# NONCOMMUTATIVE LOCALIZATION IN ALGEBRA AND TOPOLOGY

Andrew Ranicki (Edinburgh)

http://www.maths.ed.ac.uk/~aar

Heidelberg, 17th December, 2008

## Noncommutative localization

- Localizations of noncommutative rings such as group rings $\mathbb{Z}[\pi]$ are rings with complicated properties in algebra and interesting applications to topology.
- The applications are to spaces $X$ with **infinite** fundamental group $\pi_1(X)$, e.g. amalgamated free products and *HNN* extensions, such as occur when $X$ is a knot or link complement.
- The surgery classification of high-dimensional manifolds and Poincaré complexes, finite domination, fibre bundles over $S^1$, open books, circle-valued Morse theory, Morse theory of closed 1-forms, rational Novikov homology, codimension 1 and 2 splitting, homology surgery, knots and links.
- **High-dimensional knot theory**, Springer (1998)
- Survey: e-print AT.0303046 in **Noncommutative localization in algebra and topology**, LMS Lecture Notes 330, Cambridge University Press (2006)

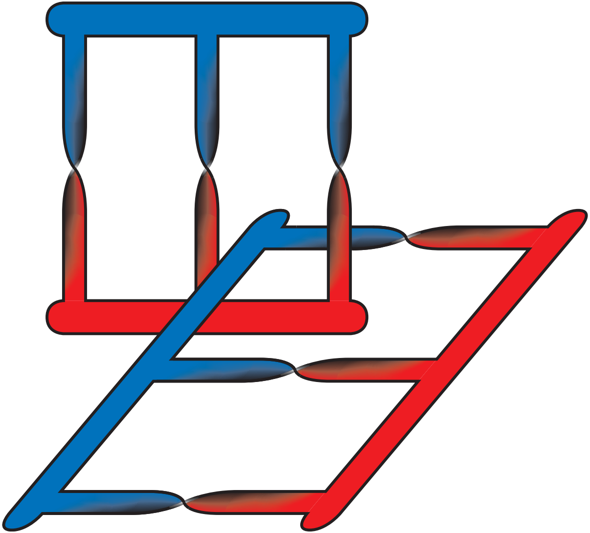
## The cobordism/concordance groups of boundary links

- An $n$-**dimensional** $\mu$-**component boundary link** is a link

$$\ell \ : \ \bigsqcup_{\mu} S^n \subset S^{n+2}$$

  such that there exists a $\mu$-component Seifert surface $M^{n+1} = \bigsqcup_{i=1}^{\mu} M_i \subset S^{n+2}$ with $\partial M = \ell(\bigsqcup_{\mu} S^n) \subset S^{n+2}$.
- Boundary condition equivalent to the existence of a surjection $\pi_1(S^{n+2} \backslash \ell(\sqcup_{\mu} S^n)) \to F_{\mu}$ sending the $\mu$ meridians to $\mu$ generators of the free group $F_{\mu}$ of rank $\mu$.
- Let $C_n(F_{\mu})$ be the cobordism group of $n$-dimensional $\mu$-component boundary links.
- A 1-component boundary link is a knot $k : S^n \subset S^{n+2}$, and $C_n(F_1) = C_n$ is the knot cobordism group.
- **Problem** Compute $C_n(F_{\mu})$ !

## A 2-component boundary link $\ell : S^1 \sqcup S^1 \subset S^3$



## Brief history of the knot cobordism groups $C_*$

- (Fox-Milnor 1957) Definition of $C_1$.
- (Kervaire 1966) Definition of $C_*$ for $* > 1$ and
$$C_{2*} \;=\; 0\;.$$
- (Levine 1969) $C_* = C_{*+4}$ for $* > 1$. Computation of $C_{2*+1}$ for $* > 0$, using Seifert forms over $\mathbb{Z}$, $S^{-1}\mathbb{Z} = \mathbb{Q}$ and signatures
$$C_{2*+1} \;=\; \bigoplus_{\infty}\mathbb{Z} \oplus \bigoplus_{\infty}\mathbb{Z}_2 \oplus \bigoplus_{\infty}\mathbb{Z}_4\;.$$
- (Kearton 1975) Expression of $C_{2*+1}$ for $* > 0$, using a commutative localization $S^{-1}\mathbb{Z}[z, z^{-1}]$ and $S^{-1}\mathbb{Z}[z, z^{-1}]/\mathbb{Z}[z, z^{-1}]$-valued Blanchfield forms.
- (Casson-Gordon 1976) $\ker(C_1 \to C_5) \neq 0$ using commutative localization.
- (Cochran-Orr-Teichner 2003) Near-computation of $C_1$, using noncommutative Ore localization of group rings and $L^2$-signatures.

