space $BTOP(k)$, and τ_M, ν_M for manifolds M.
- $(k-1)$-spherical fibrations $S^{k-1} \to E \to X$ have classifying space $BG(k)$. Forgetful maps $BO(k) \to BTOP(k) \to BG(k)$, and fibration

$$G(k)/TOP(k) \to BTOP(k) \to BG(k) \to B\big(G(k)/TOP(k)\big) \ .$$

The Spivak normal fibration

- **Theorem** (Spivak 1965, Wall 1969, R. 1980)
  A finite subcomplex $X \subset S^{n+k}$ is an n-dimensional geometric Poincaré complex if and only if for any closed regular neighbourhood $(W, \partial W) \subset S^{n+k}$
  $$\text{homotopy fibre}(\partial W \subset W) \ \simeq \ S^{k-1} \ .$$
- This is the **Spivak normal fibration**
  $$\nu_X \ : \ S^{k-1} \to \partial W \to W \simeq X \ .$$
- The **Thom space** $T(\nu_X) = W/\partial W$ has a degree 1 map
  $$\rho_X \ : \ S^{n+k} \to S^{n+k}/(S^{n+k} \backslash W) \ = \ W/\partial W \ = \ T(\nu_X)$$
  with the Hurewicz image the fundamental class $[X] \in H_n(X)$
  $$h : \pi_{n+k}(T(\nu_X)) \to \widetilde{H}_{n+k}(T(\nu_X)) = H_n(X) \ ; \ \rho_X \mapsto [X] \ .$$
- The Spivak normal fibration of a manifold M is the sphere bundle $J\nu_M : M \to BG(k)$ of $\nu_M : M \to BTOP(k)$.

The Browder-Novikov construction of normal maps, and the Wall surgery obstruction

- X = n-dimensional geometric Poincaré complex.
- If $\nu_X : X \to BG(k)$ has a topological reduction $\widetilde{\nu}_X : X \to BTOP(k)$ and $n > 4$ can make

$$\rho_X \ : \ S^{n+k} \to T(\nu_X) = T(\widetilde{\nu}_X)$$

  topologically transverse at the zero section $X \subset T(\widetilde{\nu}_X)$, with

$$\rho_X| \ = \ (f, b) \ : \ (M, \nu_M) \ = \ (\rho_X)^{-1}(X) \to (X, \widetilde{\nu}_X)$$

  a **degree 1 normal map** from an n-dimensional manifold M.
- The **Wall surgery obstruction** $\sigma_*(f, b) \in L_n(\mathbb{Z}[\pi_1(X)])$ is such that for $n > 4$ $\sigma_*(f, b) = 0$ if and only if (f, b) is normal bordant to a homotopy equivalence.

The fundamental answer according to BNSW

- Browder-Novikov-Sullivan-Wall surgery theory (1960's) for differentiable and PL manifolds, extended in 1970 by Kirby-Siebenmann to topological manifolds.
- **Fundamental answer** For $n > 4$ an n-dimensional geometric Poincaré complex X is homotopy equivalent to a manifold if and only if
  - (a) the Spivak normal fibration $\nu_X : X \to BG(k)$ (k large) admits a TOP reduction $\widetilde{\nu}_X : X \to BTOP(k)$, in which case there exists a normal map

    $$(f, b) \;=\; \rho_X| \;:\; (M, \nu_M) \;=\; (\rho_X)^{-1}(X) \to (X, \widetilde{\nu}_X)$$

    with Wall surgery obstruction $\sigma_*(f, b) \in L_n(\mathbb{Z}[\pi_1(X)])$.
  - (b) there exists $\widetilde{\nu}_X$ for which $\sigma_*(f, b) = 0$.
- (a) gives the primary obstruction in $\nu_X \in [X, B(G/TOP)]$, and (b) gives the secondary obstruction in $L_n(\mathbb{Z}[\pi_1(X)])$, defined only when the primary one vanishes.

