THE WORK OF C.T.C. WALL
IN TOPOLOGY

ANDREW RANICKI

- 90+ papers, 2+ books

- Topics covered: cobordism groups, Steenrod algebra, homological algebra, manifolds of dimensions 3, 4, ≥ 5, quadratic forms, finiteness obstruction, embeddings, bundles, Poincaré complexes, surgery obstruction theory, homology of groups, 2-dimensional complexes, topological space form problem, computations of $K$- and $L$-groups, ...

- MR 57Q12 Wall finiteness obstruction for CW-complexes

- MR 57R67 Surgery obstructions, Wall groups
Wall’s manifold classifications

1. All manifolds at once
   - cobordism (1959-1961)

2. One manifold at a time
   - diffeomorphism (1962-1966)

3. Within a homotopy type
   - surgery (1967-1977)
Cobordism

• A **cobordism** between closed $m$-dimensional manifolds $M, N$ is an $(m + 1)$-dimensional manifold $W$ with boundary $\partial W = M \cup N$

  ![Diagram of cobordism](image)

• $\Omega_m =$ abelian group of cobordism classes of oriented closed $m$-dimensional manifolds, addition by disjoint union

• $\Omega_* = \sum_{m=0}^{\infty} \Omega_m$ **oriented cobordism ring**, multiplication by cartesian product.
Computation of oriented cobordism

- Thom: expressed $\Omega_\ast$ as homotopy groups, computed $\Omega_\ast \otimes \mathbb{Q}$
  - no odd-primary torsion (Milnor).

- Wall: **Determination of the cobordism ring** Annals of Mathematics 72, 292–311 (1960)

- Calculation of 2-primary torsion.

- **Theorem** (Wall) Two oriented manifolds are cobordant if and only if they have the same Stiefel and Pontrjagin numbers
  - ultimate achievement of pioneering phase of cobordism theory.
Handles and surgery

• Given $m$-manifold $M$ and $S^r \times D^{m-r} \subset M$ define elementary cobordism $(W; M, N)$ by attaching an $(r + 1)$-handle to $M \times I$

\[
W = M \times I \cup D^{r+1} \times D^{m-r}
\]

• $N = (M \setminus S^r \times D^{m-r}) \cup D^{r+1} \times S^{m-r-1}$ manifold obtained from $M$ by surgery on $S^r \times D^{m-r} \subset M$

• Handles are the building blocks of manifolds
  
  -- need surgeries to attach handles
Structure of manifolds

- Every cobordism \((W; M, N)\) is a union of elementary cobordisms.

- \(h\)-cobordism = cobordism \((W; M, N)\) with \(M \subset W, N \subset W\) homotopy equivalences

- \(h\)-cobordism theorem (Smale): every simply-connected \(h\)-cobordism with \(\dim(W) \geq 6\) is diffeomorphic to \(M \times (I; \{0\}, \{1\})\)
  - needs Whitney trick for removing double points in dimensions \(> 4\)

- \(s\)-cobordism theorem is non-simply-connected version \(\pi_1(W) \neq \{1\}\)

- possible rearrangements of handles governed by algebraic \(K\)-theory (Whitehead torsion)
Intersection form

- $M = \text{oriented } 2n\text{-dimensional manifold.}$

- **Intersection form:** $(-)^n$-symmetric pairing
  $$H_n(M) \times H_n(M) \to \mathbb{Z}$$

- Isomorphism class of form is an oriented homotopy invariant.

- **Signature** defined for even $n$, an oriented cobordism invariant.

- The boundary of an $(n-1)$-connected $2n$-dimensional manifold $M$ with unimodular intersection form is a homotopy sphere $\partial M = \Sigma^{2n-1}$, with a potentially exotic differential structure for $n \geq 4$ (Milnor).
Classification of highly-connected manifolds

• Wall: **Classification of** $(n-1)$-connected $2n$-**manifolds** Annals of Mathematics 75, 163–189 (1962)

• **Theorem** (Wall) For $n \geq 3$ the diffeomorphism classes of differentiable $(n-1)$-connected $2n$-manifolds with boundary an exotic sphere $= \text{the isomorphism classes of } \mathbb{Z}\text{-valued } (\approx)^n \text{-symmetric forms with a quadratic refinement in } \pi_n(BSO(n))$

• Classification of handlebodies by homotopy theory, subsequently generalized to other cases:
4-manifolds

- Simply-connected 4-manifolds are homotopy equivalent if and only if intersection forms are isomorphic (Milnor).

- **Theorem (Wall)** Simply-connected 4-manifolds are $h$-cobordant if and only if intersection forms are isomorphic.