## Brief history of the boundary link cobordism groups $C_*(F_\mu)$

- (Cappell-Shaneson 1980) Geometric expression of $C_*(F_\mu)$ for $* > 1$ as relative Γ-groups, and

$$C_{2*}(F_\mu) \;=\; 0 \;.$$

- (Duval 1984) Algebraic expression of $C_{2*+1}(F_\mu)$ for $* > 0$, using a noncommutative localization $\Sigma^{-1}\mathbb{Z}[F_\mu]$ and $\Sigma^{-1}\mathbb{Z}[F_\mu]/\mathbb{Z}[F_\mu]$-valued Blanchfield forms.
- (Ko 1989) Algebraic expression of $C_{2*+1}(F_\mu)$ for $* > 0$, using Seifert forms over $\mathbb{Z}[F_\mu]$.
- (Sheiham 2003) Computation of $C_{2*+1}(F_\mu)$ for $* > 0$, using noncommutative signatures

$$C_{2*+1}(F_\mu) \;=\; \bigoplus_{\infty}\mathbb{Z} \oplus \bigoplus_{\infty}\mathbb{Z}_2 \oplus \bigoplus_{\infty}\mathbb{Z}_4 \oplus \bigoplus_{\infty}\mathbb{Z}_8 \;.$$

- **Wishful thinking** Compute $C_1(F_\mu)$ for $\mu > 1$ using noncommutative localization.

## Alexander duality in $H_*$ and $H^*$ but not in $\pi_1$

- Want to investigate knotting properties of submanifolds $N^n \subset M^m$, especially in codimension $m - n = 2$, using the complement $P = M \backslash N$.
- **Alexander duality for $H_*, H^*$.** The homology and cohomology of $M, N, P$ are related by $\mathbb{Z}$-module isomorphisms

$$H^*(M, P) \cong H_{m-*}(N) \, , \; H_*(M, P) \cong H^{m-*}(N) \, .$$

- **Failure of Alexander duality for $\pi_1$.** The group morphisms $\pi_1(P) \to \pi_1(M)$ induced by $P \subset M$ are isomorphisms for $n - m \geqslant 3$, but not in general for $n - m = 1$ or $2$.
- The $\mathbb{Z}[\pi_1(P)]$-module homology $H_*(\widetilde{P})$ of the universal cover $\widetilde{P}$ depends on the knotting of $N \subset M$, whereas the $\mathbb{Z}$-module homology $H_*(P)$ does not.

## Change of rings

- For a ring $A$ let $\mathrm{Mod}(A)$ be the category of left $A$-modules.
- Given a ring morphism $\phi : A \to B$ regard $B$ as a $(B,A)$-bimodule by

$$B \times B \times A \to B \ ; \ (b,x,a) \mapsto b.x.\phi(a) \ .$$

  Use this to define the **change of rings** a functor

$$\phi_* \ = \ B \otimes_A - \ : \ \mathrm{Mod}(A) \to \mathrm{Mod}(B) \ ; \ M \mapsto B \otimes_A M \ .$$

- An $A$-module chain complex $C$ is $B$-**contractible** if the $B$-module chain complex $B \otimes_A C$ is contractible.

## Knotting and unknotting

- **Slogan** The fundamental group $\pi_1$ detects knotting for $n - m = 1$ or 2, whereas $\mathbb{Z}$-coefficient homology and cohomology do not.
- The applications of algebraic $K$- and $L$-theory to knots and links use the chain complexes of the universal covers of the complements. They involve the algebraic $K$- and $L$-theory of $B$-contractible $A$-module chain complexes for augmentations

$$\phi = \epsilon : A = \mathbb{Z}[\pi_1] \to B = \mathbb{Z} \; ; \; \sum_{g \in \pi_1} n_g g \mapsto \sum_{g \in \pi_1} g \; .$$

- In favourable circumstances (e.g. $\pi_1 = F_\mu$) there exists a 'stably flat noncommutative localization' $A \hookrightarrow \Sigma^{-1}A$ such that an $A$-module chain complex $C$ is $B$-contractible if and only if $C$ is $\Sigma^{-1}A$-contractible. The algebraic $K$- and $L$-theory of such $C$ can be then described entirely in terms of $A$.