The algebraic surgery exact sequence

- Let $\mathbf{L}_\bullet(\mathbb{Z})$ be the 1-connective spectrum of quadratic forms over $\mathbb{Z}$ with homotopy groups the simply-connected surgery obstruction groups

$$\pi_n(\mathbf{L}_\bullet(\mathbb{Z})) = L_n(\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n \equiv 0 (\bmod 4) \\ \mathbb{Z}_2 & \text{if } n \equiv 2 (\bmod 4) \\ 0 & \text{if } n \equiv 1,3 (\bmod 4) \end{cases} .$$

- $\mathbf{L}_\bullet(\mathbb{Z})_0 \simeq G/TOP$, the homotopy fibre of $BTOP \to BG$.
- Roughly speaking, the **generalized homology groups** $H_*(X;\mathbf{L}_\bullet(\mathbb{Z}))$ are the cobordism groups of sheaves over X of quadratic forms over $\mathbb{Z}$.
- The **algebraic surgery exact sequence** is

$$\cdots \to H_n(X;\mathbf{L}_\bullet(\mathbb{Z})) \xrightarrow{A} L_n(\mathbb{Z}[\pi_1(X)]) \to \mathcal{S}_n(X) \to H_{n-1}(X;\mathbf{L}_\bullet(\mathbb{Z})) \to \dots .$$

The correspondence between the 2 BNSW obstructions and the total surgery obstruction

- Let X be an n-dimensional Poincaré complex, with total surgery obstruction $s(X) \in \mathcal{S}_n(X)$.
- The *TOP* **reduction obstruction**

$$t(X) \ = \ [s(X)] \in \mathrm{im}(\mathcal{S}_n(X) \to H_{n-1}(X;\mathbf{L}_\bullet(\mathbb{Z})))$$

  is such that $t(X)=0$ if and only if $\nu_X : X \to BG(k)$ lifts to $\widetilde{\nu}_X : X \to BTOP(k)$. (In fact, $8t(X)=0$).
- $t(X)=0$ if and only if

$$s(X) \in \ker(\mathcal{S}_n(X) \to H_{n-1}(X;\mathbf{L}_\bullet(\mathbb{Z}))) \ = \ \mathrm{im}(L_n(\mathbb{Z}[\pi_1(X)]) \to \mathcal{S}_n(X))$$

  with $s(X) = [\sigma_*((f,b):(M,\nu_M) \to (X,\widetilde{\nu}_X))]$.
- $s(X)=0$ if and only if there exists $\widetilde{\nu}_X$ with $\sigma_*(f,b)=0$.
  For $n>4$ this is equivalent to X being homotopy equivalent to a manifold.

The converse of the Hirzebruch signature theorem

- (Browder, 1962) For $k > 1$ a $4k$-dimensional geometric Poincaré complex X with $\pi_1(X) = \{1\}$ is homotopy equivalent to a differentiable manifold if and only if there exists a vector bundle reduction $\widetilde{\nu}_X : X \to BO(k)$ of $\nu_X : X \to BTOP(k)$ with

$$\begin{aligned}\sigma_*(f,b) &= \frac{1}{8}(\text{signature}(X) - \langle \mathcal{L}(-\widetilde{\nu}_X), [X]\rangle) \\ &= 0 \in L_{4k}(\mathbb{Z}) = \mathbb{Z}\end{aligned}$$

  with $\mathcal{L}(-\widetilde{\nu_X}) \in H^{4*}(X;\mathbb{Q})$ the Hirzebruch $\mathcal{L}$-genus.
- For those familiar with symmetric L-theory: for any n-dimensional geometric Poincaré complex X $8s(X) = 0 \in \mathcal{S}_n(X)$ if and only if the symmetric signature $\sigma^*(X) = (C(\widetilde{X}), \phi) \in L^n(\mathbb{Z}[\pi_1(X)])$ is in the image of the assembly map $A : H_n(X; \mathbf{L}^\bullet(\mathbb{Z})) \to L^n(\mathbb{Z}[\pi_1(X)])$.

