Wu classes and unoriented bordism classes
of certain manifolds

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§1. Introduction

Let $M$ be a closed manifold, and let $w_i$ and $v_i$ be the $i$th Stiefel-Whitney class and the $i$th Wu class of $M$, respectively. Then, the Wu formula means that they are related by the equality

\[(1.1) \quad v_n = \sum_{i=1}^{\lfloor n/2 \rfloor} \theta^{n-i} w_i \]

(cf. Proposition 3.2), where $\theta^l = c(Sq^l) \in \mathcal{A}(2)$ is the conjugation of $Sq^l$ given in [7, II, §4] and is defined inductively by

\[\theta^1 = Sq^1 + \sum_{j=1}^{\lfloor \frac{l-1}{2} \rfloor} Sq^j \theta^{l-j} \quad (l \geq 0).\]

The main purpose of this paper is to study the Wu classes by using (1.1).

To do this, we study the element $\theta^l$ in §2, and prove the following basic formula (Theorem 2.4), where we use always the notation

\[t' = 2^{t-1}\]

for any positive integer $t$:

\[(1.2) \quad \text{If } n = 2^k - 1, \text{ then} \]

\[\theta^n = Sq^k Sq^{(k-1)} \cdots Sq^1;\]

and if $n = 2^k - 1 - t_1 - \cdots - t_l$ with $k \geq t_1 > \cdots > t_l \geq 1$, then

\[\theta^n = \sum_{1 \leq p_1 < \cdots < p_l \leq k} Sq^{I(p_1, \ldots, p_l)},\]

where $I(p_1, \ldots, p_l) = (i_1, \ldots, i_k)$ is given by

\[i_p = (k - p + 1)' - t'_s \quad (s = 1, \ldots, l), \quad i_p = (k - p + 1)' \quad (p \neq p_1, \ldots, p_l),\]

and $Sq^{I(p_1, \ldots, p_l)} = Sq^{i_1} \cdots Sq^{i_k}$ with $Sq^0 = 1$ and $Sq^i = 0$ for $i < 0$.

As an application of this formula, we see the well known formula

\[\theta^{2n+1} = \theta^n Sq^1\]

(Corollary 2.14) and the one given by D. M. Davis [2, Th. 2] (Corollary 2.16). By using the former, we can reduce the equality (1.1) to the form given in Theorem 3.9, and we obtain the equality
v_{2n+1} = \sum_{i \geq 1} (w_i)^{2i-1} v_{2n+2-2i} \tag{Theorem 3.10}

We notice that this equality implies immediately the well known result that the odd dimensional Wu class \( v_{2n+1} \) of an oriented manifold \( M \) vanishes.

In § 4, we are concerned with a closed manifold \( M \) whose total Stiefel-Whitney class \( wM \) satisfies the condition

\[
(1.3) \quad wM = 1 + \sum_{b \geq 1} w_{b'} \quad (b' = 2^{b-1}).
\]

For such a manifold, by noticing that \( w_{b'}w_{c'} = 0 \) if \( c \geq b + 2 \) (Proposition 4.2) and by using (1.2), we can reduce (1.1) to the following explicit form (Theorem 4.3):

\[
\begin{align*}
(1.4) \quad v_i &= \sum_{a=1}^{\infty} (w_{b'})^{(a-b+1)}' \quad \text{if } i = a' \geq 1, \\
&= \sum_{a_1, a_2 = 1} w_{b'}^{(-j')/b'} (w_{2b'})^{(j-b)'} \quad \text{if } i = a_1 + a_2' \text{ with } a_1 > a_2 \geq 1, \\
v_i &= 0 \quad \text{otherwise}.
\end{align*}
\]

Some examples of manifolds satisfying (1.3) are given at the end of § 4.