## Algebraic $K$-theory I.

- Let $A$ be an associative ring with 1.
- The **projective class group** $K_0(A)$ is the abelian group with one generator $[P]$ for each isomorphism class of f.g. projective $A$-modules $P$, and relations

$$[P \oplus Q] \;=\; [P] + [Q] \in K_0(A)\ .$$

- A finite f.g. projective $A$-module chain complex $C$ has a chain homotopy invariant **projective class**

$$[C] \;=\; \sum_{i=0}^{\infty} (-)^i [C_i] \in K_0(A)\ .$$

- **Example** $K_0(\mathbb{Z}) = \mathbb{Z}$. The projective class of a finite f.g. free $A$-module chain complex is just the Euler characteristic the projective class

$$[C] \;=\; \chi(C) \;=\; \sum_{i=0}^{\infty} (-)^i \dim_A(C_i) \in \mathrm{im}(K_0(\mathbb{Z}) \to K_0(A))\ .$$

**Algebraic $K$-theory II.**

- The **Whitehead group** $K_1(A)$ is the abelian group with one generator $\tau(f)$ for each automorphism $f : P \to P$ of a f.g. projective $A$-module $P$, and relations

$$\tau(f \oplus f') ~=~ \tau(f) + \tau(f') ~,~ \tau(gfg^{-1}) ~=~ \tau(f) \in K_1(A) ~.$$

- The **Whitehead torsion** of a contractible finite based f.g. free $A$-module chain complex $C$ is

$$\tau(C) ~=~ \tau(d + \Gamma : C_{odd} \to C_{even}) \in K_1(A)$$

with $\Gamma : 0 \simeq 1 : C \to C$ any chain contraction

$$d\Gamma + \Gamma d ~=~ 1 ~:~ C_r \to C_r ~.$$

- Can generalize $K_0(A), K_1(A)$ to $K_*(A)$ for all $* \in \mathbb{Z}$.

## Change of rings in algebraic $K$-theory

- A ring morphism $\phi : A \to B$ induces an exact sequence of algebraic $K$-groups

$$\dots \to K_n(A) \xrightarrow{\phi_*} K_n(B) \to K_n(\phi) \to K_{n-1}(A) \to \dots$$

- A $B$-contractible finite f.g. free $A$-module chain complex $C$ with $\chi(C) = 0 \in \mathbb{Z}$ has a **Reidemeister torsion**

$$\begin{aligned} \tau[C] \in \quad & \ker(K_1(\phi) \to K_0(A)) \\ = \quad & \operatorname{im}(K_1(B) \to K_1(\phi)) \\ = \quad & \operatorname{coker}(\phi_* : K_1(A) \to K_1(B)) \end{aligned}$$

  given by $\tau(B \otimes_A C) \in K_1(B)$ for any choice of bases for $C$.
- (Milnor 1966) Whitehead torsion interpretation of the Reidemeister torsion of a knot using the augmentation $\phi : A = \mathbb{Z}[z, z^{-1}] \to B = F^\bullet$ for any field $F$.

## Commutative localization

- The **localization** of a commutative ring $A$ inverting a multiplicatively closed subset $S \subset A$ of non-zero divisors with $1 \in S$ is the ring $S^{-1}A$ of fractions $a/s$ $(a \in A, s \in S)$, where

$$a/s \;=\; b/t \text{ if and only if } at \;=\; bs \;.$$

- Usual addition and multiplication

$$a/s + b/t \;=\; (at + bs)/(st) \;,\; (a/s)(b/t) \;=\; (as)/(bt)$$

and canonical embedding $A \hookrightarrow S^{-1}A; a \mapsto a/1$.

- For an integral domain $A$ and $S = A - \{0\}$

$$S^{-1}A \;=\; \text{quotient field}(A) \;.$$

- **Example** If $A = \mathbb{Z}$ then $S^{-1}A = \mathbb{Q}$.

## The standard example $k : S^n \subset S^{n+2}$ I.

- The **exterior** of an $n$-dimensional knot $k$ is an $(n+2)$-dimensional manifold with boundary

$$(X, \partial X) \;=\; (\mathrm{cl.}(S^{n+2} \backslash (k(S^n) \times D^2)), S^n \times S^1)$$

with $X \subset S^{n+2} \backslash S^n$ a deformation retract of the complement.

