NONCOMMUTATIVE LOCALIZATION IN ALGEBRA AND TOPOLOGY

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- Papers by Beachy, Cohn, Dwyer, Linnell, Neeman, Ranicki, Reich, Sheiham and Skoda.
Noncommutative localization

Given a ring $A$ and a set $\Sigma$ of elements, matrices, morphisms, . . . , it is possible to construct a new ring $\Sigma^{-1}A$

the localization of $A$ inverting $\Sigma$.

In general, $A$ and $\Sigma^{-1}A$ are noncommutative.

Original algebraic motivation: construction of noncommutative analogues of the classical localization

$A = \text{integral domain} \iff \Sigma^{-1}A = \text{fraction field}$

with $\Sigma = A - \{0\} \subset A$.


Topological applications use the algebraic $K$- and $L$-theory of $A$ and $\Sigma^{-1}A$, with $A$ a group ring or a triangular matrix ring.
Ore localization

The Ore localization $\Sigma^{-1}A$ is defined for a multiplicatively closed subset $\Sigma \subset A$ with $1 \in \Sigma$, and such that for all $a \in A$, $s \in \Sigma$ there exist $b \in A$, $t \in \Sigma$ with $ta = bs \in A$.

E.g. central, $sa = as$ for all $a \in A$, $s \in \Sigma$.

The Ore localization is the ring of fractions

$$\Sigma^{-1}A = (\Sigma \times A) / \sim,$$

$(s, a) \sim (t, b)$ iff there exist $u, v \in A$ with

$$us = vt \in \Sigma, \ ua = vb \in A.$$

An element of $\Sigma^{-1}A$ is a noncommutative fraction

$$s^{-1}a = \text{equivalence class of } (s, a) \in \Sigma^{-1}A$$

with addition and multiplication more or less as usual.
Ore localization is flat

- An Ore localization $\Sigma^{-1}A$ is a flat $A$-module, i.e. the functor

  \[ \{ A\text{-modules} \} \to \{ \Sigma^{-1}A\text{-modules} \} ; M \mapsto \Sigma^{-1}A \otimes_A M = \Sigma^{-1}M \]

  is exact.

- For an Ore localization $\Sigma^{-1}A$ and any $A$-module $M$

  \[ \text{Tor}_i^A(\Sigma^{-1}A, M) = 0 \quad (i \geq 1). \]

- For an Ore localization $\Sigma^{-1}A$ and any $A$-module chain complex $C$

  \[ H_*(\Sigma^{-1}C) = \Sigma^{-1}H_*(C). \]
The universal localization of P.M.Cohn

- \( A = \) ring, \( \Sigma = \) a set of morphisms \( s : P \to Q \) of f.g. projective \( A \)-modules.
- A ring morphism \( A \to B \) is \( \Sigma \)-inverting if the \( B \)-module morphisms
  \[
  1 \otimes s : B \otimes_A P \to B \otimes_A Q \quad (s \in \Sigma)
  \]
  are isomorphisms.
- The universal localization \( \Sigma^{-1}A \) is a ring with a \( \Sigma \)-inverting morphism \( A \to \Sigma^{-1}A \) such that any \( \Sigma \)-inverting morphism \( A \to B \) has a unique factorization \( A \to \Sigma^{-1}A \to B \).
- The universal localization \( \Sigma^{-1}A \) exists (and it is unique); but it could be \( 0 \) – e.g if \( 0 \in \Sigma \).
- In general, \( \Sigma^{-1}A \) is not a flat \( A \)-module. \( \Sigma^{-1}A \) is a flat \( A \)-module if and only if \( \Sigma^{-1}A \) is an Ore localization (Beachy, Teichner (2003)).
The normal form (I)

(Gerasimov, Malcolmson (1981)) Every element $x \in \Sigma^{-1}A$ is of the form $x = fs^{-1}g$ for some $(s : P \to Q) \in \Sigma$, $f : P \to A$, $g : A \to Q$. Need to assume $\Sigma$ consists of all the morphisms $s : P \to Q$ of f.g. projective $A$-modules such that $1 \otimes s : \Sigma^{-1}P \to \Sigma^{-1}Q$ is a $\Sigma^{-1}A$-module isomorphism. (Can enlarge any $\Sigma$ to have this property).