These equalities are applied in § 5 to study some sufficient conditions that the unoriented bordism class \([M]\) of \( M \) with (1.3) vanishes. In fact, under (1.3) and the condition that \( \dim M \) is not equal to a power of 2, we can show that almost all the Stiefel-Whitney numbers of \( M \) vanish by using (1.4) and the fact that \( v_i = 0 \) for \( i > \dim M/2 \); and we obtain the following results (Theorems 5.1 and 5.4):

**Theorem.** Let \( M \) be a closed manifold. Then, the unoriented bordism class \([M]\) of \( M \) is 0, if one of the following three conditions holds:

1. The total Stiefel-Whitney class \( wM \) satisfies (1.3), and \( \dim M = p_1 + \cdots + p_k + 1 \) with \( p_1 > \cdots > p_k > 1 \) and \( k \geq 2 \), \( (p' = 2^{p-1}) \).

2. \( wM = 1 + w_b + w_c \) for some \( b \) and \( c \) with \( c > b \geq 1 \) in (1.3), and \( \dim M \) is not a power of 2.

3. \( wM = 1 + w_i \) for some \( i \geq 1 \).

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**§ 2. Some relations in the mod 2 Steenrod algebra**

Let \( \mathcal{A}(2) \) be the mod 2 Steenrod algebra. For any sequence \( I = (i_1, \ldots, i_k) \) of positive integers, put

\[ Sq^I = Sq^{i_1} \cdots Sq^{i_k} \in \mathcal{A}(2), \quad |I| = i_1 + \cdots + i_k; \]
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and define the element $\theta^n \in \mathcal{A}(2)$ by

$$\theta^0 = 1, \quad \theta^n = \sum_{|I|=n} Sq^I \quad (n \geq 1).$$

Then, we have clearly the relations

$$(2.1) \quad \theta^n = Sq^n + \sum_{l=1}^n Sq^l \theta^{n-l} = Sq^n + \sum_{j=1}^n \theta^{n-j} Sq^j \quad (n \geq 0),$$

which give the inductive definition of $\theta^n$. Thus, it is easily seen that $\theta^n$ is equal to $c(Sq^n)$ in [7, p. 26] or $\chi(Sq^n)$ in [2].

To study $\theta^n$, we use the following notation:

Let $I=(i_1, \ldots, i_k)$ and $T=(t_1, \ldots, t_l)$ be sequences of positive integers. Put

$$S_q^I - (T) = \sum_{1 \leq p_1 < \cdots < p_l \leq k} Sq^{I-T(p_1, \ldots, p_l)}$$

where $I-T(p_1, \ldots, p_l)=(j_1, \ldots, j_k)$ is given by

$$j_{p_s} = i_{p_s} - t_s \quad (s=1, \ldots, l), \quad j_p = i_p \quad (p \neq p_1, \ldots, p_l),$$

and $Sq^{(j_1, \ldots, j_k)} = Sq^{j_1} \cdots Sq^{j_k}$ under the convention that

$$(*) \quad Sq^0 = 1 \quad \text{and} \quad Sq^l = 0 \quad \text{if} \quad j < 0.$$  

Then, $Sq^I-(T)$ can be defined inductively on the lengths $k$ of $I$ and $l$ of $T$ by

$$(2.2) \quad Sq^I - (T) = \sum_{1 \leq p_1 < \cdots < p_l \leq k} Sq^{I-T(p_1, \ldots, p_l)}$$

where $I-T(p_1, \ldots, p_l)=(j_1, \ldots, j_k)$ is given by

$$j_{p_s} = i_{p_s} - t_s \quad (s=1, \ldots, l), \quad j_p = i_p \quad (p \neq p_1, \ldots, p_l),$$

and $Sq^{(j_1, \ldots, j_k)} = Sq^{j_1} \cdots Sq^{j_k}$ under the convention $(*)$, where $J_s=(j_1, \ldots, j_{s-1}, j_{s+1}, \ldots, j_m)$ for $J=(j_1, \ldots, j_m)$.

