ON SIGNATURES OF KNOTS

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For Cherry Kearton

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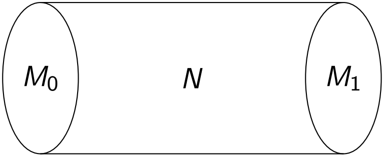
Signatures of knots

- A knot $k : S^1 \subset S^3$ determines a finite collection of quadratic forms.
- Each of these quadratic forms has a signature in $\mathbb{Z}$.
- Hence k has a finite collection of signatures.
- These signatures are the main algebraic tools in the cobordism classification of knots and their high-dimensional analogues $k : S^{2j-1} \subset S^{2j+1}$ for all $j \geqslant 1$.
- The signatures of knots have been studied for 50+ years by many mathematicians, including Kearton.
- The plan today is to take a walk in the knot garden, and explore some of the more elementary properties of the signatures of knots.
- A design for a Tudor knot garden:

Cobordism of manifolds

- Manifolds M to be oriented, with $-M$ the same manifold with the opposite orientation.
- A **cobordism** of closed n-dimensional manifolds M_0, M_1 is an $(n+1)$-dimensional manifold N with boundary $\partial N = M_0 \sqcup -M_1$.



- The set of cobordism classes of closed n-dimensional manifolds is an abelian group Ω_n, with addition by disjoint union, and the cobordism class of the empty manifold $\emptyset$ as the zero element.
- Thom etc. (1950's) Computation of Ω_n, starting with the signature

$$\sigma \ : \ \Omega_{4i} \to \mathbb{Z} \ .$$

- σ is an isomorphism for $i = 1$, a surjection for $i \geqslant 2$.
- Each Ω_n is finitely generated.

The signature of a manifold

- **Terminology**: the **dual** of a real vector space V is $V^* = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$.
- The **intersection form** of a closed oriented $2j$-dimensional manifold M^{2j} is the nonsingular $(-)^j$-symmetric form

$$\phi = (-)^j \phi^* \ : \ H_j(M;\mathbb{R}) \xrightarrow{\cong} H^j(M;\mathbb{R}) \xrightarrow{\cong} H_j(M;\mathbb{R})^*$$

  given by Poincaré duality, with $\dim_{\mathbb{R}} H_j(M;\mathbb{R}) < \infty$.
- Signature for j even. No signature for j odd.
- The **signature** of M^{4i} is the signature of the intersection form

$$\sigma(M) \ = \ \sigma(H_{2i}(M;\mathbb{R}), \phi) \in \mathbb{Z} \ .$$

- The signature is a cobordism invariant: if $M_0 \sqcup -M_1 = \partial N$ then

$$L \ = \ \ker(H_{2i}(\partial N;\mathbb{R}) \to H_{2i}(N;\mathbb{R})) \subset H_{2i}(\partial N;\mathbb{R})$$

  is a lagrangian subspace of $(H_{2i}(M_0;\mathbb{R}), \phi_0) \oplus (H_{2i}(M_1;\mathbb{R}), -\phi_1)$

$$L^{\perp} \ = \ L \subset H_{2i}(\partial N;\mathbb{R}) \ = \ H_{2i}(M_0;\mathbb{R}) \oplus H_{2i}(M_1;\mathbb{R})$$

  and $\sigma(M_0) = \sigma(M_1) \in \mathbb{Z}$.

The cobordism of knots

- An n-**knot** is an embedding

$$k \ : \ S^n \subset S^{n+2} \ .$$

- k is **unknotted** if $k(S^n) = \partial D^{n+1}$ for $D^{n+1} \subset S^{n+2}$.
- A **cobordism** of n-knots $k_0, k_1 : S^n \subset S^{n+2}$ is an embedding

$$\ell \ : \ S^n \times I \subset S^{n+2} \times I$$

  such that

$$\ell(x, i) \ = \ (k_i(x), i) \ \ (x \in S^n, i \in \{0, 1\}) \ .$$

- k is **slice** if $k(S^n) = \partial D^{n+1}$ for $D^{n+1} \subset D^{n+3}$.
- The trivial knot

$$k_0 \ : \ S^n \subset S^{n+2} \ ; \ (x_0, x_1, \dots, x_n) \mapsto (x_0, x_1, \dots, x_n, 0, 0)$$

  is both unknotted and slice.
- (Fox-Milnor, 1957 for $n = 1$, Kervaire for $n \geqslant 2$). The set of cobordism classes of n-knots is an abelian group C_n, with addition by connected sum. $k = 0 \in C_n$ if and only if k is slice.