Fibred products

- The construction of the algebraic L-theory assembly map A involves chain complex analogues of fibred products.
- The **fibred product** of maps $f : M \to X$, $g : N \to X$ is

$$M \times_X N \ = \ \{(x,y) \in M \times N \,|\, f(x) = g(y) \in X\} \subseteq M \times N \ ,$$

$$\begin{array}{ccc} M \times_X N & \longrightarrow & M \\ \Big\downarrow & & \Big\downarrow f \\ N & \xrightarrow{\ g\ } & X \end{array}$$

- **Example 1** If $f : M \to X$, $g : N \to X$ are the inclusions of subspaces $M, N \subseteq X$ the fibred product is the intersection

$$M \times_X N \ = \ M \cap N \subseteq X \ .$$

- **Example 2** For a regular covering $g : N = \widetilde{X} \to X$ with group of covering translations π the **pullback covering** of M is

$$\widetilde{M} \ = \ f^* \widetilde{X} \ = \ M \times_X \widetilde{X} \to M \ ; \ (x,y) \mapsto x \ .$$

The assembly map in $\mathbb{Z}_2$-equivariant homotopy theory

- For any map $f : M \to X$ let $\mathbb{Z}_2$ act on $M \times_X M$ by

$$T \ : \ M \times_X M \to M \times_X M \ ; \ (x, y) \mapsto (y, x) \ .$$

- The **assembly** map with respect to any regular covering $\widetilde{X} \to X$ with group of covering translations π is the $\mathbb{Z}_2$-equivariant map

$$A \ : \ M \times_X M \to \widetilde{M} \times_\pi \widetilde{M} \ ; \ (x, y) \mapsto [(x, z), (y, z)]$$

quotienting out the diagonal π-action on $\widetilde{M}$, using any

$$z \in p^{-1}(f(x)) = p^{-1}(f(y)) \subset \widetilde{X} \ .$$

- **Example** For $f = 1 : M = X \to X$

$$A \ = \ \Delta \ : \ X \times_X X \ = \ X \to \widetilde{X} \times_\pi \widetilde{X} \ ; \ x \mapsto [\widetilde{x}, \widetilde{x}] \ .$$

If X is an n-dimensional geometric Poincaré complex the Poincaré duality is the assembly $A[X] \in H_n(\widetilde{X} \times_\pi \widetilde{X})$ of the fundamental class $[X] \in H_n(X)$.

The combinatorial method

- The algebraic surgery exact sequence of the polyhedron of a **simplicial complex** X was described entirely combinatorially using the $(\mathbb{Z}, X)$-module category with chain duality, in:
  (i) (R.+Weiss) **Chain complexes and assembly**, Math. Z., 1990
  (ii) (R.) **Algebraic L-theory and topological manifolds**, CUP, 1992
  (iii) (R.) **Singularities, double points, controlled topology and chain duality**, Doc. Math., 1999.
- Chain duality: the dual of an object is a chain complex, as in Verdier duality.
- Key observation: for a reasonable (e.g. simplicial) map $f : M \to X$ the chain complex $C(M)$ is "X-controlled", and a homology class $\phi \in H_n(M \times_X M)$ can be regarded as a chain homotopy class of "X-controlled" chain maps $\phi : C(M)^{n-*} \to C(M)$. The assembly $A(\phi) \in H_n(\widetilde{M} \times_{\pi_1(X)} \widetilde{M})$ is a chain homotopy class of $\mathbb{Z}[\pi_1(X)]$-module chain maps $A(\phi) : C(\widetilde{M})^{n-*} \to C(\widetilde{M})$.

Categories with chain duality I.

- The assembly maps

$$A \;:\; H_*(X;\mathbf{L}_\bullet(\mathbb{Z})) \to L_*(\mathbb{Z}[\pi_1(X)])$$

  are constructed using the L-theory of additive categories with chain duality.