Furthermore, put $Sq^I-(t)=Sq^I-(J(t))$ and

$$\theta^n-(t) = \sum_{|I|=n} \{Sq^I - (t)\} \quad \text{for} \quad n, t \geq 0,$$

where $J(t)=(2^{t-1}, 2^{t-2}, \ldots, 1)$. Then we see the following

**Proposition 2.3.** $\theta^n-(t) = \left\{ \begin{array}{ll} 0 & \text{for} \quad n < 2^t - 1, \\ \theta^{n-2^{t+1}} & \text{for} \quad n \geq 2^{t-1} \geq 0. \end{array} \right.$

**Proof.** The equality for $n < 2^t - 1$ or $t=0$ is seen immediately by definition. We prove the equality for $n \geq 2^t - 1 \geq 1$ by the induction on $n$. By (2.1)', (2.2)' and the above definition, we see that

$$\theta^n-(t) = \sum_{l=1}^n Sq^{I-t'}(\theta^{n-l}-(t-1)) + \sum_{t'=1}^n Sq^I(\theta^{n-t}-(t)) \quad (t'=2^{t-1}).$$

By the equality for $n < 2^t - 1$, the inductive assumption and (2.1)', this is equal to

$$\sum_{l=t'+1}^n Sq^{I-t'} \theta^{n-l-t'+1} + \sum_{l=1}^{t'+1} Sq^l \theta^{n-l-2t'+1} = \theta^{n-2t'+1},$$
as desired. q.e.d.

Now, the main purpose in this section is to prove the following theorem, where we use always the notation

\[ t' = 2^{t-1} \quad \text{for any positive integer } t. \]

**Theorem 2.4.** (i) Let \( n = 2^k - 1 \). Then

\[ \theta^n = Sq^{J(k)} \quad (J(k) = (k', (k-1)', \ldots, 1)) \]

(ii) Let

\[ n = 2^k - 1 - t_1' - \cdots - t_l' = 2^k - 1 - |T| \]

for \( T = (t_1', \ldots, t_l') \) with \( k \geq t_1 > \cdots > t_l \geq 1 \) and \( l \geq 1 \). Then,

\[ \theta^n = Sq^{J(k)} - (T) \quad (J(k) = (k', (k-1)', \ldots, 1)), \]

where the right hand side is given by (2.2).

By this theorem and (2.2)', we have the following

**Corollary 2.5.** For \( n \) in (ii) of the above theorem with \( k > t_1 \),

\[ \theta^n = Sq^a \theta^{n-a} + Sq^{k'} \theta^{n-k'} \quad \text{where } a = k' - t_1'. \]

**Proof.** By the above theorem and (2.2)', \( \theta^n \) is equal to

\[ Sq^{J(k)} - (T) = Sq^a \{ Sq^{J(k-1)} - (T_1) \} + Sq^{k'} \{ Sq^{J(k-1)} - (T) \} \]

\[ = Sq^a \theta^{k-1-|T_1|} + Sq^{k'} \theta^{k-1-|T|} \quad (T_1 = (t_2', \ldots, t_l')), \]

which is equal to the right hand side of the desired equality. q.e.d.

To prove Theorem 2.4, we prepare several results.

Let \( P (= RP^\infty) \) be the \( \infty \)-dimensional real projective space and \( P^m \) be the \( m \)-fold Cartesian product of \( P \). Let \( u \) be the generator of \( H^1(P; Z_2) = Z_2 \), and consider the cohomology class

\[ u_1 \times \cdots \times u_m \in H^m(P^m; Z_2) \quad (u_1 = \cdots = u_m = u). \]

Furthermore for any sequence \( A = (a_1, \ldots, a_m) \) of positive integers, we consider the cohomology class

\[ u(A) = u_1(a_1) \times \cdots \times u_m(a_m) \in H^*(P^m; Z_2) \quad (u(a) = u^{a'}, \ a' = 2^{a-1}). \]

Then, we have the following proposition, where \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_m) \) is a sequence with \( \varepsilon_i = 0 \) or 1 and \( A + \varepsilon = (a_1 + \varepsilon_1, \ldots, a_m + \varepsilon_m) \) and \( \|A\| = a'_1 + \cdots + a'_m \), for \( A = (a_1, \ldots, a_m) \):

