THE GEOMETRIC HOPF INVARIANT

Michael Crabb (Aberdeen)
Andrew Ranicki (Edinburgh)

- When is a k-stable map $F : \Sigma^k X \to \Sigma^k Y$ homotopic to the k-fold suspension $\Sigma^k F_0$ of an unstable map $F_0 : X \to Y$?
- The geometric Hopf invariant is the stable $\mathbb{Z}_2$-equivariant map
$$h_\infty(F) = (F \wedge F)\Delta_X - \Delta_Y F : X \to Y \wedge Y$$
with $\Delta_X(x) = (x, x)$, $T(x) = x$, $T(y_1, y_2) = (y_2, y_1)$.
- *The stable $\mathbb{Z}_2$-equivariant homotopy class of $h_\infty(F)$ is the primary obstruction to the k-fold desuspension of F.*
- Need non-simply-connected $h_\infty(F)$ for surgery theory.

The Hopf invariant H, the suspension map E and the EHP sequence

- (Hopf 1931) Isomorphism $H : \pi_3(S^2) \cong \mathbb{Z}$ via linking numbers of $S^1 \sqcup S^1 \hookrightarrow S^3$.
- (Freudenthal 1937) Suspension map for pointed space X

  $$E \ : \ \pi_n(X) \to \pi_{n+1}(\Sigma X)\,;\ (f : S^n \to X) \mapsto (\Sigma f : S^{n+1} \to \Sigma X)\,.$$

  (E for *Einhängung*). If X is $(m-1)$-connected then E is an isomorphism for $n \leqslant 2m-2$ and surjective for $n = 2m-1$.
- (G.W.Whitehead 1950) EHP exact sequence

  $$\cdots \longrightarrow \pi_n(X) \xrightarrow{E} \pi_{n+1}(\Sigma X) \xrightarrow{H} \pi_n(X \wedge X) \xrightarrow{P} \pi_{n-1}(X) \longrightarrow \cdots$$

  for any $(m-1)$-connected space X, with $n \leqslant 3m-2$.
  For $X = S^m$, $n = 2m$ H is a Hopf invariant map

  $$H \ : \ \pi_{2m+1}(S^{m+1}) \to \pi_{2m}(S^m \wedge S^m) = \mathbb{Z}\ .$$

k-fold desuspension = compression into $X \subset \Omega^k\Sigma^k X$

- For any connected pointed space X and $k \geqslant 1$ the k-fold suspension map is induced by the inclusion $X \subset \Omega^k\Sigma^k X$

$$E^k : \pi_n(X) \xrightarrow{E} \pi_{n+1}(\Sigma X) \xrightarrow{E} \pi_{n+2}(\Sigma^2 X) \xrightarrow{E} \cdots \xrightarrow{E} \pi_{n+k}(\Sigma^k X) = \pi_n(\Omega^k\Sigma^k X) .$$

- A map $F : \Sigma^k Y \to \Sigma^k X$ is homotopic to $\Sigma^k F_0$ for $F_0 : Y \to X$ if and only if the adjoint map

$$\mathrm{adj}(F) \ : \ Y \to \Omega^k\Sigma^k X \ ; \ y \mapsto (s \mapsto F(s,y))$$

can be factored up to homotopy through $X \subset \Omega^k\Sigma^k X$.

The combinatorial models for $\Omega^k \Sigma^k X$

- Theorem (James 1955) For any pointed space X map

  $$X \times X \to \Omega\Sigma X \ .$$

  For connected X a stable homotopy decomposition

  $$\Omega\Sigma X \ \simeq_s \bigvee_{j=1}^{\infty} (\bigwedge_j X) \ .$$

- (Dyer-Lashof 1962) For any pointed space X and $k \geqslant 1$ map

  $$S^{k-1} \times_{\mathbb{Z}_2} (X \times X) \to \Omega^k \Sigma^k X \ .$$

  (Snaith 1974) Stable homotopy decomposition

  $$\Omega^\infty \Sigma^\infty X \ \simeq_s \bigvee_{j=1}^{\infty} (E\Sigma_j)^+ \wedge_{\Sigma_j} (\bigwedge_j X) \ \ (k = \infty)$$

  for connected X. Also $\Omega^k \Sigma^k X$ with $1 \leqslant k < \infty$ (May 1975). Also for disconnected $X = Y^+$ (Barratt, Quillen, 1970's).