Where do signatures of knots comes from?

- The signatures of an n-knot $k : S^n \subset S^{n+2}$ can be defined using two alternative methods:
  (a) The **Seifert surfaces** $F^{n+1} \subset S^{n+2}$ with
  $$\partial F = k(S^n) \subset S^{n+2} .$$
  For $n = 2i - 1$ get signatures from the Seifert matrix of linking numbers on $H_i(F)$ (Levine) but F is non-unique.
  (b) The **exterior** of an n-knot $k : S^n \subset S^{n+2}$ is the $(n + 2)$-dimensional manifold with boundary
  $$(X, \partial X) = (\text{closure}(S^{n+2} \backslash k(S^n) \times D^2), S^n \times S^1) .$$
  Unique but $H_*(X) = H_*(S^1)$. Canonical surjection
  $$p : \pi_1(X) \to \mathbb{Z} ; \ (S^1 \subset X) \mapsto \text{linking no.}(S^1, k(S^n) \subset S^{n+2})$$
  classifying an infinite cyclic cover $\overline{X} \to X$. For $n = 2i - 1$ get signatures from Poincaré duality on $H_i(\overline{X}; \mathbb{R})$ (Milnor) or the Blanchfield duality on $H_i(\overline{X})$ (Kearton). As $\overline{X}$ is non-compact $\dim_{\mathbb{Z}} H_i(\overline{X})$/torsion could be infinite, although $\dim_{\mathbb{R}} H_i(\overline{X}; \mathbb{R})$ is finite.

C_n is infinitely generated

- (Kervaire, 1966) (i) For $n \geqslant 2$ every n-knot $k : S^n \subset S^{n+2}$ is cobordant to one with $p : \pi_1(X) \to \mathbb{Z}$ an isomorphism.
  (ii) $C_{2j} = 0$ for $j \geqslant 1$.
  (iii) Maps $C_{2j-1} \to C_{2j+3}$: isomorphisms for $j \geqslant 2$, surjection for $j = 1$.
  (iv) $C_n = C_{n+4}$ for all $n \geqslant 2$.
- Write C_{n+4*} for the high-dimensional groups.
- (Milnor, Levine 1969) Algebraic computation for $j \geqslant 2$

$$C_{2j-1} \;=\; \bigoplus_{\infty} \mathbb{Z} \oplus \bigoplus_{\infty} \mathbb{Z}_4 \oplus \bigoplus_{\infty} \mathbb{Z}_2 \text{ (countable } \infty\text{'s)}~.$$

- **Algebraic number** = complex number which is a root of a $\mathbb{Z}$-coefficient polynomial $p(z) \in \mathbb{Z}[z, z^{-1}]$.
- A $(2j-1)$-knot $k : S^{2j-1} \subset S^{2j+1}$ has an Alexander polynomial $\Delta_k(z) \in \mathbb{Z}[z, z^{-1}]$. k has one $\mathbb{Z}$-valued signature for each root $e^{i\theta} \in S^1$ of $\Delta_k(z)$ with $0 < \theta < \pi$.
- For $j \geqslant 2$ the cobordism class $k \in C_{2j-1}$ is determined by primary invariants of the signature type.

C_1 is much bigger than C_{1+4*}

- (Casson-Gordon, 1975) The surjection $C_1 \to C_{1+4*}$ has non-trivial kernel, detected by the signatures of quadratic forms over cyclotomic fields. Need to consider $k : S^1 \subset S^3$ with $p : \pi_1(X) \to \mathbb{Z}$ not an isomorphism; k is unknotted if and only if $p : \pi_1(X) \cong \mathbb{Z}$, by Dehn's lemma.
- (Cochran-Orr-Teichner, 2003) A geometric filtration

$$\cdots \subset \mathcal{F}_{(n+1)/2} \subset \mathcal{F}_{n/2} \subset \ldots \mathcal{F}_{0.5} \subset \mathcal{F}_0 \subset C_1$$

with $C_1/\mathcal{F}_{0.5} = C_{1+4*} \to C_1/\mathcal{F}_0 = \mathbb{Z}_2$ the Arf invariant map and $\mathcal{F}_{0.5}/\mathcal{F}_{1.5}$ partially detected by the Casson-Gordon signatures. The higher quotients $\mathcal{F}_m/\mathcal{F}_{m+1/2}$ are partially detected by signatures of forms over skewfields and the $\mathbb{R}$-valued $L^{(2)}$-signatures of quadratic forms over von Neumann algebras. These are all secondary invariants, depending on the vanishing of the invariants in the lower filtration quotients.