For f.g. projective $A$-modules $M, N$ every $\Sigma^{-1}A$-module morphism $x : \Sigma^{-1}M \to \Sigma^{-1}N$ is of the form $x = fs^{-1}g$ for some $(s : P \to Q) \in \Sigma$, $f : P \to N$, $g : M \to Q$.

Addition by

$$fs^{-1}g + f's'^{-1}g' = (f \oplus f')(s \oplus s')^{-1}(g \oplus g') : \Sigma^{-1}M \to \Sigma^{-1}N.$$ 

Similarly for composition.
The normal form (II)

For f.g. projective $M, N$, a $\Sigma^{-1}A$-module morphism $f_{s^{-1}}g : \Sigma^{-1}M \to \Sigma^{-1}N$ is such that $f_{s^{-1}}g = 0$ if and only if there is a commutative diagram of $A$-module morphisms

$$
\begin{pmatrix}
    s & 0 & 0 & g \\
    0 & s_1 & 0 & 0 \\
    0 & 0 & s_2 & g_2 \\
    f & f_1 & 0 & 0
\end{pmatrix}
$$

$$
P \oplus P_1 \oplus P_2 \oplus M \quad \xto{L} \quad Q \oplus Q_1 \oplus Q_2 \oplus N
$$

with $s, s_1, s_2, (p \quad p_1 \quad p_2), (q \quad q_1 \quad q_2)^T \in \Sigma$.

(Exercise: diagram $\Rightarrow f_{s^{-1}}g = 0$).
The $K_0$-$K_1$ localization exact sequence

- Assume each $(s : P \to Q) \in \Sigma$ is injective and $A \to \Sigma^{-1}A$ is injective. The torsion exact category $T(A, \Sigma)$ has objects $A$-modules $T$ with $\Sigma^{-1}T = 0$, hom. dim. $(T) = 1$. E.g., $T = \text{coker}(s)$ for $s \in \Sigma$.

- **Theorem** (Bass (1968) for central $\Sigma^{-1}A$, Schofield (1985) for universal $\Sigma^{-1}A$). Exact sequence

$$K_1(A) \xrightarrow{\partial} K_1(\Sigma^{-1}A) \xrightarrow{\partial} K_0(T(A, \Sigma)) \xrightarrow{} K_0(A) \xrightarrow{} K_0(\Sigma^{-1}A)$$

$$\partial(\tau(fs^{-1}g : \Sigma^{-1}M \to \Sigma^{-1}N))$$

$$= [\text{coker}(\begin{pmatrix} f & 0 \\ s & g \end{pmatrix} : P \oplus M \to N \oplus Q)] - [\text{coker}(s : P \to Q)]$$

($(M, N$ based f.g. free $A$-modules).

- **Theorem** (Quillen (1972), Grayson (1980)) Higher $K$-theory localization exact sequence for Ore localization $\Sigma^{-1}A$, by flatness.
Universal localization is not flat

In general, if $M$ is an $A$-module and $C$ is an $A$-module chain complex

$$\text{Tor}^A_*(\Sigma^{-1}A, M) \neq 0, \quad H_*^{}(\Sigma^{-1}C) \neq \Sigma^{-1} H_*^{}(C).$$

Equality for Ore localization $\Sigma^{-1}A$, by flatness.

Example The universal localization $\Sigma^{-1}A$ of $A = \mathbb{Z}\langle x_1, x_2 \rangle$ inverting $\Sigma = \{x_1\}$ is not flat. The 1-dimensional f.g. free $A$-module chain complex

$$d_C = (x_1 \quad x_2) : \quad C_1 = A \oplus A \rightarrow C_0 = A$$

is a resolution of $H_0^{}(C) = \mathbb{Z}$ and

$$H_1^{}(\Sigma^{-1}C) = \text{Tor}_1^A(\Sigma^{-1}A, H_0^{}(C)) = \Sigma^{-1}A$$

$$\neq \Sigma^{-1}H_1^{}(C) = 0.$$
The lifting problem for chain complexes

- A lift of a f.g. free $\Sigma^{-1}A$-module chain complex $D$ is a f.g. projective $A$-module chain complex $C$ with a $\Sigma^{-1}A$-module chain equivalence
  \[ \Sigma^{-1}C \simeq D. \]