- A **symmetric product** on $\mathcal{A}$ is a covariant additive functor

$$\otimes \;:\; \mathcal{A} \times \mathcal{A} \to \{\mathbb{Z}\text{-modules}\} \;;\; (M,N) \mapsto M \otimes_{\mathcal{A}} N$$

  with natural isomorphisms

$$T_{M,N} \;:\; M \otimes_{\mathcal{A}} N \to N \otimes_{\mathcal{A}} M$$

  such that $T_{N,M} = (T_{M,N})^{-1}$.

- Let $B(\mathcal{A})$ be the additive category of finite chain complexes in $\mathcal{A}$. For C, D in $B(\mathcal{A})$ can define a $\mathbb{Z}$-module chain complex $C \otimes_{\mathcal{A}} D$ with an isomorphism $T_{C,D} : C \otimes_{\mathcal{A}} D \to D \otimes_{\mathcal{A}} C$.

Categories with chain duality II.

- A **chain duality** on an additive category $\mathcal{A}$ with a symmetric product $(\otimes, T)$ is a contravariant functor

$$* \ : \ B(\mathcal{A}) \to B(\mathcal{A}) \ ; \ C \mapsto C^{-*}$$

with a natural $\mathbb{Z}$-module chain map

$$C \otimes_{\mathcal{A}} D \to \mathrm{Hom}_{\mathcal{A}}(C^{-*}, D)$$

inducing isomorphisms

$$H_n(C \otimes_{\mathcal{A}} D) \ \cong \ H_0(\mathrm{Hom}_{\mathcal{A}}(C^{n-*}, D)) \ (n \in \mathbb{Z}) \ .$$

- An element $\phi \in H_n(C \otimes_{\mathcal{A}} D)$ is a chain homotopy class of chain maps $\phi : C^{n-*} \to D$.

The quadratic Q-groups

- Let W be the standard free $\mathbb{Z}[\mathbb{Z}_2]$-module resolution of $\mathbb{Z}$

$$W \; : \; \dots \longrightarrow \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1+T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}_2]$$

- The **quadratic Q-groups** of a finite chain complex C in $\mathcal{A}$ are

$$Q_n(C) \; = \; H_n(\mathbb{Z}_2; C \otimes_{\mathcal{A}} C) \; = \; H_n(W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} (C \otimes_{\mathcal{A}} C))$$

with $\mathbb{Z}_2$ acting by the involution

$$T \; = \; T_{C,C} \; : \; C \otimes_{\mathcal{A}} C \to C \otimes_{\mathcal{A}} C \; .$$

- An element $\psi \in Q_n(C)$ is represented by a collection of chains

$$\{\psi_s \in (C \otimes_{\mathcal{A}} C)_{n-s} \,|\, s \geqslant 0\}$$

such that

$$d(\psi_s) \; = \; \psi_{s+1} + (-)^{s+1} T(\psi_{s+1}) \in (C \otimes_{\mathcal{A}} C)_{n-s-1} \; .$$

- $\phi = (1+T)\psi_0 \in H_n(C \otimes_{\mathcal{A}} C) = H_0(\mathrm{Hom}_{\mathcal{A}}(C^{n-*}, C))$ is a chain homotopy class of chain maps $\phi : C^{n-*} \to C$.

The quadratic L-groups of $\mathcal{A}$

- An n-**dimensional quadratic Poincaré complex** (C, ψ) is a finite chain complex C in $\mathcal{A}$ with $\psi \in Q_n(C)$ such that $(1+T)\psi_0 : C^{n-*} \to C$ is a Poincaré duality chain equivalence.
- There is a corresponding notion of an $(n+1)$-**dimensional quadratic Poincaré pair** $(f : C \to D, (\delta\psi, \psi))$ with a Poincaré-Lefschetz duality chain equivalence
$$\mathcal{C}(f)^{n+1-*} \simeq D ,$$
with $\mathcal{C}(f)$ the algebraic mapping cone of f.
- The n-dimensional quadratic Poincaré complexes (C, ψ), (C', ψ') are **cobordant** if there exists an $(n+1)$-dimensional quadratic Poincaré pair $((f\ \ f') : C \oplus C' \to D, (\delta\psi, \psi \oplus -\psi'))$.
- The **quadratic** L-**group** $L_n(\mathcal{A})$ is the cobordism group of n-dimensional quadratic Poincaré complexes in $\mathcal{A}$.

