THE POINCARÉ DUALITY THEOREM
AND ITS CONVERSE I.
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FESTIVE OPENING COLLOQUIUM
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Local to global and, if possible, global to local

- There are many theorems in TOPOLOGY of the type
  \[
  \text{local input} \implies \text{global output}
  \]

- Theorems of the type
  \[
  \text{global input} \implies \text{local output}
  \]
  are even more interesting, and correspondingly harder to prove! This frequently requires ALGEBRA.

- *Algebra is a pact one makes with the devil!* (Sir Michael Atiyah)

- *I rather think that algebra is the song that the angels sing!* (Barry Mazur)

- *One thing I’ve learned about algebra … don’t take it too seriously* (Peanuts cartoon)
Poincaré duality and its converse

The Poincaré duality of an $n$-dimensional topological manifold $M$

$$H^*(M) \cong H_{n-*}(M)$$

is a local $\Longrightarrow$ global theorem.

**Theorem** Let $n \geq 5$. A space $X$ with $n$-dimensional Poincaré duality $H^*(X) \cong H_{n-*}(X)$ is homotopy equivalent to an $n$-dimensional topological manifold if and only if $X$ has sufficient local Poincaré duality.

Modern take on central result of the Browder-Novikov-Sullivan-Wall high-dimensional surgery theory for differentiable and PL manifolds, and its Kirby-Siebenmann extension to topological manifolds (1962-1970)

Will explain "sufficient" over the course of the lectures!
The Seifert-van Kampen Theorem and its converse

- Local $\implies$ global. The fundamental group of a union

\[ X = X_1 \cup_Y X_2, \ Y = X_1 \cap X_2 \]

is an amalgamated free product

\[ \pi_1(X) = \pi_1(X_1) \ast_{\pi_1(Y)} \pi_1(X_2). \]

- Global $\implies$ local. Let $n \geq 6$. If $X$ is an $n$-dimensional manifold such that $\pi_1(X) = G_1 \ast_H G_2$ then $X = X_1 \cup_Y X_2$ for codimension 0 submanifolds $X_1, X_2 \subset X$ with

\[ \partial X_1 = \partial X_2 = Y = (n-1)\text{-dimensional manifold}, \]

\[ \pi_1(X_1) = G_1, \ \pi_1(X_2) = G_2, \ \pi_1(Y) = H. \]
The Vietoris Theorem and its converses

- **Theorem** If $f : X \to Y$ is a surjection of compact metric spaces such that for each $y \in Y$ the restriction
  
  $$f| : f^{-1}(y) \to \{y\}$$
  
  induces an isomorphisms in homology
  
  $$H_*(f^{-1}(y)) \cong H_*(\{y\})$$
  
  then $f$ induces isomorphisms in homology
  
  $$f_* : H_*(X) \cong H_*(Y).$$

- **Local input:** each $f^{-1}(y)$ ($y \in Y$) is acyclic
  
  $$\tilde{H}_*(f^{-1}(y)) = 0.$$  

- **Global output:** $f_*$ is an isomorphism.

- Would like to have converses of the Vietoris theorem! For example, under what conditions is a homotopy equivalence homotopic to a homeomorphism?
Manifolds and homology manifolds

- An $n$-dimensional topological manifold is a topological space $M$ such that each $x \in M$ has an open neighbourhood homeomorphic to $\mathbb{R}^n$.

- An $n$-dimensional homology manifold is a topological space $M$ such that the local homology groups of $M$ at each $x \in M$ are isomorphic to the local homology groups of $\mathbb{R}^n$ at 0

$$H_*(M, M \setminus \{x\}) \cong H_*(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) = \begin{cases} \mathbb{Z} & \text{if } * = n \\ 0 & \text{if } * \neq n \end{cases}$$

- A topological manifold is a homology manifold.

- A homology manifold need not be a topological manifold.

- Will only consider compact $M$ which can be realized as a subspace $M \subset \mathbb{R}^{n+k}$ for some large $k \geq 0$, i.e. a compact ENR. This is automatically the case for topological manifolds.
The triangulation of manifolds

A triangulation of a space $X$ is a simplicial complex $K$ together with a homeomorphism

$$X \cong |K|$$

with $|K|$ the polyhedron of $K$.

$X$ is compact if and only if $K$ is finite.