- For an Ore localization $\Sigma^{-1}A$ one can lift every $n$-dimensional f.g. free $\Sigma^{-1}A$-module chain complex $D$, for any $n \geq 0$.
- For a universal localization $\Sigma^{-1}A$ one can only lift for $n \leq 2$ in general.
- For $n \geq 3$ there are lifting obstructions in $\text{Tor}_i^A(\Sigma^{-1}A, \Sigma^{-1}A)$ for $i \geq 2$.
- $\text{Tor}_1^A(\Sigma^{-1}A, \Sigma^{-1}A) = 0$. 
Chain complex lifting = algebraic transversality

Typical example: the boundary map in the Schofield exact sequence

\[ \partial : K_1(\Sigma^{-1}A) \to K_0(T(A, \Sigma)) ; \tau(D) \mapsto [C] \]

sends the Whitehead torsion \( \tau(D) \) of a contractible based f.g. free \( \Sigma^{-1}A \)-module chain complex \( D \) to the projective class \( [C] \) of any lift f.g. projective \( A \)-module chain complex \( C \) with a \( \Sigma^{-1}A \)-module chain equivalence

\[ \Sigma^{-1}C \simeq D . \]

“Algebraic and combinatorial codimension 1 transversality”,
Stable flatness

- A universal localization $\Sigma^{-1}A$ is stably flat if
  \[
  \text{Tor}^A_i(\Sigma^{-1}A, \Sigma^{-1}A) = 0 \quad (i \geq 2).
  \]

- For stably flat $\Sigma^{-1}A$ have stable exactness:
  \[
  H_*(\Sigma^{-1}C) = \lim_{\longrightarrow} \Sigma^{-1}H_*(B)
  \]
  with $A$-module chain maps $C \to B$ such that $\Sigma^{-1}C \simeq \Sigma^{-1}B$.

- Flat $\iff$ stably flat.
  If $\Sigma^{-1}A$ is flat (i.e. an Ore localization) then
  \[
  \text{Tor}^A_i(\Sigma^{-1}A, M) = 0 \quad (i \geq 1)
  \]
  for every $A$-module $M$. The special case $M = \Sigma^{-1}A$ gives that $\Sigma^{-1}A$ is stably flat.
A localization which is not stably flat

Given a ring extension \( R \subset S \) and an \( S \)-module \( M \) define the \( S \)-module
\[
K(M) = \ker(S \otimes_R M \to M).
\]
For \( n \geq 1 \) let \( K^n(M) = K(K(\ldots K(M)\ldots)) \).

Theorem (Neeman, R. and Schofield)

(i) The universal localization of the ring
\[
A = \begin{pmatrix} R & 0 & 0 \\ S & R & 0 \\ S & S & R \end{pmatrix} = P_1 \oplus P_2 \oplus P_3 \text{ (columns)}
\]
inverting \( \Sigma = \{P_3 \subset P_2, P_2 \subset P_1\} \) is \( \Sigma^{-1}A = M_3(S) \).

(ii) If \( S \) is a flat \( R \)-module then
\[
\text{Tor}^A_{n-1}(\Sigma^{-1}A, \Sigma^{-1}A) = M_n(K^n(S)) \text{ (} n \geq 3 \).
\]

(iii) If \( R \) is a field and \( \dim_R(S) = d \) then \( K^n(S) = R^{(d-1)^nd} \).
If \( d \geq 2 \), e.g. \( S = R[x]/(x^d) \), then \( \Sigma^{-1}A \) is not stably flat.

Theorem of Neeman + R.

If $A \to \Sigma^{-1}A$ is injective and stably flat there is a ‘fibration sequence of exact categories’

$$T(A, \Sigma) \to P(A) \to P(\Sigma^{-1}A)$$

with $P(A)$ the category of f.g. projective $A$-modules, and every finite f.g. free $\Sigma^{-1}A$-module chain complex can be lifted.