**Proposition 2.6.** In \( H^*(P^m; Z_2) \), there hold the equalities
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\[(i) \quad Sq^u(A) = \sum_{\|A + \varepsilon\| = \|A\|} u(A + \varepsilon),\]
\[(ii) \quad \theta^n(u_1 \times \cdots \times u_m) = \sum_{\|A\| = n + m} u(A).\]

**Proof.** Let \(x\) be any 1-dimensional cohomology class. Then, the equality
\[Sq^l x = \binom{k}{i} x^{k+i}\] of \([7, I, Lemma 2.4]\) implies
\[(2.7) \quad Sq^l(x(a)) = \begin{cases} x(a + \varepsilon) & \text{if } i = \pm a', \varepsilon = 0 \text{ or } 1, \\ 0 & \text{otherwise}, \end{cases}\]
where \(x(b) = x^{b'} (b' = 2^{b-1})\). Thus, we see by definition that
\[(2.7)' \quad Sq^l x = \begin{cases} x(l) & \text{if } l = J(l - 1), \\ 0 & \text{otherwise}; \end{cases}\]
\[(2.8) \quad \theta^n x = \begin{cases} x^{n+1} = x(l) & \text{if } n = l' - 1 \geq 0, \\ 0 & \text{otherwise.} \end{cases}\]

(i) follows immediately from (2.7) and the Cartan formula.

(ii) By the Cartan formula and (2.7)', we see easily that
\[(2.9) \quad Sq^l(u_1 \times u_2 \times \cdots \times u_m) = \sum_{t \geq 1} u_1(t) \times (Sq^l - (l - 1))(u_2 \times \cdots \times u_m).\]

Therefore, by (2.1) and Proposition 2.3,
\[\theta^n(u_1 \times u_2 \times \cdots \times u_m) = \sum_{t \geq 1} u_1(t) \times \theta^{n-t'+1}(u_2 \times \cdots \times u_m)\]
\[= \cdots = \sum u_1(a_1) \times \cdots \times u_{m-1}(a_{m-1}) \times \theta^{n-a}(u_m),\]
where \(a = (a'_1 - 1) + \cdots + (a'_{m-1} - 1)\). Hence, we see the equality (ii) by (2.8)
q.e.d.

For the case \(m = n\) in (ii) of the above proposition, we have the following lemma, where \(A\) and \(B\) are sequences of \(n\) positive integers and \(\varepsilon\) and \(\rho\) are sequences of \(n\) integers consisting of 0 or 1:

**Lemma 2.10.** (i) If \(n = 2^k - 1 \geq 1\), then
\[\theta^n(u_1 \times \cdots \times u_n) = \sum_{\|A\| = k' + n - 1} \sum_{\|A + \varepsilon\| = 2n} u(A + \varepsilon).\]

(ii) If \(n = 2^k - t' - s\) with \(k > t \geq 1\) and \(1 \leq s \leq t'\), then
\[\theta^n(u_1 \times \cdots \times u_n) = \sum_{\|A\| = k' - s + n} \sum_{\|A + \varepsilon\| = 2n} u(A + \varepsilon)\]
\[+ \sum_{\|B\| = k' - t' - s + n} \sum_{\|B + \rho\| = 2n} u(B + \rho).\]

**Proof.** (ii) Let \(C = (c_1, \ldots, c_a)\) be a sequence of positive integers with \(\|C\| = 2n\), and assume that \(u(C)\) appears \(a\) and \(b\) times in the first and the second
summations in the right hand side of the equality in (ii), respectively. Then, by (ii) of the above proposition, it is sufficient to prove that

\[ a + b = \text{odd}. \]