Fibred knots

- An n-knot $k : S^n \subset S^{n+2}$ is **fibred** if $p : \pi_1(X) \to \mathbb{Z}$ is represented by a fibre bundle, meaning that

$$X \ = \ T(\zeta : F \to F) \ = \ F \times I/\{(x,0) \sim (\zeta(x),1) \,|\, x \in F\}$$

  is the mapping torus of the monodromy automorphism $\zeta : F \to F$ of a Seifert surface $F^{n+1} \subset S^{n+2}$ such that $\zeta| = 1 : \partial F = k(S^n) \to \partial F$, and

$$p \ : \ X \ = \ T(\zeta) \to S^1 \ = \ I/(0 \sim 1) \ ; \ [x,t] \to [t] \ .$$

- For a fibred knot the generating covering translation is

$$A \ : \ \overline{X} \ = \ F \times \mathbb{R} \to F \times \mathbb{R} \ ; \ (x,t) \mapsto (\zeta(x), t+1) \ ,$$

$$A_* \ = \ \zeta_* \ : \ H_*(\overline{X}) \ = \ H_*(F) \to H_*(F) \ ,$$

$$\dim_{\mathbb{Z}} H_*(\overline{X})/\text{torsion} \ = \ \dim_{\mathbb{Z}} H_*(F)/\text{torsion} < \infty.$$

- The n-knots arising from function theory and algebraic geometry (e.g. the torus knots $T_{p,q}$) are fibred. But there are many non-fibred n-knots.

Cherry + ζ

With $\mathbb{R}$-coefficients every knot $k : S^n \subset S^{n+2}$ is fibred!

- The infinite cyclic cover $\overline{X}$ of the knot exterior X is a non-compact $(n+2)$-dimensional manifold with boundary $S^n \times \mathbb{R}$.
- (Milnor, 1968) k has the $\mathbb{R}$-coefficient homology properties of a fibred knot with fibre $\overline{X}$ and monodromy a generating covering translation $A : \overline{X} \to \overline{X}$, with

$$\dim_{\mathbb{R}} H_*(\overline{X};\mathbb{R}) < \infty \ , \ H_j(\overline{X};\mathbb{R}) \ \cong \ H_{n+1-j}(\overline{X};\mathbb{R})^* \ .$$

- The projection

$$T(A : \overline{X} \to \overline{X}) \to \overline{X}/A \ = \ X$$

is a homotopy equivalence. Exact sequence

$$\dots \to H_r(\overline{X};\mathbb{R}) \xrightarrow{I-A} H_r(\overline{X};\mathbb{R}) \to H_r(X;\mathbb{R}) \to H_{r-1}(\overline{X};\mathbb{R}) \to \dots$$

with $H_r(X;\mathbb{R}) = H_r(S^1;\mathbb{R}) = \begin{cases} \mathbb{R} & \text{if } r = 0,1 \\ 0 & \text{if } r \neq 0,1 \ . \end{cases}$

- In practice, $H_*(\overline{X};\mathbb{R})$ and A are computed from a Seifert matrix for k.

The Alexander polynomial of $k : S^{2j-1} \subset S^{2j+1}$

- Two polynomials $p(z), q(z) \in \mathbb{R}[z, z^{-1}]$ are **equivalent** if

  $$p(z) \;=\; az^b q(z) \in \mathbb{R}[z, z^{-1}]$$

  with $a \neq 0 \in \mathbb{R}$, $b \in \mathbb{Z}$. Written $p(z) \sim q(z)$.
- The **Alexander polynomial** of k is the characteristic polynomial of the monodromy on $H = H_j(\overline{X}; \mathbb{R})$

  $$\Delta_k(z) \;=\; \mathrm{ch}_z(A) \;=\; \det(z - A : H[z, z^{-1}] \to H[z, z^{-1}]) \in \mathbb{R}[z, z^{-1}] \;.$$