It follows that there are long exact localization sequences in algebraic $K$- and $L$-theory

$$\cdots \to K_n(A) \to K_n(\Sigma^{-1}A) \to K_{n-1}(T(A, \Sigma)) \to \cdots$$

$$\cdots \to L_n(A) \to L_n(\Sigma^{-1}A) \to L_n(T(A, \Sigma)) \to \cdots .$$

Quadratic $L$-theory $L_\ast$ sequence obtained by Vogel (1982) without stable flatness; symmetric $L$-theory $L^\ast$ needs stable flatness.
Noncommutative localization in topology

- Applications to spaces $X$ with infinite fundamental group $\pi_1(X)$, e.g. amalgamated free products and $HNN$ extensions.
- The surgery classification of high-dimensional manifolds and Poincaré complexes, finite domination, fibre bundles over $S^1$, open books, circle-valued Morse theory, Morse theory of closed 1-forms, rational Novikov homology, codimension 1 and 2 splitting, homology surgery, knots and links.
The splitting problem in topology

- A homotopy equivalence $h : V \rightarrow W$ splits at a subspace $X \subset W$ if the restriction $h| : h^{-1}(X) \rightarrow X$ is also a homotopy equivalence. In general homotopy equivalences do not split, not even up to homotopy.

- For a homotopy equivalence of $n$-dimensional manifolds $h : V \rightarrow W$ and a codimension 1 submanifold $X \subset W$ there are algebraic $K$- and $L$-theory obstructions to splitting $h$ at $X$ up to homotopy. For $n \geq 6$ splitting up to homotopy is possible if and only if these obstructions are zero.

- For connected $X, W$ and injective $\pi_1(X) \rightarrow \pi_1(W)$ the splitting obstructions can be recovered from the algebraic $K$- and $L$-theory exact sequences of appropriate universal localizations expressing $\mathbb{Z}[\pi_1(W)]$ in terms of $\mathbb{Z}[\pi_1(X)]$ and $\mathbb{Z}[\pi_1(W - X)]$. 
Generalized free products

Seifert-van Kampen Theorem For any space

\[ W = X \times [0, 1] \cup_{X \times \{0,1\}} Y \]

such that \( W \) and \( X \) are connected the complement \( Y \) has either 1 or 2 components, and the fundamental group \( \pi_1(W) \) is a generalized free product:

(A) If \( Y \) is connected then \( \pi_1(W) \) is an HNN extension

\[
\pi_1(W) = \pi_1(Y) \ast_{i_1, i_2} \{ z \} = \pi_1(Y) \ast \{ z \} / \{ i_1(x)z = zi_2(x) \mid x \in \pi_1(X) \}
\]

with \( i_1, i_2 : \pi_1(X) \to \pi_1(Y) \) induced by the two inclusions \( i_1, i_2 : X \to Y \).

(B) \( Y \) is disconnected, \( Y = Y_1 \cup_X Y_2 \), then \( \pi_1(W) \) is an amalgamated free product

\[
\pi_1(W) = \pi_1(Y_1) \ast_{\pi_1(X)} \pi_1(Y_2)
\]

with \( i_1 : \pi_1(X) \to \pi_1(Y_1), \ i_2 : \pi_1(X) \to \pi_1(Y_2) \) induced by the inclusions \( i_1 : X \to Y_1, \ i_2 : X \to Y_2 \).
Mayer-Vietoris in homology and $K$-theory

Let $W = X \times [0, 1] \cup Y$. The homology groups fit into the Mayer-Vietoris exact sequence

$$
\cdots \to H_n(X) \xrightarrow{i_1 - i_2} H_n(Y) \to H_n(W) \xrightarrow{\partial} H_{n-1}(X) \to \cdots .
$$

The algebraic $K$-groups of $\mathbb{Z}[\pi_1(W)]$ for $W = X \times [0, 1] \cup Y$ with $\pi_1(X) \to \pi_1(W)$ injective fit into almost-Mayer-Vietoris exact sequence (Waldhausen (1972))

$$
\cdots \to K_n(\mathbb{Z}[\pi_1(X)]) \xrightarrow{i_1 - i_2} K_n(\mathbb{Z}[\pi_1(Y)]) \to K_n(\mathbb{Z}[\pi_1(W)]) \xrightarrow{\partial} \widetilde{\text{Nil}}_{n-1} \oplus K_{n-1}(\mathbb{Z}[\pi_1(X)]) \to \cdots .
$$

Also $L$-theory: UNil-groups (Cappell (1974)).