The quadratic L-groups of R

- For any ring R with involution

$$R \to R \; ; \; r \mapsto \bar{r}$$

let $\mathcal{A}(R)$ be the additive category of f.g. free left R-modules, with symmetric product, transposition and duality given by

$$\otimes \;:\; \mathcal{A}(R) \times \mathcal{A}(R) \to \{\mathbb{Z}\text{-modules}\} \; ;$$

$$(K, L) \mapsto K \otimes_R L = K \otimes_{\mathbb{Z}} L / \{rx \otimes y - x \otimes \bar{r}y\} \; ,$$

$$T \;:\; K \otimes_R L \to L \otimes_R K \; ; \; x \otimes y \mapsto y \otimes x \; ,$$

$$K^* \;=\; \mathrm{Hom}_R(K, R) \; , \; R \times K^* \to K^* \; ; \; (r, f) \mapsto (x \mapsto f(x).\bar{r}) \; ,$$

$$K \otimes_R L \xrightarrow{\;\cong\;} \mathrm{Hom}_R(K^*, L) \; ; \; x \otimes y \mapsto (f \mapsto \overline{f(x)}.y) \; .$$

- **Proposition** The Wall surgery obstruction groups of R are the quadratic L-groups of $\mathcal{A}(R)$

$$L_*(R) \;=\; L_*(\mathcal{A}(R)) \; .$$

The $(\mathbb{Z}, X)$-module category

- Let X be a finite simplicial complex.
- The $(\mathbb{Z}, X)$-**module category** $\mathcal{A}(\mathbb{Z}, X)$ has objects f.g. free $\mathbb{Z}$-modules K with a direct sum decomposition
$$K = \sum_{\sigma \in X} K(\sigma) \ .$$
- The morphisms in $\mathcal{A}(\mathbb{Z}, X)$ are the $\mathbb{Z}$-module morphisms $f : K \to L$ such that
$$f(K(\sigma)) \subseteq \sum_{\tau \geqslant \sigma} L(\tau) \ (\sigma \in X) \ .$$
- f is an isomorphism in $\mathcal{A}(\mathbb{Z}, X)$ if and only if each diagonal component $f(\sigma, \sigma) : K(\sigma) \to L(\sigma)$ $(\sigma \in X)$ is an isomorphism in $\mathcal{A}(\mathbb{Z})$.
- $\mathcal{A}(\mathbb{Z}, X)$ has product, transposition and chain duality
$$K \otimes_{\mathcal{A}(\mathbb{Z},X)} L = \sum_{\sigma,\tau \in X, \sigma \cap \tau \neq \emptyset} K(\sigma) \otimes_{\mathbb{Z}} L(\tau) \ , \ \ T_{K,L}(x \otimes y) = y \otimes x \ ,$$
$$K^{-*}(\sigma)_{-r} = \sum_{\tau \geqslant \sigma} K(\tau)^* \text{ if } \sigma \in X^{(r)} \ .$$

Dissections

- **Definition** Let X be a finite simplicial complex. An X-**dissection** of a space M is a collection of subspaces $\{M(\sigma) \subseteq M \mid \sigma \in X\}$ such that

$$M = \bigcup_{\sigma \in X} M(\sigma) \ , \ M(\sigma) \cap M(\tau) = \begin{cases} M(\sigma \cup \tau) & \text{if } \sigma \cup \tau \in X \\ \emptyset & \text{otherwise .} \end{cases}$$

Write $\partial M(\sigma) = \bigcup_{\tau > \sigma} M(\tau) \subseteq M(\sigma)$.