- **Theorem (Wall)** $h$-cobordant simply-connected 4-manifolds $M, N$ are stably diffeomorphic

$$M \#_k S^2 \times S^2 \cong N \#_k S^2 \times S^2$$

for some $k \geq 0$. $\#$ = connected sum
**CW complexes**

- $X$ space, $f : S^r \to X$ map

- $X \cup_f D^{r+1} =$ space obtained from $X$ by attaching an $(r + 1)$-cell

- **CW complex** = space obtained from $\emptyset$ by attaching cells

- When is a space homotopy equivalent to a finite CW complex?
Finite domination

• A space $X$ is finitely dominated if it is a homotopy retract of a finite $CW$ complex $K$, i.e. if there exist maps $f : X \to K$, $g : K \to X$ and a homotopy $gf \simeq 1 : X \to X$.

• Is a finitely dominated space homotopy equivalent to a finite $CW$ complex?

• Every compact $ANR$, e.g. a topological manifold, is finitely dominated (Borsuk).

• A finite group $\pi$ with cohomology of period $q$ acts freely on an infinite $CW$ complex $Y$ homotopy equivalent to $S^{q-1}$, with $Y/\pi$ finitely dominated (Swan).
Finiteness obstruction

- **Wall**: Finiteness conditions for $CW$-complexes
  Annals of Mathematics 81, 56–89 (1965)

- **Wall finiteness obstruction** $[X] \in \tilde{K}_0(\mathbb{Z}[\pi_1(X)])$
  of finitely dominated space $X$
  - fundamental algebraic invariant of non-compact topology.

- **Theorem** (Wall) $X$ is homotopy equivalent to finite $CW$ complex if and only if $[X] = 0$

- Many applications to topology of manifolds
  - Siebenmann end obstruction for closing tame ends of open manifolds
  - Topologically stratified sets
The surgery method

- Standard method for classifying manifolds within a homotopy type.

- An $m$-dimensional manifold $M$ has Poincaré duality $H^{m-*}(M) \cong H_*(M)$.

- Is a space $X$ with $m$-dimensional Poincaré duality $H^{m-*}(X) \cong H_*(X)$ homotopy equivalent to an $m$-dimensional manifold?

- Is a homotopy equivalence of manifolds homotopic to a diffeomorphism?
  
  — relative version of previous question

- Formulation by Browder, Novikov, Sullivan in terms of normal maps $(f, b) : M \to X$ from manifolds to Poincaré duality spaces, with $f$ degree 1 and $b$ a bundle map.
Wall surgery theory

- **Wall**: *Surgery on compact manifolds*
  LMS Monograph 1, Academic Press (1970)
  - the surgeon’s bible
  - algebraic $L$-groups $L_*(\mathbb{Z}[\pi])$ of group ring
    $\mathbb{Z}[\pi] = \text{quadratic algebraic } K$-groups
  - surgery obstruction of normal map $(f, b) : M \to X$
    
    $$\sigma_*(f, b) \in L_m(\mathbb{Z}[\pi_1(X)])$$

- **Theorem** (Wall) For $m \geq 5$ an $m$-dimensional
  Poincaré duality space $X$ is homotopy equivalent to an $m$-dimensional manifold if and only if there exists a normal map $(f, b) : M \to X$ with $\sigma_*(f, b) = 0$. 

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Properties of Wall groups $L_m(\mathbb{Z}[\pi])$

- Quadratic forms over $\mathbb{Z}[\pi]$ for $m$ even
- Automorphisms of forms for $m$ odd
- Govern existence and effects of surgeries on $m$-dimensional manifolds with fundamental group $\pi$
- Computations for finite $\pi$ using algebra
- Computations for infinite $\pi$ using topology
- Many, many applications to both algebra and topology
The topological space form problem

- Wall: The topological space-form problem, pp 319-351 in Topology of manifolds, Markham, 1970

- Wall: Free actions of finite groups on spheres, pp 115-124 in Proc Symp in Pure Math 32, AMS 1978

- +3 further papers (with Madsen and Thomas)

- Complete classification of finite groups $\pi$ which have a free topological action on $S^m$ for $m \geq 5$, using:
  - group cohomology
  - homotopy theory
  - algebraic $K$- and $L$-theory of $\mathbb{Z}[\pi]$. 
**PL structures on tori**

- **Wall:** *On homotopy tori and the annulus theorem* Bulletin LMS 1, 95–97 (1969)

- Uses geometric computation of $L_*(\mathbb{Z}[\mathbb{Z}^m])$ to classify PL manifolds homotopy equivalent to $m$-torus $T^m$ for $m \geq 5$

- Applied by Kirby to prove the *annulus theorem* for $m \geq 5$: if $D^m \subset \text{int}(D^m)$ is an embedding then $D^m \setminus \text{int}(D^m)$ is homeomorphic to $S^{m-1} \times I$

- Crucial ingredient of Kirby-Siebenmann handlebody theory of topological manifolds of dimension $\geq 5$

- Now know as much about topological manifolds as about differentiable manifolds.