The almost-Mayer-Vietoris sequences are the localization exact sequences for the “Mayer-Vietoris localizations” $\Sigma^{-1}A$ of triangular matrix rings $A$. 
The Seifert-van Kampen localization (I)

Let \( W = X \times [0, 1] \cup Y \). The expression of \( \pi_1(W) \) as generalized free product motivates an expression of the \( k \times k \) matrix ring of \( \mathbb{Z}[\pi_1(W)] \) as a universal localization

\[
M_k(\mathbb{Z}[\pi_1(W)]) = \Sigma^{-1}A \quad (k = 2 \text{ or } 3)
\]

of a triangular matrix ring \( A \).

(A) If \( Y \) is connected take \( k = 2 \),

\[
A = \begin{pmatrix}
\mathbb{Z}[\pi_1(X)] & 0 \\
\mathbb{Z}[\pi_1(Y)]_1 \oplus \mathbb{Z}[\pi_1(Y)]_2 & \mathbb{Z}[\pi_1(Y)]
\end{pmatrix}
\]

(\( \Sigma \) defined in “HNN extensions” below).

(B) If \( Y = Y_1 \cup Y_2 \) is disconnected take \( k = 3 \),

\[
A = \begin{pmatrix}
\mathbb{Z}[\pi_1(X)] & 0 & 0 \\
\mathbb{Z}[\pi_1(Y_1)] & \mathbb{Z}[\pi_1(Y_1)] & 0 \\
\mathbb{Z}[\pi_1(Y_2)] & 0 & \mathbb{Z}[\pi_1(Y_2)]
\end{pmatrix}
\]

(\( \Sigma \) defined in “Amalgamated free products”).
The Seifert-van Kampen localization (II)

▶ A map \( h : V^n \to W = X \times [0, 1] \cup Y \) on an \( n \)-manifold \( V \) is \underline{transverse} at \( X \subset W \) if

\[
T^{n-1} = h^{-1}(X), \quad U^n = h^{-1}(Y) \subset V^n
\]

are submanifolds, so \( V = T \times [0, 1] \cup U \).

▶ The localization functor

\[
\{ A\text{-modules} \} \to \{ \Sigma^{-1}A\text{-modules} \} ; M \mapsto \Sigma^{-1}M
\]

is an algebraic analogue of the forgetful functor

\[
\{ \text{transverse maps } V \to W \} \to \{ \text{maps } V \to W \}.
\]

▶ For any map \( V \to W \) \( C(\tilde{V}) \) is a \( \Sigma^{-1}A \)-module chain complex, up to Morita equivalence. For a transverse map \( h : V = T \times [0, 1] \cup U \to W \) the Mayer-Vietoris presentation of \( C(\tilde{V}) \) is an \( A \)-module chain complex \( \Gamma \) with assembly \( \Sigma^{-1}\Gamma = C(\tilde{V}) \).
Morita theory

▶ For any ring $R$ and $k \geq 1$ let $M_k(R)$ be the ring of $k \times k$ matrices in $R$.

▶ Proposition The functors

\[
\text{(1)} \quad \{R\text{-modules}\} \to \{M_k(R)\text{-modules}\} ; \quad M \mapsto \begin{pmatrix} R & & \\ & R & \\ & & \ddots \end{pmatrix} \otimes_R M,
\]

\[
\text{(2)} \quad \{M_k(R)\text{-modules}\} \to \{R\text{-modules}\} ; \quad N \mapsto (R R \ldots R) \otimes_{M_k(R)} N
\]

are inverse equivalences of categories.

▶ Proposition $K_*(M_k(R)) = K_*(R)$. 
Algebraic $K$-theory of triangular rings

- Given rings $A_1, A_2$ and an $(A_2, A_1)$-bimodule $B$ define the triangular matrix ring $A = \begin{pmatrix} A_1 & 0 \\ B & A_2 \end{pmatrix}$. The f.g. projective $A$-modules $P_1 = \begin{pmatrix} A_1 \\ B \end{pmatrix}$, $P_2 = \begin{pmatrix} 0 \\ A_2 \end{pmatrix}$ are such that $A = P_1 \oplus P_2$.