Triangulation of $n$-dimensional topological manifolds:

- Exists and is unique for $n \leq 3$
- Known: may not exist for $n = 4$
- Unknown: if exists for $n \geq 5$
- Differentiable and PL manifolds are triangulated for all $n \geq 0$

Triangulation of $n$-dimensional homology manifolds:

- Exists and is unique for $n \leq 3$
- Known: may not exist for $n \geq 4$. 
The naked homeomorphism

- Poincaré, for one, was emphatic about the importance of the naked homeomorphism - when writing philosophically - yet his memoirs treat DIFF or PL manifolds only. in L. Siebenmann’s 1970 ICM lecture on topological manifolds.

- ... topological manifolds bear the simplest possible relation to their underlying homotopy types. This is a broad statement worth testing. (ibid.)

- Will describe how surgery theory manufactures the homotopy theory of topological manifolds of dimension $> 4$ from Poincaré duality spaces and chain complexes.

- Poincaré duality is the most important property of the algebraic topology of manifolds.
The original statement of Poincaré duality

- **Analysis Situs and its Five Supplements (1892–1904)**

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\[ \text{Par conséquent, pour une variété fermée, les nombres de Betti également distants des extrèmes sont égaux.} \]
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Ce théorème n’a, je crois, jamais été énoncé; il était cependant connu de plusieurs personnes qui en ont même fait des applications.

- Originally proved for a differentiable manifold \( M \), but long since established for topological and homology manifolds.
- \( h = n \), the dimension of \( M \).
- \( P_p = \dim_{\mathbb{Z}} H_p(M) \), the \( p \)th Betti number of \( M \).
- Happy birthday! 2011 is the 100th anniversary of Brouwer’s proof that homeomorphic manifolds have the same dimension. Also true for homology manifolds.
Orientation

A local fundamental class of an $n$-dimensional homology manifold $M$ at $x \in M$ is a choice of generator

$$[M]_x \in \{1, -1\} \subset H_n(M, M\{x\}) = \mathbb{Z}.$$  

The local Poincaré duality isomorphisms are defined by

$$[M]_x \cap - : H^*(\{x\}) \cong H_{n-*}(M, M\{x\}).$$

An $n$-dimensional homology manifold $M$ is orientable if there exists a fundamental homology class $[M] \in H_n(M)$ such that for each $x \in M$ the image

$$[M]_x \in H_n(M, M\{x\}) = \mathbb{Z}$$

is a local fundamental class.

We shall only consider manifolds which are orientable, together with a choice of fundamental class $[M] \in H_n(M)$. 


Poincaré duality in modern terminology

- **Theorem** For an $n$-dimensional manifold $M$ the cap products with the orientation $[M] \in H_n(M)$ are Poincaré duality isomorphisms

  $$[M] \cap - : H^*(M) \cong H_{n-*}(M).$$

- **Idea of proof** Glue together the local Poincaré duality isomorphisms

  $$[M]_x \cap - : \ H^*(\{x\}) \cong H_{n-*}(M, M\{x\}) \ (x \in M)$$

  to obtain the global Poincaré duality isomorphisms

  $$[M] \cap - = \lim_{\leftarrow x \in M} [M]_x \cap - :$$

  $$H^*(M) = \lim_{\leftarrow x \in M} H^*(\{x\}) \cong H_{n-*}(M) = \lim_{\leftarrow x \in M} H_{n-*}(M, M\{x\}).$$

- Need to work on the chain level, rather than directly with homology.
Poincaré duality spaces

Definition An $n$-dimensional Poincaré duality space $X$ is a finite $CW$ complex $X$ with a homology class $[X] \in H_n(X)$ such that cap product with $[X]$ defines Poincaré duality isomorphism

$$[X] \cap - : H^*(X; \mathbb{Z}[\pi_1(X)]) \cong H_{n-*}(X; \mathbb{Z}[\pi_1(X)]) .$$

In the simply-connected case $\pi_1(X) = \{1\}$ just

$$[X] \cap - : H^*(X) \cong H_{n-*}(X) .$$

Homotopy invariant: any finite $CW$ complex homotopy equivalent to an $n$-dimensional Poincaré duality space is an $n$-dimensional Poincaré duality space.

A triangulable $n$-dimensional homology manifold is an $n$-dimensional Poincaré duality space.

A nontriangulable $n$-dimensional homology manifold is homotopy equivalent to an $n$-dimensional Poincaré duality space.
Floer’s Diplom thesis

- Floer’s 1982 Bochum Diplom thesis (under the supervision of Ralph Stöcker) was on the homotopy-theoretic classification of \((n - 1)\)-connected \((2n + 1)\)-dimensional Poincaré duality spaces for \(n > 1\).

Klassifikation hochzusammenhängender Poincaré-Räume

Andreas Floer

Diplomarbeit

Ruhr-Universität Bochum

Abteilung für Mathematik

1982
(Existence) When is an $n$-dimensional Poincaré duality space homotopy equivalent to an $n$-dimensional topological manifold?