  Roots of $\Delta_k(z) =$ complex eigenvalues of A.
- $\Delta_k(z^{-1}) \sim \Delta_k(z)$ (Poincaré duality): if $\omega \in \mathbb{C}$ is an eigenvalue then so are $\omega^{-1}, \overline{\omega} \in \mathbb{C}$.
- Seifert matrices show $\Delta_k(z) \sim p(z)$ for some $p(z) \in \mathbb{Z}[z, z^{-1}]$, so roots of $\Delta_k(z)$ are algebraic numbers $\omega \in \mathbb{C} \backslash \{-1, 0, 1\}$.
- degree $\Delta_k(z) = \dim_{\mathbb{R}} H$, $\Delta_k(1), \Delta_k(-1) \neq 0 \in \mathbb{R}$.
- If k is slice then $\Delta_k(z) \sim q(z)q(z^{-1})$ for some $q(z) \in \mathbb{Z}[z, z^{-1}]$.

Fibred automorphisms

- An automorphism $A : H \to H$ is **fibred** if neither 1 nor -1 is an eigenvalue, so that $A - A^{-1} : H \to H$ is an automorphism.
- For any $k : S^{2j-1} \subset S^{2j+1}$ the Poincaré-Milnor duality of $\overline{X}$ defines a nonsingular $(-1)^j$-symmetric form
$$\phi \ = \ (-)^j \phi^* \ : \ H \ = \ H_j(\overline{X};\mathbb{R}) \to H^* \ , \ \dim_{\mathbb{R}} H < \infty \ .$$
- The various knot signatures only depend on the fibred automorphism $A : (H,\phi) \to (H,\phi)$ with eigenvalues the roots of the Alexander polynomial $\Delta_k(z)$.
- The function
$$\{\text{fibred automorphisms } A \text{ of } (-1)\text{-symmetric forms } (H,\phi^-)\}$$
$$\to \{\text{fibred automorphisms } A \text{ of } (+1)\text{-symmetric forms } (H,\phi^+)\} \ ;$$
$$(H,\phi^-,A) \mapsto (H,\phi^+,A) \ , \ \phi^+ \ = \ \phi^-(A - A^{-1})$$
is a one-one correspondence, so from now on need only consider the case j odd, say $j = 1$.

The signature of a knot

- (Trotter 1962, Murasugi 1965) The **signature** of $k : S^1 \subset S^3$ is the signature of the nonsingular symmetric form $(H, \phi(A - A^{-1}))$

$$\sigma(k) \ = \ \sigma(H, \phi(A - A^{-1})) \in \mathbb{Z} \ .$$

- Originally defined using a Seifert matrix.
- The signature is a knot cobordism invariant.
- The nonsingular symmetric form $(H, \phi(A - A^{-1}))$ related to the linking properties of the 3-manifold

$$M_r \ = \ \overline{X}/A^r \cup S^1 \times D^2 \ (r \geqslant 2)$$

  the r-fold cover of S^3 branched over k.
- Viro (1984) identified $(H, \phi(A - A^{-1}))$ with the intersection form of a 4-manifold N with boundary $\partial N = M_2$ which is a double cover of D^4 branched over a Seifert surface for k.

The Milnor signatures of $k : S^1 \subset S^3$

- Let $\omega_\ell = e^{i\theta_\ell} \in S^1$ $(\ell = 1, 2, \dots, 2n)$ be the eigenvalues of A on the unit circle, with

$$0 < \theta_1 < \cdots < \theta_n < \pi < \theta_{n+1} = 2\pi - \theta_n < \cdots < \theta_{2n} = 2\pi - \theta_1 < 2\pi$$

  so that $\omega_\ell = \overline{\omega}_{2n+1-\ell}$.
- The other eigenvalues do not contribute signatures, so may be ignored.
- (1968) The **Milnor signatures** of k are knot cobordism invariants

$$\tau_{\omega_\ell}(k) \ = \ \begin{cases} \sigma(H^{\omega_\ell}, \phi(A - A^{-1})|) & \text{if } 1 \leqslant \ell \leqslant n \\ -\tau_{\omega_{2n+1-\ell}}(k) & \text{if } n+1 \leqslant \ell \leqslant 2n \end{cases}$$

  with $H^{\omega_\ell} = \bigcup\limits_{s=1}^{\infty} \{x \in H \,|\, (A^2 - 2\cos\theta_\ell\, A + I)^s(x) = 0\}$.
- $(H, \phi, A) = \sum\limits_{\ell=1}^{n} (H^{\omega_\ell}, \phi|, A|)$, so $\sigma(k) = \sum\limits_{\ell=1}^{n} \tau_{\omega_\ell}(k) \in \mathbb{Z}$.
- In essence, (H, ϕ, A) is a sum of 2-dimensional components.