The stable $\mathbb{Z}_2$-equivariant homotopy groups

- Given pointed spaces X, Y let $[X, Y]$ be the set of homotopy classes of maps $X \to Y$. The stable homotopy group is

$$\{X; Y\} = \varinjlim_{V} [V^\infty \wedge X, V^\infty \wedge Y]$$

with V running over finite-dimensional real vector spaces, and $V^\infty = V \cup \{\infty\}$ the one-point compactification.

- Given pointed $\mathbb{Z}_2$-spaces X, Y let $[X, Y]_{\mathbb{Z}_2}$ be the set of homotopy classes of maps $X \to Y$. The stable $\mathbb{Z}_2$-equivariant homotopy group is

$$\{X; Y\}_{\mathbb{Z}_2} = \varinjlim_{U} \varinjlim_{V} [U^\infty \wedge LV^\infty \wedge X, U^\infty \wedge LV^\infty \wedge Y]_{\mathbb{Z}_2}$$

where $LV = V$ with the $\mathbb{Z}_2$-action

$$T : LV \to LV ; v \mapsto -v .$$

The quadratic construction

- The quadratic construction on a pointed space X is defined for any inner product space V to be the pointed space

$$Q_V(X) \ = \ S(LV)^+ \wedge_{\mathbb{Z}_2} (X \wedge X)$$

with $S(LV) = \{v \in LV \mid \|v\| = 1\}$, $S(LV)^+ = S(LV) \cup \{*\}$,

$$T \ : \ X \wedge X \to X \wedge X \ ; \ (x, y) \mapsto (y, x) \ .$$

The projection $\widetilde{Q}_V(X) = S(LV)^+ \wedge (X \wedge X) \to Q_V(X)$ is a double cover away from the base point. For $1 \leqslant k \leqslant \infty$ write

$$Q_k(X) \ = \ Q_{\mathbb{R}^k}(X) \ , \ \widetilde{Q}_k(X) \ = \ \widetilde{Q}_{\mathbb{R}^k}(X) \ .$$

In particular, $Q_\infty(X) = \varinjlim_k Q_k(X) = Q_{\mathbb{R}^\infty}(X)$.

- Example $Q_0(X) = \{\text{pt.}\}$, $Q_1(X) \ = \ X \wedge X$.
- Example $Q_k(S^0) = (S^{k-1})^+/\mathbb{Z}_2 = (\mathbb{RP}^{k-1})^+$.

The relative difference

- For any inner product space V there is a cofibration

$$S^0 = 0^\infty \to V^\infty \to V^\infty/0^\infty = \Sigma S(V)^+ \to S^1 \to \dots$$

with a homeomorphism

$$\Sigma S(V)^+ \to V^\infty/0^\infty \ ; \ (t,u) \mapsto [t,u] \ = \ \frac{tu}{1-t} \ .$$

- Given maps $p,q : V^\infty \wedge X \to Y$ such that

$$p(0,x) \ = \ q(0,x) \in Y \ \ (x \in X)$$

define the relative difference map

$$\delta(p,q) \ : \ \Sigma S(V)^+ \wedge X \to Y \ ;$$
$$(t,u,x) \mapsto \begin{cases} p([1-2t,u],x) & \text{if } 0 \leqslant t \leqslant 1/2 \\ q([2t-1,u],x) & \text{if } 1/2 \leqslant t \leqslant 1 \ . \end{cases}$$

The homotopy class of $\delta(p,q)$ is the obstruction to the existence of a rel $0^\infty \wedge X$ homotopy $p \simeq q : V^\infty \wedge X \to Y$.