- Proposition (i) The category of $A$-modules is equivalent to the category of triples $M = (M_1, M_2, \mu : B \otimes_{A_1} M_1 \rightarrow M_2)$ with $M_i$ an $A_i$-module and $\mu$ an $A_2$-module morphism.
  (ii) $K_\ast(A) = K_\ast(A_1) \oplus K_\ast(A_2)$.
  (iii) If $A \rightarrow S$ is a ring morphism such that there is an $S$-module isomorphism $S \otimes_A P_1 \cong S \otimes_A P_2$ then $S = M_2(R)$ with $R = \text{End}_S(S \otimes_A P_1)$, inducing the assembly functor

$$\{A\text{-modules}\} \rightarrow \{S\text{-modules}\} \cong \{R\text{-modules}\} ; \quad M \mapsto (R \oplus R) \otimes_A M = \text{coker}(R \otimes_{A_2} B \otimes_{A_1} M_1 \rightarrow R \otimes_{A_1} M_1 \oplus R \otimes_{A_2} M_2).$$
The stable flatness theorem

Theorem Let $A = \begin{pmatrix} A_1 & 0 \\ B & A_2 \end{pmatrix} \rightarrow \Sigma^{-1}A = M_2(R)$ be the noncommutative localization inverting a set $\Sigma$ of $A$-module morphisms $s : P_2 = \begin{pmatrix} 0 \\ A_2 \end{pmatrix} \rightarrow P_1 = \begin{pmatrix} A_1 \\ B \end{pmatrix}$ with $R = \text{End}(\Sigma^{-1}P_i) \ (i = 1, 2)$. If $B$ and $R$ are flat $A_1$-modules and $R$ is a flat $A_2$-module then $\Sigma^{-1}A$ is stably flat.

Proof The $A$-module $M = \begin{pmatrix} R \\ R \end{pmatrix}$ has a 1-dimensional flat $A$-module resolution

$$0 \rightarrow \begin{pmatrix} 0 \\ B \end{pmatrix} \otimes A_1 R \rightarrow \begin{pmatrix} A_1 \\ B \end{pmatrix} \otimes A_1 R \oplus \begin{pmatrix} 0 \\ A_2 \end{pmatrix} \otimes A_2 R \rightarrow M \rightarrow 0$$

and hence so does $\Sigma^{-1}A = M \oplus M$.

Remark $\text{Tor}_1^A((0 \ A_2), E) = \ker(B \otimes A_1 R \rightarrow R)$, so in general $\Sigma^{-1}A$ is not flat.
**HNN extensions**

The **HNN extension** of ring morphisms $i_1, i_2 : R \to S$ is

$$S *_{i_1, i_2} \{z\} = S * \mathbb{Z}/\{ i_1(x)z = zi_2(x) \mid x \in R \}.$$  

Let $S_j = S$ with $(S, R)$-bimodule structure

$$S \times S_j \times R \to S_j ; (s, t, u) \mapsto st_i(u).$$

The Seifert-van Kampen localization of $A = \begin{pmatrix} R & 0 \\ S_1 \oplus S_2 & S \end{pmatrix}$ inverts $\Sigma = \{\sigma_1, \sigma_2\}$ with

$$\sigma_1, \sigma_2 : \begin{pmatrix} 0 \\ S \end{pmatrix} \to \begin{pmatrix} R \\ S_1 \oplus S_2 \end{pmatrix}$$

the inclusions, and

$$\Sigma^{-1}A = M_2(S *_{i_1, i_2} \{z\}).$$
The Nil-groups are the torsion $K$-groups of a noncommutative localization.