Assume that a positive integer \( l \) appears \( \alpha_i \) times in \( C \). Then

\[
(2.11) \quad \alpha_i \geq 0, \quad \sum_{l \geq 1} \alpha_i = n \quad \text{and} \quad \sum_{l \geq 1} l \alpha_i = 2n.
\]

Furthermore, in the first summation in the right hand side of the equality in (ii), the equality \( A + \varepsilon = C \) holds if and only if \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \) satisfies the condition that

\[ (\ast) \quad 0 \leq p_l \leq \alpha_i (l \geq 2), \quad \sum_{l \geq 2} (l-1) p_l = 2n - (k' - s + n) = k' - t', \]

where \( p_i \) is the number of elements of \( \{ i \mid c_i = l, \varepsilon_i = 1 \} \). Thus

\[ a = \sum_{(\ast)} \left( \begin{array}{c} \alpha_2 \\ p_2 \end{array} \right) \cdots \left( \begin{array}{c} \alpha_i \\ p_i \end{array} \right) \cdots , \]

which is equal to the coefficient of \( x^{k' - t'} \) in the polynomial \((1 + x)^{\alpha_i/2} \cdots (1 + x^{(l-1)})^{\alpha_l} \cdots \). By (2.11), this polynomial is congruent to \((1 + x)^{n - \alpha_i/2} \mod 2\). Hence

\[ a \equiv \left( \frac{n - \alpha_i/2}{k' - t'} \right) \mod 2. \]

By the same way, since \( 2n - (k' - t' - s + n) = k' \), we see that

\[ b \equiv \left( \frac{n - \alpha_i/2}{k'} \right) \mod 2. \]

On the other hand, by using the well known formula

\[
(2.12) \quad \left( \frac{\beta}{\alpha} \right) \equiv \prod_i \left( \frac{b_i}{a_i} \right) \mod 2 \quad \text{for} \quad \alpha = \sum_i a_i 2^i, \beta = \sum_i b_i 2^i (0 \leq a_i, b_i \leq 1),
\]

we see easily that

\[ \left( \frac{n - \alpha_i/2}{k' - t'} \right) \equiv \begin{cases} 1 \mod 2 & (n/2 \leq n - \alpha_i/2 \leq k' - 1) \\
0 \mod 2 & (k' \leq n - \alpha_i/2 \leq n), \end{cases} \]

\[ \left( \frac{n - \alpha_i/2}{k'} \right) \equiv \begin{cases} 0 \mod 2 & (n/2 \leq n - \alpha_i/2 \leq k' - 1) \\
1 \mod 2 & (k' \leq n - \alpha_i/2 \leq n), \end{cases} \]

since \( n = 2k' - t' - s \geq \alpha_i \geq 0 \) with \( k' > t' \geq s \geq 1 \). Thus \( a + b \equiv 1 \mod 2 \), and (ii) is proved.
(i) can be proved similarly by noticing \(2n - (k' + n - 1) = k'\) and \(\binom{n - a_1/2}{k'}\) \(\equiv 1 \mod 2\) for \(n = 2k' - 1 \geq a_1 \geq 0\).

By using the above results, we can prove Theorem 2.4.

**Proof of Theorem 2.4.** (i) Since \(\theta^1 = Sq^1\), we see (i) for \(k = 1\). Assume inductively that (i) holds for \(k - 1\). Then

\[
Sq^{j(k)} = Sq^k \theta^{k-1}.
\]

On the other hand, by Proposition 2.6 and Lemma 2.10(i), we see that

\[
Sq^k \theta^{k-1} (u_1 \times \cdots \times u_n) = Sq^k \sum_{\|A\| = k-1 + n} u(A)
\]

\[
= \sum_{\|A\| = k-1 + n} \sum_{\|A + \varepsilon\| = 2n} u(A + \varepsilon) = \theta^n (u_1 \times \cdots \times u_n) \quad (n = 2k' - 1).
\]

Therefore \(Sq^k \theta^{k-1} = \theta^n\) by the following fundamental result in [7, I, Cor. 3.3]:

\[ (2.13) \quad \text{The homomorphism } \mathfrak{a}(2) \rightarrow H^*(P^n; \mathbb{Z}_2) \text{ given by } Sq^1 \rightarrow Sq^1(u_1 \times \cdots \times u_n) \text{ is a monomorphism in degree } \leq m. \]

Thus, we obtain \(\theta^n = Sq^{j(k)}\) as desired.