The Levine-Tristram signatures of $k : S^1 \subset S^3$

- (Levine, Tristram 1969) The **Levine-Tristram signatures** of k are the knot cobordism invariants defined for non-eigenvalues $\omega \in S^1$ of A

$$\sigma_\omega(k) \;=\; \sigma(\mathbb{C} \otimes_{\mathbb{R}} H, (1-\omega)\phi(I-A)^{-1} + (1-\overline{\omega})(I-A^*)^{-1}\phi^*) \in \mathbb{Z} \;.$$

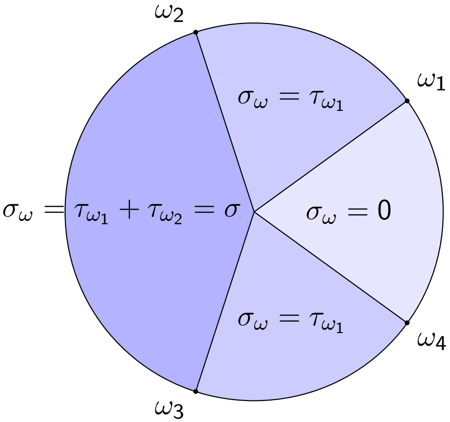
- $\sigma_1(k) = 0$, $\sigma_{-1}(k) = \sigma(k)$.
- (Levine 1969, Matumoto 1972) (i) The Levine-Tristram signature function

$$S^1 \backslash \{\text{eigenvalues of } A\} \to \mathbb{Z} \;;\; \omega \mapsto \sigma_\omega(k)$$

  is constant on each of the open arcs between successive eigenvalues. (ii) The jump at an eigenvalue $\omega = e^{i\theta} \in S^1$ of A is the Milnor signature of k at ω

$$\lim_{\delta \to 0^+} (\sigma_{\omega e^{i\delta}}(k) - \sigma_{\omega e^{-i\delta}}(k)) \;=\; \tau_\omega(k) \in \mathbb{Z} \;.$$

Jumping round the circle



The $L^{(2)}$-signature of $k : S^1 \subset S^3$

- (Reich 2001, Cochran-Orr-Teichner 2003) The $L^{(2)}$-**signature** of k is the knot cobordism invariant defined by the average of the Levine-Tristram signatures

$$\sigma^{(2)}(k) \;=\; \frac{1}{2\pi}\int_{\omega\in S^1}\sigma_\omega(k)d\omega \in \mathbb{R}\ .$$

- Also called the ρ-**invariant** of k. Related to Atiyah-Singer index theory.
- **Proposition** The $L^{(2)}$-signature is expressed in terms of the Milnor signatures by

$$\begin{aligned}\sigma^{(2)}(k) \;&=\; \frac{1}{2\pi}\sum_{\ell=1}^{2n-1}\sigma_{(\theta_\ell+\theta_{\ell+1})/2}(k)(\theta_{\ell+1}-\theta_\ell)\\ &=\; \sum_{\ell=1}^{n}\tau_{\omega_\ell}(k)(1-\theta_\ell/\pi)\in\mathbb{R}\ .\end{aligned}$$

- In order to capture $k \in C_1$ also need to consider the $L^{(2)}$-signatures of k which arise from factorizations $p : \pi_1(X) \to \Gamma \to \mathbb{Z}$ through representations $\rho : \pi_1(X) \to \Gamma$ other than p.