- The **dual cells** of the barycentric subdivision X' define an X-dissection $\{D(\sigma, X) \subseteq X' \mid \sigma \in X\}$ of X', with

$$D(\sigma, X) = \{\widehat{\sigma}_0 \widehat{\sigma}_1 \dots \widehat{\sigma}_r \mid \sigma \leqslant \sigma_0 < \sigma_1 < \dots < \sigma_r\} \ ,$$
$$\partial D(\sigma, X) = \{\widehat{\sigma}_0 \widehat{\sigma}_1 \dots \widehat{\sigma}_r \mid \sigma < \sigma_0 < \sigma_1 < \dots < \sigma_r\} \ .$$

The dual cells $D(\sigma, X)$ are contractible. X is an n-dimensional homology manifold if and only if each $(D(\sigma, X), \partial D(\sigma, X))$ is an $(n - |\sigma|)$-dimensional Poincaré pair, if and only if

$$H_*(\partial D(\sigma, X)) \cong H_*(S^{n-|\sigma|-1}) \ .$$

Fibred products in $\mathcal{A}(\mathbb{Z},X)$

- **Proposition** (i) For any map $f : M \to X'$ the inverse images

$$M(\sigma) \;=\; f^{-1}D(\sigma,X) \subseteq M$$

define an X-dissection $\{M(\sigma)\,|\,\sigma \in X\}$ of M with

$$\partial M(\sigma) \;=\; f^{-1}\partial D(\sigma,X)\ .$$

- (ii) For a finite simplicial complex M and a simplicial map $f : M \to X'$ the simplicial chain complex $C(M)$ is a finite chain complex in $\mathcal{A}(\mathbb{Z},X)$, with

$$C(M)(\sigma) \;=\; C(M(\sigma),\partial M(\sigma))\ (\sigma \in X)\ .$$

- (iii) If $f : M \to X'$, $g : N \to X'$ are two simplicial maps as in (ii) then up to chain equivalence in $\mathcal{A}(\mathbb{Z})$

$$C(M \times_X N) \;=\; C(M) \otimes_{\mathcal{A}(\mathbb{Z},X)} C(N)\ .$$

A homology class $\phi \in H_n(M \times_X N)$ is a chain homotopy class of chain maps $\phi : C(M)^{n-*} \to C(N)$ in $\mathcal{A}(\mathbb{Z},X)$.

The algebraic L-theory assembly map

- Let $p : \widetilde{X} \to X$ be the universal cover of the simplicial complex X. $\widetilde{X}$ is a simplicial complex with a free $\pi_1(X)$-action.
- **Proposition** (i) The assembly functor of additive categories with chain duality

$$A : \mathcal{A}(\mathbb{Z}, X) \to \mathcal{A}(\mathbb{Z}[\pi_1(X)]) \; ; \; K = \sum_{\sigma \in X} K(\sigma) \mapsto A(K) = \sum_{\widetilde{\sigma} \in \widetilde{X}} K(p(\widetilde{\sigma}))$$

  induces the assembly maps in quadratic L-theory

$$A : L_*(\mathcal{A}(\mathbb{Z}, X)) = H_*(X; \mathbf{L}_\bullet(\mathbb{Z})) \to L_*(\mathcal{A}(\mathbb{Z}[\pi_1(X)])) = L_*(\mathbb{Z}[\pi_1(X)]) \; .$$

- (ii) The relative group $\mathcal{S}_n(X)$ in the algebraic surgery exact sequence

$$\cdots \to H_n(X; \mathbf{L}_\bullet(\mathbb{Z})) \xrightarrow{A} L_n(\mathbb{Z}[\pi_1(X)]) \to \mathcal{S}_n(X) \to H_{n-1}(X; \mathbf{L}_\bullet(\mathbb{Z})) \to \dots$$

  is the cobordism group of $(n-1)$-dimensional quadratic Poincaré complexes (C, ψ) in $\mathcal{A}(\mathbb{Z}, X)$ such that the assembly $A(C)$ is a contractible chain complex in $\mathcal{A}(\mathbb{Z}[\pi_1(X)])$.