$\mathbb{Z}_2$-equivariant stable homotopy theory = fixed-point + fixed-point-free

- Theorem For any pointed spaces X, Y there is an exact sequence of abelian groups

$$0 \to \{X; Q_\infty(Y)\} \xrightarrow{1+T} \{X; Y \wedge Y\}_{\mathbb{Z}_2} \xrightarrow{\rho} \{X; Y\} \to 0$$

with

$$\{X; Q_\infty(Y)\} = \varinjlim_V [\Sigma S(LV)^+ \wedge X, LV^\infty \wedge Y \wedge Y]_{\mathbb{Z}_2} \text{ (S-duality)} .$$

- ρ is given by the $\mathbb{Z}_2$-fixed points, with nonadditive section

$$\sigma : \{X; Y\} \to \{X; Y \wedge Y\}_{\mathbb{Z}_2} ; F \mapsto (F \wedge F)\Delta_X .$$

- The injection $1 + T$ is induced by projection $(S^\infty)^+ \to 0^\infty$

$$1 + T : \{X; Q_\infty(Y)\} = \{X; \widetilde{Q}_\infty(Y)\}_{\mathbb{Z}_2} \to \{X; Y \wedge Y\}_{\mathbb{Z}_2} ,$$

with nonadditive projection

$$h : \{X; Y \wedge Y\}_{\mathbb{Z}_2} \to \{X; Q_\infty(Y)\} ; G \mapsto \delta(\sigma\rho(G), G) .$$

The stable geometric Hopf invariant $h_\infty(F)$

- The stable geometric Hopf invariant of a stable map $F : \Sigma^k X \to \Sigma^k Y$ is

$$h_\infty(F) \ = \ (F \wedge F)\Delta_X - \Delta_Y F \ = \ \delta(\Delta_Y F, (F \wedge F)\Delta_X)$$
$$\in \ker(\rho : \{X; Y \wedge Y\}_{\mathbb{Z}_2} \to \{X; Y\})$$
$$= \ \mathrm{im}(1 + T : \{X; Q_\infty(Y)\} \hookrightarrow \{X; Y \wedge Y\}_{\mathbb{Z}_2}) \ .$$

- Proposition (i) The function

$$h_\infty : \{X; Y\} \to \{X; Q_\infty(Y)\} \ ; \ F \mapsto h_\infty(F)$$

is nonadditive, being quadratic in nature:

$$h_\infty(F + G) = h_\infty(F) + h_\infty(G) + (F \wedge G)\Delta_X$$

(ii) If $F \in \mathrm{im}([X, Y] \to \{X; Y\})$ then $h_\infty(F) = 0$.
- Example If $X = Y = S^0$, $k = 1$, $d \in \mathbb{Z}$, $F = d : \Sigma X = S^1 \to \Sigma Y = S^1$ then

$$h_\infty(F) = d(d-1)/2 \in \{0^\infty; Q_\infty(0^\infty)\} = \mathbb{Z}$$

Double points

- The ordered double point set of a map $f : M \to N$ is the free $\mathbb{Z}_2$-set

$$\widetilde{D}_2(f) = \{(x, y) \,|\, x \neq y \in M,\ f(x) = f(y) = N\}$$

with $\mathbb{Z}_2$ acting by $T(x, y) = (y, x)$.
- The unordered double point set is

$$D_2(f) \ = \ \widetilde{D}_2(f)/\mathbb{Z}_2 \ .$$

- f is an embedding if and only if $D_2(f) = \emptyset$.
- *The geometric Hopf invariant is the primary homotopy theoretic method of capturing $D_2(f)$.*

Immersions of spaces

- Definition A map $f : M \to N$ is an immersion of spaces if there exists an open embedding of the type

$$g = (e, f) : V \times M \hookrightarrow V \times N \; ; \; (v, x) \mapsto (e(v, x), f(x))$$

with V finite dimensional, for some map $e : V \times M \to V$, so that there is defined a commutative diagram

$$\begin{array}{ccc} V \times M & \stackrel{g}{\hookrightarrow} & V \times N \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & N \end{array}$$

The Umkehr map of g is the stable map

$$F : (V \times N)^\infty = V^\infty \wedge N^\infty \to (V \times M)^\infty = V^\infty \wedge M^\infty \; ;$$

$$(w, y) \mapsto \begin{cases} (v, x) & \text{if } (w, y) = g(v, x) \\ \infty & \text{if } (w, y) \notin \text{im}(g) \; . \end{cases}$$

- Example A codimension 0 immersion of manifolds.