- The generator $1 \in H^1(X) = \mathbb{Z}$ is realized by a homology equivalence $(f, \partial f) : (X, \partial X) \to (X_0, \partial X_0)$ with $(X_0, \partial X_0)$ the exterior of the trivial knot

$$k_0 \;:\; S^n \subset S^{n+2} \;=\; S^n \times D^2 \cup D^{n+1} \times S^1$$

with $X_0 = D^{n+1} \times S^1 \simeq S^1$, and $\partial f$ a homeomorphism.

- **Theorem** (Dehn+P. for $n = 1$, Kervaire+Levine for $n \geqslant 2$) $k$ is unknotted if and only if $f$ is a homotopy equivalence.
- The circle $S^1$ has universal cover $\widetilde{S}^1 = \mathbb{R}$, with $\pi_1(S^1) = \mathbb{Z}$, $\mathbb{Z}[\pi_1(S^1)] = \mathbb{Z}[z, z^{-1}]$. The homology equivalence $f : X \to S^1$ lifts to a $\mathbb{Z}$-equivariant map $\overline{f} : \overline{X} \to \mathbb{R}$ with $\overline{X} = f^*\mathbb{R}$ the pullback infinite cyclic cover of $X$.

## The standard example $k : S^n \subset S^{n+2}$ II.

- The **Blanchfield localization** $S^{-1}A$ of $A = \mathbb{Z}[z, z^{-1}]$ inverts $S = \epsilon^{-1}(1) \subset A$, with $\epsilon : A \to \mathbb{Z}; z \mapsto 1$ the augmentation.
- The cellular $A$-module chain map $\overline{f} : C(\overline{X}) \to C(\mathbb{R})$ induces a chain equivalence

$$f = 1 \otimes \overline{f} : \mathbb{Z} \otimes_A C(\overline{X}) = C(X) \to \mathbb{Z} \otimes_A C(\mathbb{R}) = C(S^1).$$

- The algebraic mapping cone $C = \mathcal{C}(\overline{f})$ is a finite f.g. free $A$-module chain complex such that

$$H_*(\mathbb{Z} \otimes_A C) \ = \ 0 \ , \ S^{-1}H_*(C) \ = \ 0 \ , \ \chi(C) \ = \ 0 \ .$$

  The Reidemeister torsion is an isotopy invariant

$$\begin{aligned} \tau[C] \ &= \ (1 - \phi(z))/\Delta(k) \\ &\in K_1(\phi) = \text{coker}(\phi_* : K_1(A) \to K_1(S^{-1}A)) = (S^{-1}A)^\bullet/A^\bullet \end{aligned}$$

  with $\Delta(k) \in S$ the Alexander polynomial of $k$.
- The localization $\phi : A \hookrightarrow S^{-1}A$ first used by Blanchfield (1957) in the study of the duality properties of $H_*(\overline{X})$.

## The noncommutative Ore localization

- (Ore 1931) The **Ore localization** $S^{-1}A$ is defined for a multiplicatively closed subset $S \subset A$ with $1 \in S$, and such that for all $a \in A$, $s \in S$ there exist $b \in A$, $t \in S$ with $ta = bs \in A$.
- E.g. central, $sa = as$ for all $a \in A$, $s \in S$.
- The Ore localization is the ring of fractions

  $$S^{-1}A \;=\; (S \times A)/\!\sim \;\;,$$

  with $(s,a) \sim (t,b)$ if and only if there exist $u, v \in A$ with

  $$us \;=\; vt \in S \;,\; ua \;=\; vb \in A \;.$$

- An element of $S^{-1}A$ is a noncommutative fraction

  $$s^{-1}a \;=\; \text{equivalence class of } (s,a) \in S^{-1}A$$

  with addition and multiplication more or less as usual.
- **Example** A commutative localization is an Ore localization.