The generic example: elliptic $A \in SL_2(\mathbb{R})$

- For a 2-dimensional fibred (H, ϕ, A) can take

$$H = \mathbb{R} \oplus \mathbb{R}\ ,\ \phi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\ ,\ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

  with $A \in SL_2(\mathbb{R})$ $(ad - bc = 1)$ and $\det(I \pm A) = 2 \pm (a + d) \neq 0$.
- A is **elliptic** if $|\mathrm{trace}(A)| = |a + d| < 2$. Only elliptic A have a signature. Let $(a + d)/2 = \cos\theta$ $(\theta \in (0, \pi))$. Note that $c \neq 0$.
- The Alexander polynomial and signatures of (H, ϕ, A) are given by

$$\mathrm{ch}_z(A) = \begin{vmatrix} z - a & -b \\ -c & z - d \end{vmatrix} = z^2 - (a + d)z + 1 = (z - e^{i\theta})(z - e^{-i\theta}),$$

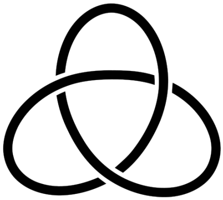
$$\sigma(H, \phi(A - A^{-1})) = \tau_{e^{i\theta}}(H, \phi, A) = \sigma \begin{pmatrix} 2c & d - a \\ d - a & -2b \end{pmatrix} = 2\,\mathrm{sgn}(c) \in \mathbb{Z}\ ,$$

$$\sigma_{e^{i\psi}}(H, \phi, A) = \begin{cases} 0 & \text{if } 0 \leqslant \psi < \theta \text{ or } 2\pi - \theta < \psi < 2\pi \\ 2\,\mathrm{sgn}(c) & \text{if } \theta < \psi < 2\pi - \theta\ , \end{cases}$$

$$\sigma^{(2)}(H, \phi, A) = \frac{1}{2\pi} \int_0^{2\pi} \sigma_{e^{i\psi}}(H, \phi, A)\,d\psi = 2\,\mathrm{sgn}(c)(1 - \theta/\pi) \in \mathbb{R}\ .$$

The trefoil knot I.

- The trefoil knot $k : S^1 \subset S^3$:



- k is a fibred knot with fibre a punctured torus, and

$$H = H_1(\overline{X};\mathbb{R}) = \mathbb{R} \oplus \mathbb{R} .$$

- The monodromy and Alexander polynomial of k are

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} : (H,\phi) = (\mathbb{R} \oplus \mathbb{R}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) \to (H,\phi) ,$$

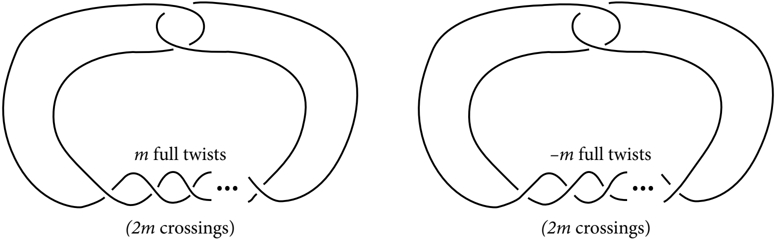
$$\Delta_k(z) = \begin{vmatrix} z & -1 \\ 1 & z-1 \end{vmatrix} = z^2 - z + 1 .$$

The trefoil knot II.

- Algebraically, k is the generic example with $\theta = \pi/3$

$$\begin{aligned}
\Delta_k(z) \ &= \ z^2 - z + 1 \ = \ (z - e^{\pi i/3})(z - e^{5\pi i/3}) \ , \\
\sigma(k) \ &= \ \tau_{e^{\pi i/3}}(k) \ = \ \sigma(H, \phi(A - A^{-1})) \\
&= \ \sigma(\mathbb{R} \oplus \mathbb{R}, \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}) \ = \ -2 \ , \\
\sigma_{e^{i\psi}}(k) \ &= \ \begin{cases} 0 & \text{if } 0 \leqslant \psi < \pi/3 \text{ or } 5\pi/3 < \psi < 2\pi \\ -2 & \text{if } \pi/3 < \psi < 5\pi/3 \ , \end{cases} \\
\sigma^{(2)}(k) \ &= \ \tau_{e^{\pi i/3}}(k)(1 - \theta/\pi) \ = \ \frac{-4}{3} \ .
\end{aligned}$$