(Uniqueness) When is a homotopy equivalence of $n$-dimensional topological manifolds homotopic to a homeomorphism?

There are also versions of these questions for differentiable and $PL$ manifolds, and also for homology manifolds.

But it is the topological manifold version which is the most interesting! Both intrinsically, and because most susceptible to algebra, at least for $n > 4$. 

Manifold structures in the homotopy type of a Poincaré duality space
Surfaces

- Surface = 2-dimensional topological manifold.
- Every orientable surface is homeomorphic to the standard surface $\Sigma_g$ of genus $g \geq 0$.
- Every 2-dimensional Poincaré duality space is homotopy equivalent to a surface.
- A homotopy equivalence of surfaces is homotopic to a homeomorphism.
- In general, the analogous statements for false for $n$-dimensional manifolds with $n > 2$. 
## Bundle theories

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- Forgetful maps downwards. Difference between the first two rows = finite (but non-zero) = exotic spheres (Milnor).
- An \(n\)-dimensional differentiable manifold \(M\) has a tangent bundle \(\tau_M : M \to BO(n)\) and a stable normal bundle \(\nu_M : M \to BO\).
- Similarly for a topological manifold \(M\), with \(BTOP(n)\).
- An \(n\)-dimensional Poincaré duality space \(X\) has a Spivak normal fibration \(\nu_X : X \to BG\).
The Hirzebruch signature theorem

- The **signature** of a $4k$-dimensional Poincaré duality space $X$ is

  \[ \sigma(X) = \text{signature}(H^{2k}(X), \text{intersection form}) \in \mathbb{Z} \]

- The **Hirzebruch $L$-genus** of a vector bundle $\eta$ over a space $X$ is a certain polynomial $L(\eta) \in H^{4*}(X; \mathbb{Q})$ in the Pontrjagin classes $p_*(\eta) \in H^{4*}(M)$.

- **Signature Theorem** (1953) If $M$ is a $4k$-dimensional differentiable manifold then

  \[ \sigma(M) = \langle L(\tau_M), [M] \rangle \in \mathbb{Z} \]

- There have been many extensions of the theorem since 1953!
The Browder converse of the Hirzebruch signature theorem

Theorem (Browder, 1962) For $k > 1$ a simply-connected $4k$-dimensional Poincaré duality space $X$ is homotopy equivalent to a $4k$-dimensional differentiable manifold $M$ if and only if $\nu_X : X \to BG$ lifts to a vector bundle $\eta : X \to BO$ such that

$$\sigma(X) = \langle \mathcal{L}(-\eta), [X] \rangle \in \mathbb{Z}.$$ 

Novikov (1962) initiated the complementary theory of necessary and sufficient conditions for a homotopy equivalence of simply-connected differentiable manifolds to be homotopic to a diffeomorphism.

Many developments in the last 50 years, including versions for topological manifolds and homeomorphisms.
The Browder-Novikov-Sullivan-Wall surgery theory I.

- Is an $n$-dimensional Poincaré duality space $X$ homotopy equivalent to an $n$-dimensional topological manifold?

- The surgery theory provides a 2-stage obstruction for $n > 4$, working outside of $X$, involving normal maps $(f, b) : M \to X$ from manifolds $M$, with $b$ a bundle map.

- Primary obstruction in the topological $K$-theory of vector bundles to the existence of a normal map $(f, b) : M \to X$.

- Secondary obstruction $\sigma(f, b) \in L_n(\mathbb{Z}[\pi_1(X)])$ in the Wall surgery obstruction group, depending on the choice of $(f, b)$ in resolving the primary obstruction. The algebraic $L$-groups defined algebraically using quadratic forms over $\mathbb{Z}[\pi_1(X)]$.

- The mixture of topological $K$-theory and algebraic $L$-theory not very satisfactory!
The Browder-Novikov-Sullivan-Wall surgery theory II.

- Is a homotopy equivalence $f : M \to N$ of $n$-dimensional topological manifolds homotopic to a homeomorphism?
- For $n > 4$ similar 2-stage obstruction theory for deciding if $f$ is homotopic to a homeomorphism.
- The mapping cylinder of $f$

$$L = M \times [0, 1] \cup_{(x, 1) \sim f(x)} N$$

defines an $(n + 1)$-dimensional Poincaré pair $(L, M \sqcup N)$ with manifold boundary. The 2-stage obstruction for uniqueness is the 2-stage obstruction for relative existence.
- Again, the mixture of topological $K$-theory and algebraic $L$-theory not very satisfactory!