## Ore localization is flat

- The Ore localization $S^{-1}A$ is a flat $A$-module, i.e. the functor

  $$S^{-1} \; : \; \mathrm{Mod}(A) \to \mathrm{Mod}(S^{-1}A) \; ; \; M \mapsto S^{-1}M = S^{-1}A \otimes_A M$$

  is exact.
- For any $A$-module $M$

  $$\mathrm{Tor}_i^A(S^{-1}A, M) \; = \; 0 \;\; (i \geqslant 1) \; .$$

- For any $A$-module chain complex $C$

  $$H_*(S^{-1}C) \; = \; S^{-1}H_*(C) \; .$$

- **Proposition** For any finite f.g. free $S^{-1}A$-module chain complex $D$ there exists a finite f.g. free $A$-module chain complex $C$ with an $S^{-1}A$-module isomorphism $S^{-1}C \cong D$.
- **Proof** Clear denominators!

## Universal localization I.

- Given a ring $A$ and a set $\Sigma$ of elements, matrices, morphisms, ..., it is possible to construct a new ring $\Sigma^{-1}A$, the localization of $A$ inverting all the elements in $\Sigma$.
- In general, $A$ and $\Sigma^{-1}A$ are noncommutative, and $A \to \Sigma^{-1}A$ is not injective.
- Original algebraic motivation: construction of noncommutative analogues of the quotient field of an integral domain.
- Topological applications to knots and links use the algebraic $K$- and $L$-theory of $A$ and $\Sigma^{-1}A$, in two separate situations:
  - Given a ring morphism $\phi : A \to B$ there exists a factorization $\phi : A \to \Sigma^{-1}A \to B$ such that a free $A$-module chain complex $C$ is $B$-contractible if and only if $C$ is $\Sigma^{-1}A$-contractible.
  - If a ring $R$ is an amalgamated free product or an *HNN* extension then for $k = 2$ or 3 the matrix ring $M_k(R)$ is $\Sigma^{-1}A$ for a triangular matrix ring $A \subset M_k(R)$ : gives all known decomposition theorems for $K_*(R)$ and $L_*(R)$.

## Universal localization II.

- $A$ = ring, $\Sigma$ = a set of morphisms $s : P \to Q$ of f.g. projective $A$-modules.
- A ring morphism $A \to B$ is **$\Sigma$-inverting** if each $1 \otimes s : B \otimes_A P \to B \otimes_A Q$ $(s \in \Sigma)$ is a $B$-module isomorphism.
- (P.M. Cohn 1970) The **universal localization** $\Sigma^{-1}A$ is a ring with a $\Sigma$-inverting morphism $A \to \Sigma^{-1}A$ such that any $\Sigma$-inverting morphism $A \to B$ has a unique factorization $A \to \Sigma^{-1}A \to B$.
- The universal localization $\Sigma^{-1}A$ exists (and it is unique); but it could be 0 – e.g if $0 \in \Sigma$.
- In general, $\Sigma^{-1}A$ is not a flat $A$-module. $\Sigma^{-1}A$ is a flat $A$-module if and only if $\Sigma^{-1}A$ is an Ore localization (Beachy, Teichner, 2003).

## The normal form I.

- (Gerasimov, Malcolmson 1981) Assume $\Sigma$ consists of all the morphisms $s : P \to Q$ of f.g. projective $A$-modules such that $1 \otimes s : \Sigma^{-1}P \to \Sigma^{-1}Q$ is a $\Sigma^{-1}A$-module isomorphism. (Can enlarge any $\Sigma$ to have this property). Every element $x \in \Sigma^{-1}A$ is of the form $x = fs^{-1}g$ for some

$$(s : P \to Q) \in \Sigma \ , \ f : P \to A \ , \ g : A \to Q \ .$$

- For f.g. projective $A$-modules $M, N$ every $\Sigma^{-1}A$-module morphism $x : \Sigma^{-1}M \to \Sigma^{-1}N$ is of the form $x = fs^{-1}g$ for some $(s : P \to Q) \in \Sigma$, $f : P \to N$, $g : M \to Q$

$$M \xrightarrow{\ g\ } P \xleftarrow{\ s\ } Q \xrightarrow{\ f\ } N \ .$$

- Addition by

$$fs^{-1}g + f's'^{-1}g' = (f \oplus f')(s \oplus s')^{-1}(g \oplus g') : \Sigma^{-1}M \to \Sigma^{-1}N$$

Similarly for composition.