Capturing double points with homotopy theory

- Let $f : M \to N$ be an immersion of spaces, with embedding $g = (e, f) : V \times M \hookrightarrow V \times N$. The $\mathbb{Z}_2$-equivariant product embedding

$$g \times g \ : \ V \times V \times M \times M \hookrightarrow V \times V \times N \times N$$

restricts to a $\mathbb{Z}_2$-equivariant embedding

$$g \times g| \ : \ V \times V \times D_2(f) \hookrightarrow V \times V \times N$$

with $\mathbb{Z}_2$-equivariant Umkehr map

$$G \ : \ V^\infty \wedge V^\infty \wedge N^\infty \to V^\infty \wedge V^\infty \wedge D_2(f)^+ \ .$$

- Define also the $\mathbb{Z}_2$-equivariant map

$$H \ : \ D_2(f)^+ \to \widetilde{Q}_V(M^\infty) \ = \ S(LV)^+ \wedge (M^\infty \wedge M^\infty) \ ;$$

$$(x, y) \mapsto \Big(\frac{e(0,x) - e(0,y)}{\|e(0,x) - e(0,y)\|}, x, y\Big) \ .$$

Double points of immersions of manifolds

- The double point set $D_2(f)$ of a generic immersion $f : M^m \looparrowright N^n$ with normal bundle $\nu_f : M \to BO(n-m)$ is a $(2m-n)$-dimensional manifold. For $k \geqslant 2m-n+1$ there exists a map $e : M \to \mathbb{R}^k$ such that

$$g = (e, f) : M \hookrightarrow \mathbb{R}^k \times N \; ; \; x \mapsto (e(x), f(x))$$

  is an embedding with normal bundle

$$\nu_g = \nu_f \oplus \epsilon^k : M \to BO(n-m+k) .$$

- By the tubular neighbourhood theorem can approximate the product immersion $1 \times f : \mathbb{R}^k \times M \looparrowright \mathbb{R}^k \times N$ by an embedding

$$\overline{g} = (\overline{e}, \overline{f}) : \mathbb{R}^k \times E(\nu_f) \hookrightarrow \mathbb{R}^k \times N$$

  extending g, with $\overline{f} : E(\nu_f) \looparrowright N$ a codimension 0 immersion. $E(\nu_f)^\infty = T(\nu_f) =$ Thom space. Write Umkehr map of $\overline{g}$ as

$$F : \Sigma^k N^\infty \to \Sigma^k T(\nu_f) .$$

The Double Point Theorem

- Theorem If $f : M^m \looparrowright N^n$ is an immersion of manifolds with Umkehr map $F : \Sigma^k N^\infty \to \Sigma^k T(\nu_f)$ (k large) then

$$\begin{aligned} h_\infty(F) &= HG \\ &\in \ker\big(\rho : \{N^\infty; T(\nu_f) \wedge T(\nu_f)\}_{\mathbb{Z}_2} \to \{N^\infty; T(\nu_f)\}\big) \\ &\qquad = \operatorname{im}\big(\{N^\infty; Q_\infty(T(\nu_f))\} \hookrightarrow \{N^\infty; T(\nu_f) \wedge T(\nu_f)\}_{\mathbb{Z}_2}\big) \end{aligned}$$

  is a factorization of $h_\infty(F)$ through $D_2(f)^+$, with

$$N^\infty \xrightarrow{G} T(\nu_f \times \nu_f|_{D_2(f)}) \xrightarrow{H} T(\nu_f \times \nu_f) = T(\nu_f) \wedge T(\nu_f) \ .$$

- If $f : M \looparrowright N$ is regular homotopic to an embedding $f_0 : M \hookrightarrow N$ with Umkehr map $F_0 : N^\infty \to T(\nu_f)$ then F is stably homotopic to F_0, and $h_\infty(F)$ is stably null-homotopic.