**Theorem** If $i_1, i_2 : R \to S$ are split injections and $S_1, S_2$ are flat $R$-modules then

$$
A = \begin{pmatrix} R & 0 \\ S_1 \oplus S_2 & S \end{pmatrix} \to \Sigma^{-1}A = M_2(S \ast_{i_1,i_2} \{z\})
$$

is injective and stably flat. The algebraic $K$-theory localization exact sequence identifies

$$
K_n(A) = K_n(R) \oplus K_n(S),
$$
$$
K_n(\Sigma^{-1}A) = K_n(S \ast_{i_1,i_2} \{z\}),
$$
$$
K_n(T(A, \Sigma)) = K_n(R) \oplus K_n(R) \oplus \tilde{\text{Nil}}_n.
$$
Amalgamated free products

The amalgamated free product \( S_1 \ast_R S_2 \) is defined for ring morphisms \( R \to S_1, \ R \to S_2 \). The Seifert-van Kampen localization of

\[
\begin{pmatrix}
R & 0 & 0 \\
S_1 & S_1 & 0 \\
S_2 & 0 & S_2
\end{pmatrix}
\]

inverts \( \Sigma = \{ \sigma_1, \sigma_2 \} \),

\[
\sigma_1 : \begin{pmatrix} 0 \\ S_1 \\ 0 \end{pmatrix} \to \begin{pmatrix} R \\ S_1 \\ S_2 \end{pmatrix}, \quad \sigma_2 : \begin{pmatrix} 0 \\ S_2 \end{pmatrix} \to \begin{pmatrix} R \\ S_1 \\ S_2 \end{pmatrix}
\]

the inclusions,

\[
\Sigma^{-1} A = M_3(S_1 \ast_R S_2).
\]

Theorem If \( R \to S_1, \ R \to S_2 \) are split injections with \( S_1, S_2 \) flat \( R \)-modules then \( A \to \Sigma^{-1} A \) is injective and stably flat. The algebraic \( K \)-theory localization exact sequence has

\[
K_n(A) = K_n(R) \oplus K_n(S_1) \oplus K_n(S_2), \quad K_n(\Sigma^{-1} A) = K_n(S_1 \ast_R S_2),
\]

\[
K_n(T(A, \Sigma)) = K_n(R) \oplus K_n(R) \oplus \widetilde{\text{Nil}}_n.
\]

Again, the Nil-groups are the torsion \( K \)-groups of \( \Sigma^{-1} A \).
The algebraic $L$-theory of a triangular ring

- If $A_1, A_2, B$ have involutions then $A = \begin{pmatrix} A_1 & 0 \\ B & A_2 \end{pmatrix}$ may not have an involution.

- Involutions on $A_1, A_2$ and a symmetric isomorphism $\beta : B \to \text{Hom}_{A_2}(B, A_2)$ give a "chain duality" involution on the derived category of $A$-module chain complexes.

- The dual of an $A$-module $M = (M_1, M_2, \mu)$ is the $A$-module chain complex

$$d = (0, \beta^{-1} \mu^*) : C_1 = (0, M_2^*, 0) \to C_0 = (M_1^*, B \otimes_{A_1} M_1^*, 1)$$

- The quadratic $L$-groups of $A$ are just the relative $L$-groups in the sequence

$$\cdots \to L_n(A_1) \to \otimes(B, \beta) L_n(A_2) \to L_n(A) \to L_{n-1}(A_1) \to \cdots.$$
The algebraic $L$-theory of a noncommutative localization

**Theorem** Let $\Sigma^{-1}A$ be the localization of a triangular ring $A = \begin{pmatrix} A_1 & 0 \\ B & A_2 \end{pmatrix}$ with chain duality inverting a set $\Sigma$ of $A$-module morphisms $s : P_1 = \begin{pmatrix} 0 \\ A_2 \end{pmatrix} \to P_2 = \begin{pmatrix} A_1 \\ B \end{pmatrix}$, so that

$$\Sigma^{-1}A = M_2(D)$$

with $D = \text{End}(\Sigma^{-1}P_1)$. If $B$ and $D$ are flat $A_1$-modules and $D$ is a flat $A_2$-module then $\Sigma^{-1}A$ is stably flat,

$$L_*(\Sigma^{-1}A) = L_*(D) \text{ (Morita)}$$

and there is an exact sequence

$$\cdots \to L_n(A) \to L_n(D) \to L_n(T(A, \Sigma)) \to L_{n-1}(A) \to \cdots .$$
The UNil groups are the torsion $L$-groups of a noncommutative localization