## The normal form II.

- For f.g. projective $M, N$, a $\Sigma^{-1}A$-module morphism $fs^{-1}g : \Sigma^{-1}M \to \Sigma^{-1}N$ is such that $fs^{-1}g = 0$ if and only if there is a commutative diagram of $A$-module morphisms

$$P \oplus P_1 \oplus P_2 \oplus M \xrightarrow{\begin{pmatrix} s & 0 & 0 & g \\ 0 & s_1 & 0 & 0 \\ 0 & 0 & s_2 & g_2 \\ f & f_1 & 0 & 0 \end{pmatrix}} Q \oplus Q_1 \oplus Q_2 \oplus N$$

$$P \oplus P_1 \oplus P_2 \oplus M \xrightarrow{\begin{pmatrix} p & p_1 & p_2 & m \end{pmatrix}} L \xrightarrow{\begin{pmatrix} q & q_1 & q_2 & n \end{pmatrix}^T} Q \oplus Q_1 \oplus Q_2 \oplus N$$

with $s, s_1, s_2, \begin{pmatrix} p & p_1 & p_2 \end{pmatrix}, \begin{pmatrix} q & q_1 & q_2 \end{pmatrix}^T \in \Sigma$.
(Exercise: diagram $\Longrightarrow fs^{-1}g = 0$).

## Localization in algebraic $K$-theory I.

- Assume each $(s : P \to Q) \in \Sigma$ is injective and $A \to \Sigma^{-1}A$ is injective. The **torsion** exact category $T(A,\Sigma)$ has objects $A$-modules $T$ with $\Sigma^{-1}T = 0, \text{hom.dim.}(T) = 1$.
E.g., $T = \text{coker}(s)$ for $s \in \Sigma$.
- **Theorem** (Bass 1968 for central, Schofield 1985 for universal $\Sigma^{-1}A$). Exact sequence

$$K_1(A) \to K_1(\Sigma^{-1}A) \xrightarrow{\partial} K_0(T(A,\Sigma)) \to K_0(A) \to K_0(\Sigma^{-1}A)$$

with

$$\partial\big(\tau(fs^{-1}g : \Sigma^{-1}M \to \Sigma^{-1}N)\big)$$
$$= \big[\text{coker}(\begin{pmatrix} f & 0 \\ s & g \end{pmatrix} : P \oplus M \to N \oplus Q)\big] - \big[\text{coker}(s : P \to Q)\big]$$

- **Theorem** (Quillen 1972, Grayson 1980) Higher $K$-theory localization exact sequence for Ore localization $\Sigma^{-1}A$, by flatness.

## Universal localization is not flat

- In general, if $M$ is an $A$-module and $C$ is an $A$-module chain complex

$$\mathrm{Tor}_*^A(\Sigma^{-1}A, M) \neq 0~,~~ H_*(\Sigma^{-1}C) \neq \Sigma^{-1}H_*(C)~.$$

True for Ore localization $\Sigma^{-1}A$, by flatness.

- **Example** The universal localization $\Sigma^{-1}A$ of the free product

$$A ~=~ \mathbb{Z}\langle x_1, x_2\rangle ~=~ \mathbb{Z}[x_1] * \mathbb{Z}[x_2]$$

inverting $\Sigma = \{x_1\}$ is not flat. The 1-dimensional f.g. free $A$-module chain complex

$$d_C ~=~ (x_1~x_2)~:~C_1 ~=~ A \oplus A \to C_0 ~=~ A$$

is a resolution of $H_0(C) = \mathbb{Z}$ and

$$\begin{aligned} H_1(\Sigma^{-1}C) ~&=~ \mathrm{Tor}_1^A(\Sigma^{-1}A, H_0(C)) \\ &=~ \Sigma^{-1}A ~\neq~ \Sigma^{-1}H_1(C) ~=~ 0~. \end{aligned}$$

## Chain complex lifting I.

- A **lift** of a f.g. free $\Sigma^{-1}A$-module chain complex $D$ is a f.g. projective $A$-module chain complex $C$ with a chain equivalence $\Sigma^{-1}C \simeq D$.
- For an Ore localization $\Sigma^{-1}A$ one can lift every $n$-dimensional f.g. free $\Sigma^{-1}A$-module chain complex $D$, for any $n \geqslant 0$.
- For a universal localization $\Sigma^{-1}A$ one can only lift for $n \leqslant 2$ in general.
- **Proposition** (Neeman+R., 2001) For $n \geqslant 3$ there are lifting obstructions in $\mathrm{Tor}_i^A(\Sigma^{-1}A, \Sigma^{-1}A)$ for $i \geqslant 2$.
- $\mathrm{Tor}_1^A(\Sigma^{-1}A, \Sigma^{-1}A) = 0$ always.