The $(\mathbb{Z}, X)$-interpretation of $H_*(X)$

- **Proposition** (i) A homology class

$$[X] \in H_n(X) \ = \ H_0(\mathrm{Hom}_{\mathcal{A}(\mathbb{Z},X)}(C(X')^{n-*}, C(X')))$$

is a chain homotopy class of chain maps in $\mathcal{A}(\mathbb{Z}, X)$

$$\phi \ = \ [X] \cap - \ : \ C(X')^{n-*} \to C(X')$$

with diagonal components

$$\begin{aligned}\phi(\sigma,\sigma) \ = \ [X]_{\widehat{\sigma}} \ : \ C(X')^{n-*} \ &= \ C(D(\sigma, X))^{n-|\sigma|-*} \\ \to C(X')(\sigma) \ &= \ C(D(\sigma, X), \partial D(\sigma, X)) \ (\sigma \in X) \ .\end{aligned}$$

- (ii) ϕ is a chain equivalence in $\mathcal{A}(\mathbb{Z}, X)$ if and only if each $\phi(\sigma,\sigma)$ is a chain equivalence in $\mathcal{A}(\mathbb{Z})$, if and only if X is an n-dimensional homology manifold.
- (iii) The assembly $A(\phi) : C(\widetilde{X}')^{n-*} \to C(\widetilde{X'})$ is a chain equivalence in $\mathcal{A}(\mathbb{Z}[\pi_1(X)])$ if and only if X is an n-dimensional geometric Poincaré complex.

The total surgery obstruction

- A simplicial n-dimensional geometric Poincaré complex X determines an $(n-1)$-dimensional quadratic Poincaré complex (C,ψ) in $\mathcal{A}(\mathbb{Z},X)$:

$$C \;=\; \mathcal{C}(\phi : C(X')^{n-*} \to C(X'))_{*+1}\ ,$$

$$\psi \in Q_{n-1}(C) \;=\; H_{n-1}(W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} (C \otimes_{\mathcal{A}(\mathbb{Z},X)} C))\ ,$$

$$(1+T)\psi_0 \;=\; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\ :$$

$$C^{n-1-r} \;=\; C(X')^{n-r} \oplus C(X')_{r+1} \to C_r \;=\; C(X')_{r+1} \oplus C(X')^{n-r}\ .$$

- The assembly $A(C) = \mathcal{C}(A(\phi) : C(\widetilde{X}')^{n-*} \to C(\widetilde{X}'))_{*+1}$ is a contractible finite chain complex in $\mathcal{A}(\mathbb{Z}[\pi_1(X)])$, being the algebraic mapping cone of the Poincaré duality chain equivalence

$$A(\phi) \;=\; [X] \cap - \;:\; C(\widetilde{X}')^{n-*} \to C(\widetilde{X}')\ .$$

- The **total surgery obstruction** of X is defined by

$$s(X) = (C,\psi) \in \mathcal{S}_n(X)\ .$$

What next?

- In an ideal world, the algebraic surgery exact sequence would be defined for any space X using sheaves over X of chain complexes with quadratic structure. The total surgery obstruction $s(X) \in \mathcal{S}_n(X)$ would be defined for any space X with n-dimensional Poincaré duality, measuring the failure of the morphisms

$$[X]_x \ : \ H^{n-*}(\{x\}) = H_*(\mathbb{R}^n, \mathbb{R}^n \backslash \{0\}) \to H_*(X, X \backslash \{x\}) \ (x \in X)$$

  to be isomorphisms in a homotopy invariant way. Would be better for the version of the total surgery obstruction appropriate for the Quinn resolution obstruction of *ANR* homology manifolds. The paper (R.+Weiss) **On the construction and topological invariance of the Pontryagin classes**, Arxiv 0901.0819 + Geometriae Dedicata, 2009 goes some way towards a sheaf construction.
- The construction of $s(X) \in \mathcal{S}_n(X)$ using fibrewise homotopy theory, building on:
  Crabb + R. **The geometric Hopf invariant and double points** Arxiv 1002.2907 + J. of Fixed Point Theory and Applications 7 (2010).