**Theorem** Let $D = S_1 *_R S_2$ be the amalgamated free product of split injections $R \to S_1$, $R \to S_2$ of rings with involution, and let $A \to \Sigma^{-1}A = M_3(D)$ be the Seifert-van Kampen localization of $A = \begin{pmatrix} R & 0 & 0 \\ S_1 & S_1 & 0 \\ S_2 & 0 & S_2 \end{pmatrix}$. If $S_1, S_2$ are flat $R$-modules then

$$L_n(\Sigma^{-1}A) = L_n(D) = L_n(A) \oplus L_n(T(A, \Sigma)),$$

$$L_n(T(A, \Sigma)) = \text{UNil}_n(R; S_1, S_2).$$

**Similarly** for the UNil-groups of an HNN extension $D = S *_{i_1, i_2} \{z\}$ of split injective morphisms $i_1, i_2 : R \to S$ of rings with involution with $S_1$ and $S_2$ flat $R$-modules, and the $S$-vK localization $\Sigma^{-1}A = M_2(D)$. 
A polynomial extension is a noncommutative localization

- For any ring $R$ define triangular matrix ring

$$A = \begin{pmatrix} R & 0 \\ R \oplus R & R \end{pmatrix}.$$ 

An $A$-module is a quadruple $M = (K, L, \mu_1, \mu_2 : K \to L)$ with $K, L$ $R$-modules and $\mu_1, \mu_2$ $R$-module morphisms.

- The localization of $A$ inverting

$$\Sigma = \{ \sigma_1, \sigma_2 : \begin{pmatrix} 0 \\ R \end{pmatrix} \to \begin{pmatrix} R \\ R \oplus R \end{pmatrix} \}$$

is a ring morphism $A \to \Sigma^{-1}A = M_2(S)$ with $S = R[z, z^{-1}]$.

- The localization functor

$$\Sigma^{-1} : \{A\text{-modules}\} \to \{M_2(S)\text{-modules}\} \approx \{S\text{-modules}\}; M \mapsto \Sigma^{-1}M$$

sends an $A$-module $M$ to the assembly $S$-module

$$(S \quad S) \otimes_A M = \text{coker}(\mu_1 - z\mu_2 : K[z, z^{-1}] \to L[z, z^{-1}]).$$
Manifolds over $S^1$

Given a map $f : V^n \to S^1$ on an $n$-manifold $V$ which is transverse at $\{\text{pt.}\} \subset S^1$ cut $V$ along the codimension 1 submanifold $T^{n-1} = f^{-1}(\{\text{pt.}\}) \subset V$ to obtain

$$V = T \times [0, 1] \cup_{T \times \{0,1\}} U.$$ 

The cobordism $(U; T_1, T_2)$ is a fundamental domain for the infinite cyclic cover $\overline{V} = f^*\mathbb{R}$ of $V$, with $T_1$, $T_2$ copies of $T$.

Let $A = \left( \begin{array}{cc} \mathbb{Z} & 0 \\ \mathbb{Z} \oplus \mathbb{Z} & \mathbb{Z} \end{array} \right)$, $\Sigma^{-1}A = M_2(\mathbb{Z}[z, z^{-1}])$.

The $A$-module chain complex

$$\Gamma = (C(T), C(U), \mu_1, \mu_2 : C(T) \to C(U))$$

induces the assembly $\mathbb{Z}[z, z^{-1}]$-module chain complex

$$(\mathbb{Z}[z, z^{-1}] \otimes \mathbb{Z}[z, z^{-1}]) \otimes_A \Gamma$$

$$= \text{coker}(\mu_1 - z\mu_2 : C(T)[z, z^{-1}] \to C(U)[z, z^{-1}])$$

$$= C(\overline{V}) .$$
Manifolds over $\bigvee_{\mu} S^1$

The complement of a $\mu$-component boundary link $\bigcup_{\mu} S^n \subset S^{n+2}$ is equipped with a map

$$M = S^{n+2} \setminus (\bigcup_{\mu} S^n) \to \bigvee_{\mu} S^1$$

which induces a surjection $\pi_1(M) \to \pi_1(\bigvee_{\mu} S^1) = F_{\mu}$ and isomorphisms $H_\ast(M) \cong H_\ast(\bigvee_{\mu} S^1)$.

Noncommutative localization methods apply to the algebraic K- and L-theory of $A[F_{\mu}]$, for any ring $A$. For $A = \mathbb{Z}$ applications to the classification of high-dimensional boundary links.

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