## Chain complex lifting II.

- **Example** The boundary map in the Schofield exact sequence for an injective universal localization $A \to \Sigma^{-1}A$

$$\partial \ : \ K_1(\Sigma^{-1}A) \to K_0(T(A,\Sigma)) \ ; \ \tau(D) \mapsto [C]$$

  sends the Whitehead torsion $\tau(D)$ of a contractible based f.g. free $\Sigma^{-1}A$-module chain complex $D$ to the projective class $[C]$ of any f.g. projective $A$-module chain complex $C$ such that $\Sigma^{-1}C \simeq D$.

## Stable flatness

- A universal localization $\Sigma^{-1}A$ is **stably flat** if

$$\mathrm{Tor}_i^A(\Sigma^{-1}A, \Sigma^{-1}A) \; = \; 0 \quad (i \geqslant 2) \; .$$

- For stably flat $\Sigma^{-1}A$ have stable exactness:

$$H_*(\Sigma^{-1}C) \; = \; \varinjlim_B \; \Sigma^{-1}H_*(B)$$

  with maps $C \to B$ such that $\Sigma^{-1}C \simeq \Sigma^{-1}B$.
- Flat $\Longrightarrow$ stably flat. If $\Sigma^{-1}A$ is flat (i.e. an Ore localization) then

$$\mathrm{Tor}_i^A(\Sigma^{-1}A, M) \; = \; 0 \quad (i \geqslant 1)$$

  for every $A$-module $M$. The special case $M = \Sigma^{-1}A$ gives that $\Sigma^{-1}A$ is stably flat.

## A localization which is not stably flat

- Given a ring extension $R \subset S$ and an $S$-module $M$ let $K(M) = \ker(S \otimes_R M \to M)$.
- **Theorem** (Neeman, R. and Schofield)
  (i) The universal localization of the ring

$$A = \begin{pmatrix} R & 0 & 0 \\ S & R & 0 \\ S & S & R \end{pmatrix} = P_1 \oplus P_2 \oplus P_3 \text{ (columns)}$$

  inverting $\Sigma = \{P_3 \subset P_2, P_2 \subset P_1\}$ is $\Sigma^{-1}A = M_3(S)$.
  (ii) If $S$ is a flat $R$-module then

$$\mathrm{Tor}^A_{n-1}(\Sigma^{-1}A, \Sigma^{-1}A) = M_n(K^n(S)) \ (n \geqslant 3).$$

  (iii) If $R$ is a field and $\dim_R(S) = d$ then

$$K^n(S) = K(K(\ldots K(S)\ldots)) = R^{(d-1)^n d}\ .$$

  If $d \geqslant 2$, e.g. $S = R[x]/(x^d)$, then $\Sigma^{-1}A$ is not stably flat.
  (e-print RA.0205034, Math. Proc. Camb. Phil. Soc. 2004).

## Localization in algebraic $K$-theory II.

- **Theorem** (Neeman + R., 2001) If $A \to \Sigma^{-1}A$ is injective and stably flat then :
  - 'fibration sequence of exact categories'

    $$T(A,\Sigma) \to P(A) \to P(\Sigma^{-1}A)$$

    with $P(A)$ the category of f.g. projective $A$-modules, and every finite f.g. free $\Sigma^{-1}A$-module chain complex can be lifted,
  - there are exact localization sequences

    $$\cdots \to K_n(A) \to K_n(\Sigma^{-1}A) \to K_{n-1}(T(A,\Sigma)) \to K_{n-1}(A) \to \ldots$$

  - e-print RA.0109118, Geometry and Topology (2004)

## The standard example $\ell : \sqcup_\mu S^n \subset S^{n+2}$ I.

- The **exterior** of an $n$-dimensional boundary link $\ell$ is an $(n+2)$-dimensional manifold with boundary

$$(X, \partial X) \;=\; (\mathrm{cl.}(S^{n+2} \backslash ((\ell(\bigsqcup_\mu S^n) \times D^2)), S^n \times S^1)$$

  with $X \subset S^{n+2} \backslash S^n$ a deformation retract of the complement.
- (Cappell-Shaneson 1980) There is a homology equivalence

$$(f, \partial f) \;:\; (X, \partial X) \to (X_0, \partial X_0)$$

  with $(X_0, \partial X_0)$ the exterior of the trivial boundary link $\ell_0$

$$X_0 \;=\; \#_\mu (S^1 \times D^{n+1}) \simeq \vee_\mu S^1 \vee \vee_{\mu-1} S^{n+1}$$