(ii) We prove (ii) by the induction on \(k\). If \(k = 1\), then (ii) is clear, since \(\theta^0 = 1 = Sq^0\). Assume inductively that (ii) holds for \(k - 1\). Then, by (2.2)', (i) and the inductive assumption, we see that

\[
Sq^{j(k)} - (T) = Sq^a \{Sq^{j(k-1)} - (T)\} + Sq^k \{Sq^{j(k-1)} - (T)\}
\]

\[ = Sq^a \theta^{a-a} + Sq^k \theta^{n-k'} \quad (a = k' - t_1, \quad T = (t_2, \ldots, t_i)), \]

where the second terms do not appear if \(k = t_1\) by the convention (*) in (2.2)'.

If \(k = t_1\), then \(a = 0\) and we have the desired equality.

Let \(k > t_1\). Then, by Proposition 2.6 and Lemma 2.10(ii), we see that

\[
(Sq^a \theta^{n-a} + Sq^k \theta^{n-k'}) (u_1 \times \cdots \times u_n)
\]

\[ = Sq^a \sum_{\|A\| = 2n-a} u(A) + Sq^k \sum_{\|B\| = 2n-k} u(B)
\]

\[ = \sum_{\|A\| = 2n-a} \sum_{\|A + \varepsilon\| = 2n} u(A + \varepsilon) + \sum_{\|B\| = 2n-k} \sum_{\|B + \rho\| = 2n} u(B + \rho)
\]

\[ = \theta^n (u_1 \times \cdots \times u_n), \quad (n = 2k' - t_1 - s, \quad s = t_2 + \cdots + t_i + 1). \]

Therefore \(Sq^a \theta^{n-a} + Sq^k \theta^{n-k'} = \theta^n\) by (2.13).

Thus \(Sq^{j(k)} - (T) = \theta^n\), and the theorem is proved completely. q. e. d.

As applications of Theorem 2.4, we have the following known results:

**Corollary 2.14.** \(\theta^{2n+1} = \theta^{2n} Sq^1\).

**Proof.** We notice that
(2.15) \[ Sq^{2^a-1}Sq^a = \sum_{j=1}^{a-1} \left( \frac{j-a}{2a-1-2j} \right) Sq^{2^a-1-j}Sq^j = 0 \]

by the Adam relation [7, p. 2].

If \( n = 0 \), then the equality holds since \( \theta^1 = Sq^1 \).

Let \( 2n = 2^k - 1 - |T| > 0 \) for \( T = (t'_1, \ldots, t'_l) \) with \( k \geq t_1 > \cdots > t_l \geq 1 \). Then \( t'_l = 1 \). Thus, in the summation of the equality

\[ Sq^{(k)}(T) = \sum_{i \in p_i < \cdots < p_l \leq k} Sq^{(k)}(T(p_1, \ldots, p_l)) \]

of (2.2), the term for \( p_i = k - a < k \) contains \( Sq^{2^{a'-1}}Sq^{a'} \) and is 0 by (2.15). Therefore, the above sum is equal to

\[ Sq^{(k)}(T) \quad (J(k)_k = (k', (k-1)', \ldots, 2), T_i = (t'_i, \ldots, t'_{i-1})) \]

On the other hand, \( 2n + 1 = 2^k - 1 - |T| \) and

\[ Sq^{(k)}(T) = \{ Sq^{(k)}_2 - (T) \} Sq^1 \]

by definition, since \( t'_{i-1} \geq 2 \) or \( l - 1 = 0 \). Thus, we see the desired equality by Theorem 2.4. q.e.d.