The m-twist knots $K_m : S^1 \subset S^3$ for $m \in \mathbb{Z}$

- K_0 = unknot k_0
- 

K_m for $m \geqslant 1$ $\qquad$ K_{-m} for $m \geqslant 1$

- K_1 = figure 8 knot, K_{-1} = trefoil knot k.
- For $m \neq 0$ K_m has

$$(H, \phi) = (\mathbb{R} \oplus \mathbb{R}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) , \ A = \begin{pmatrix} 1 & 1 \\ 1/m & 1+1/m \end{pmatrix} ,$$

$$\Delta_{K_m}(z) = z^2 - 2(1+1/2m)z + 1 \in \mathbb{R}[z, z^{-1}] .$$

The m-twist knots K_m for $m \leqslant -1$

- (Milnor 1968) A is elliptic, the generic example with

$$\begin{aligned}
\theta &= \cos^{-1}(1+1/2m) \in (0,\pi/2)\ , \\
\Delta_{K_m}(z) &= (z-e^{i\theta})(z-e^{-i\theta}) \in \mathbb{R}[z,z^{-1}]\ , \\
\sigma(K_m) &= \tau_{e^{i\theta}}(K_m) = \sigma(\mathbb{R}\oplus\mathbb{R}, \begin{pmatrix} 2/m & 1/m \\ 1/m & -2 \end{pmatrix}) = -2\ , \\
\sigma_{e^{i\psi}}(K_m) &= \begin{cases} 0 & \text{if } 0 \leqslant \psi < \theta \text{ or } 2\pi-\theta < \psi < 2\pi \\ -2 & \text{if } \theta < \psi < 2\pi-\theta\ , \end{cases} \\
\sigma^{(2)}(K_m) &= -2(1-\theta/\pi) \neq 0\ ,\ K_m \neq 0 \in C_{1+4*}\ .
\end{aligned}$$

- (Milnor 1968) The twist knots K_m ($m \leqslant -1$) are linearly independent in C_1, so C_1 is infinitely generated.

The m-twist knots K_m for $m \geqslant 1$

- A is hyperbolic: $|\text{trace}(A)| = 2 + 1/m > 2$.
- The Milnor and Levine-Tristram signatures vanish

$$\sigma_\omega(K_m) \ = \ \sigma(K_m) \ = \ \tau_{e^{i\theta}}(K_m) \ = \ \sigma^{(2)}(K_m) \ = \ 0 \in \mathbb{Z} \ .$$

- The following conditions are equivalent:
  (i) $K_m = 0 \in C_{1+4*}$
  (ii) $\Delta_{K_m}(z) \sim q(z)q(z^{-1})$ for some $q(z) \in \mathbb{Z}[z, z^{-1}]$
  (iii) $m = n(n+1)$ for some $n \geqslant 1$.
- Casson-Gordon (1975) used higher signatures to show that for $m \geqslant 2$

$$K_m \ = \ 0 \in C_1 \text{ if and only if } m = 2 \ .$$

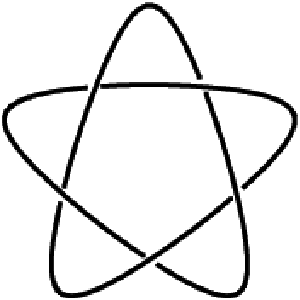
Reproved by Collins (2010) using the $L^{(2)}$-signature of the torus knots $T_{n,n+1}$. These occur as self-linking 0 simple curves on a Seifert surface F for $K_{n(n+1)}$, so that $K_{n(n+1)} \neq 0 \in C_1/\mathcal{F}_{(1.5)}$ for $n \geqslant 2$.

The (p, q)-torus knots $T_{p,q} : S^1 \subset S^3$

- For coprime $p, q \geqslant 2$ define $T_{p,q}$ to be the composite of

$$S^1 \subset S^1 \times S^1 \; ; \; s \mapsto (s^p, s^q) \text{ and } S^1 \times S^1 \subset S^3 \; .$$

- Example: $T_{2,3}$ = trefoil knot
- Example: $T_{2,5}$ = cinquefoil knot (Solomon's seal)



The signatures of the torus knots $T_{p,q}$ I.

- Studied by many authors, including:

Pham 1965
Brieskorn 1966
Hirzebruch-Mayer 1968
Hirzebruch-Zagier 1974
Goldsmith 1975
Matumoto 1977
Kauffman 1978
Kearton 1979
Litherland 1979
Gordon-Litherland-Murasugi 1981
Kirby-Melvin 1994
Nemethi 1998
Borodzik 2009
Collins 2010
Borodzik-Oleszkiewicz 2010

The signatures of the torus knots $T_{p,q}$ II.