  $\pi_1(X_0) = F_\mu$, and $\partial f$ a homeomorphism.
- The universal $F_\mu$-cover $\widetilde{X}_0$ of $X_0$ and the pullback cover $\widetilde{X} = f^* \widetilde{X}_0$ are such that $f$ lifts to an $F_\mu$-equivariant map $\widetilde{f} : \widetilde{X} \to \widetilde{X}_0$ with $C(\widetilde{f})$ a $\mathbb{Z}$-contractible f.g. free $\mathbb{Z}[F_\mu]$-module chain complex.

## The standard example $\ell : \sqcup_\mu S^n \subset S^{n+2}$ II.

- The **Blanchfield universal localization** $\Sigma^{-1}A$ of $A = \mathbb{Z}[F_\mu]$ inverts the set $\Sigma$ of all $\mathbb{Z}$-invertible square matrices in $A$.
- (Sontag-Dicks 1978, Farber-Vogel 1986) $\Sigma^{-1}A$ is stably flat.
- (R.-Sheiham 2003-) The algebraic mapping cone $C = \mathcal{C}(\widetilde{f})$ is a finite f.g. free $A$-module chain complex such that $H_*(\Sigma^{-1}C) = 0$, giving an isotopy invariant

$$\tau[C] \ = \ 1/\Delta(\ell)$$
$$\in K_1(\phi) = \text{coker}(\phi_* : K_1(A) \to K_1(S^{-1}A)) \subseteq K_0(T(A, \Sigma))$$

  with $\Delta(\ell) \in \Sigma$ the Alexander matrix of $\ell$. Isotopy invariant: mild generalization of the noncommutative Alexander polynomials of Farber (1986) and Garoufalidis-Kricker (2003).
- (R.-S.) **Blanchfield and Seifert algebra in high-dimensional boundary link theory I. Algebraic $K$-theory**, e-print AT.0508405, Geometry and Topology (2006)

## Algebraic $L$-theory

- Let $A$ be an associative ring with 1, and with an involution $A \to A; a \mapsto \bar{a}$ used to identify

$$\text{left } A\text{-modules} \;=\; \text{right } A\text{-modules} \;.$$

- **Example** A group ring $A = \mathbb{Z}[\pi]$ with $\bar{g} = g^{-1}$ for $g \in \pi$.
- The **algebraic $L$-group** $L_n(A)$ is the abelian group of cobordism classes $(C, \psi)$ of $n$-dimensional f.g. projective $A$-module chain complexes $C$ with an $n$-dimensional quadratic Poincaré duality

$$\psi \;:\; H^{n-*}(C) \;\cong\; H_*(C) \;.$$

- These are the Wall (1970) surgery obstruction groups $L_*(A)$, originally defined using quadratic forms and their automorphisms.

## Localization in algebraic $L$-theory

- **Theorem** (R. 1980 for Ore, Vogel 1982 in general) For any injective universal localization $A \to \Sigma^{-1}A$ of a ring with involution $A$ there is an exact sequence of algebraic $L$-groups

$$\cdots \to L_n(A) \to L_n(\Sigma^{-1}A) \to L_n(A,\Sigma) \to L_{n-1}(A) \to \ldots$$

with $L_n(A,\Sigma)$ the cobordism group of $\Sigma^{-1}A$-contractible $(n-1)$-dimensional quadratic Poincaré complexes $(C,\psi)$ over $A$.

- **Corollary** (Duval 1984 + R. 2008) For $n \geqslant 2$ the cobordism class of a boundary link $\ell : \bigsqcup_{\mu} S^n \subset S^{n+2}$ is the cobordism class of the $\mathbb{Z}$-contractible $(n+2)$-dimensional quadratic Poincaré complex $(\mathcal{C}(\widetilde{f}),\psi)$ over $\mathbb{Z}[F_\mu]$

$$\ell \ = \ (\mathcal{C}(\widetilde{f}),\psi) \in C_n(F_\mu) \ = \ L_{n+3}(\mathbb{Z}[F_\mu],\Sigma)$$

with $f : X \to X_0$ the homology equivalence between the exteriors of $\ell$ and $\ell_0$.