The difference of diagonals

- For any space X the diagonal map
$$\Delta_X \ : \ X \to X \wedge X \ ; \ x \mapsto (x,x)$$
is $\mathbb{Z}_2$-equivariant.
- For any f.d. inner product space V define $\mathbb{Z}_2$-equivariant homeomorphism
$$\kappa_V \ : \ LV^\infty \wedge V^\infty \to V^\infty \wedge V^\infty \ ; \ (x,y) \mapsto (x+y, -x+y) \ .$$
- Given a map $F : V^\infty \wedge X \to V^\infty \wedge Y$ define the <u>noncommutative</u> square of $\mathbb{Z}_2$-equivariant maps

$$\begin{array}{ccc} LV^\infty \wedge V^\infty \wedge X & \xrightarrow{1 \wedge \Delta_X} & LV^\infty \wedge V^\infty \wedge X \wedge X \\ {\scriptstyle 1 \wedge F}\big\downarrow & & \big\downarrow {\scriptstyle (\kappa_V^{-1} \wedge 1)(F \wedge F)(\kappa_V \wedge 1)} \\ LV^\infty \wedge V^\infty \wedge Y & \xrightarrow{1 \wedge \Delta_Y} & LV^\infty \wedge V^\infty \wedge Y \wedge Y \end{array}$$

The unstable geometric Hopf invariant $h_V(F)$

- Definition The unstable geometric Hopf invariant of a map $F : V^\infty \wedge X \to V^\infty \wedge Y$ is the $\mathbb{Z}_2$-equivariant relative difference map

$$h_V(F) = \delta(p,q) : \Sigma S(LV)^+ \wedge V^\infty \wedge X \to LV^\infty \wedge V^\infty \wedge Y \wedge Y$$

of the $\mathbb{Z}_2$-equivariant maps

$$p = (1 \wedge \Delta_Y)F \ , \ q = (\kappa_V^{-1} \wedge 1)(F \wedge F)(\kappa_V \wedge 1) : LV^\infty \wedge V^\infty \wedge X \to LV^\infty \wedge V^\infty \wedge Y \wedge Y$$

with $\Sigma S(LV)^+ = LV^\infty/0^\infty = (LV \backslash \{0\})^\infty$.
- The stable $\mathbb{Z}_2$-equivariant homotopy class of $h_V(F)$ depends only on the homotopy class of F, defining a function

$$h_V : [V^\infty \wedge X, V^\infty \wedge Y] \to \{\Sigma S(LV)^+ \wedge V^\infty \wedge X, LV^\infty \wedge V^\infty \wedge Y \wedge Y\}_{\mathbb{Z}_2} = \{X, Q_V(Y)\}$$

Some properties of $h_V(F)$

- Composition and addition formulae

$$h_V(GF) = (G \wedge G)h_V(F) + h_V(G)F \ ,$$
$$h_V(F + F') = h_V(F) + h_V(F') + (F \wedge F')\Delta \ .$$

- If $F \simeq 1_V \wedge F_0$ for some $F_0 : X \to Y$ then

$$h_V(F) = 0 \in \{X, Q_V(Y)\} \ .$$

- The Double Point Theorem has unstable version, with $h_V(F)$.
- The original Hopf invariant of a map

$$F : S^{2m+1} = \Sigma(S^{2m}) \to S^{m+1} = \Sigma(S^m)$$

is

$$H(F) = h_{\mathbb{R}}(F) \in \{S^{2m}, Q_{\mathbb{R}}(S^m)\} = \{S^{2m}, S^{2m}\} = \mathbb{Z} \ .$$

The universal example

- For any pointed space X evaluation defines a k-stable map

$$e \ : \ \Sigma^k(\Omega^k\Sigma^k X) \to \Sigma^k X \ ; \ (s,\omega) \mapsto \omega(s)$$

with $\mathrm{adj}(e) = 1 : \Omega^k\Sigma^k X \to \Omega^k\Sigma^k X$. The unstable geometric Hopf invariant of e defines a stable map

$$h_{\mathbb{R}^k}(e) \ : \ \Omega^k\Sigma^k X \to Q_k(X) = (S^{k-1})^+ \wedge_{\mathbb{Z}_2} (X \wedge X)$$

which is a stable splitting of the Dyer-Lashof map.

- For any k-stable map $F : \Sigma^k Y \to \Sigma^k X$ the stable homotopy class of the composite

$$h_{\mathbb{R}^k}(F) : Y \xrightarrow{\mathrm{adj}(F)} \Omega^k\Sigma^k X \xrightarrow{h_{\mathbb{R}^k}(e)} Q_k(X)$$

is the primary obstruction to a k-fold desuspension of F, i.e. to the compression of $\mathrm{adj}(F)$ into $X \subset \Omega^k\Sigma^k X$.