- The Alexander polynomial of $T_{p,q}$

$$\Delta_{T_{p,q}}(z) \;=\; \frac{(z^{pq}-1)(z-1)}{(z^p-1)(z^q-1)}$$

  has even degree $pq+1-p-q=(p-1)(q-1)=2n$.
- The roots of $\Delta_{T_{p,q}}(z)$ are

$$\omega_\ell \;=\; e^{2\pi i r_\ell/pq} \in S^1 \text{ for } 1 \leqslant \ell \leqslant 2n$$

  with
  $\{r_1 < r_2 < \cdots < r_{2n}\} \;=\; \{1,2,\dots,pq-1\}\backslash\{r \text{ such that } p|r \text{ or } q|r\}$.
- (Matumoto 1977, Kearton 1979) The Milnor signatures of $T_{p,q}$ are

$$\tau_{\omega_\ell}(T_{p,q}) \;=\; 2\,\mathrm{sgn}(1/2-(a_\ell/p+b_\ell/q)) \in \{-2,2\}$$

  if $r_\ell = a_\ell q + b_\ell p \,(\mathrm{mod}\, pq)$ with $1 \leqslant a_\ell < p$, $1 \leqslant b_\ell < q$.
  Number theory related to Dedekind sums.
- (Litherland 1979) The torus knots $T_{p,q}$ are linearly independent in C_1.

The signatures of the torus knots $T_{p,q}$ III.

- The Levine-Tristram signatures of $T_{p,q}$ at the non-eigenvalues of $\omega = e^{i\theta} \in S^1$ by A are given by

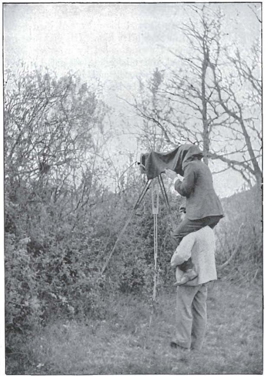
$$\sigma_\omega(T_{p,q}) = \begin{cases} \sum\limits_{j=1}^{\ell} \tau_{\omega_j}(T_{p,q}) & \text{if } r_\ell/pq < \theta < r_{\ell+1}/pq \\ 0 & \text{if } -r_1/pq < \theta < r_1/pq \ . \end{cases}$$

- (Kirby-Melvin 1994, Nemethi 1998, Borodzik 2009, Collins 2010) The $L^{(2)}$-signature of $T_{p,q}$ is

$$\sigma^{(2)}(T_{p,q}) = -(p-1/p)(q-1/q)/3 \in \mathbb{R} \ .$$

- There are inductive procedures for computing $\sigma_\omega(T_{p,q})$ and $\sigma(T_{p,q}) = \sum\limits_{j=1}^{n} \tau_{\omega_j}(T_{p,q}) \in \mathbb{Z}$ in terms of p, q, but no closed formula.
- (Borodzik-Oleszkiewicz 2010) There is no closed formula for $\sigma(T_{p,q})$ as a rational function of p, q.

How to photograph birds



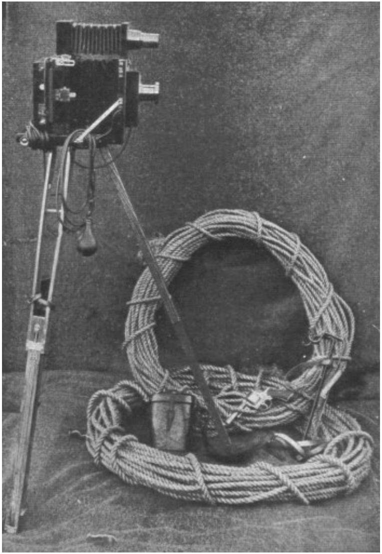
METHOD OF PHOTOGRAPHING BIRDS' NESTS SITUATED IN HIGH HEDGES.

PHOTOGRAPHING A NEST IN A TREE.

Plates from *With Nature and A Camera* (1897) by Richard and Cherry Kearton, the grandfather and great-uncle of our Cherry Kearton.

http://gdl.cdlr.strath.ac.uk/keacam

The outfit for descending cliffs



Knot theory in practice