(ii) and (iii) of the following corollary are due to Davis [2, Th. 2].

**Corollary 2.16.** (i) \( \theta^{2k'} = Sq^{2k'} + Sq^{k'}\theta^{k'} \).

(ii) \( \theta^{2k'-1} = Sq^{(k;1)}\theta^{l'-1} \quad \text{for} \quad k \geq l \geq 1, \)

where \( J(k; l) = (k', (k-1)', \ldots, l') \).

(iii) \( \theta^{2k-k-1} = Sq^{k'}\theta^{k' - k - 1} + Sq^{(k-1);(k-1)';\ldots,1} \quad \text{for} \quad k \geq 2. \)

**Proof.** By using (2.9) and (2.13), we see easily that

(2.17) \[ \sum_i S_q^l = \sum_i S_q^l \quad \text{implies} \quad \sum_i (S_q^l - (i)) = \sum_j (S_q^l - (i)). \]

(i) By Proposition 2.3, Theorem 2.4, (2.2)' and (2.17), we see that

\( \theta^{2k'} = \theta^{2k'-1} - (k) = Sq^{(k+1)} - (k) \]

\[ = Sq^{2k'-k'}(Sq^{l}(k) - (k-1)) + Sq^{2k'}(Sq^{l}(k) - (k)) = Sq^{k'}\theta^{k'} + Sq^{2k'}. \]

(ii) We prove the equality by the induction on \( l \). (ii) for \( l = 1 \) is in Theorem 2.4(i). Assume (ii) for \( l \). Then, by Proposition 2.3, (2.15) and (2.17), we see (ii) for \( l+1 (\leq k) \) as follows:

\( \theta^{2k'-l-1} = \theta^{2k'-l} - (1) = (Sq^{(k;l+1)}Sq^{l'}\theta^{l'-1}) - (1) \]

\[ = Sq^{(k;l+1)}(Sq^{l'}\theta^{l'-1} - (1)) = Sq^{(k;l+1)}(\theta^{2l'-1} - (1)) = Sq^{(k;l+1)}\theta^{2l'-l-1}. \]
(iii) By (ii), \( \theta^{2l-1} = Sq^{l} \theta^{l-1} \) for any \( l \geq 1 \). Thus,
\[
\theta^{2l-1} = \theta^{2l-1} - 1 = Sq^{l} \theta^{l-1} - 1 + Sq^{l-1} \theta^{l-1}
\]
for any \( l \geq 1 \) in the same way. By using this equality for \( l = k, k-1, \ldots, 1 \) and (2.15), we see immediately (iii). q.e.d.

The following Cartan formula for \( \theta \), which may be well-known, is used in the next section.

**Proposition 2.18.** For any cohomology classes \( x \) and \( y \),
\[
\theta^n(x) = \sum_{i+j=n} (\theta^i x)(\theta^j y).
\]

**Proof.** We can prove easily the formula by the induction on \( n \), by using (2.1)' and the Cartan formula for \( Sq \). q.e.d.

**Remark 2.19.** We remark that Proposition 2.6 (ii) can be proved by (2.8) and the Cartan formula
\[
\theta^n(x \times y) = \sum_{i+j=n} (\theta^i x) \times (\theta^j y).
\]

§3. **Odd dimensional Wu classes**

Let \( M^d \) be a closed \( d \)-manifold, and let
\[ v_i \in H^i(M^d; \mathbb{Z}_2) \]
be the \( i \)th Wu class of \( M^d \), which is defined to be the element with
\[
\langle v_i x, \mu \rangle = \langle Sq^i x, \mu \rangle \quad \text{for every} \quad x \in H^{d-i}(M^d; \mathbb{Z}_2).
\]
Here \( \mu \in H_d(M^d; \mathbb{Z}_2) \) is the fundamental homology class and \( \langle , \rangle \) is the Kronecker index. Then, the \( k \)th Stiefel-Whitney class
\[ w_k \in H^k(M^d; \mathbb{Z}_2) \]
of \( M^d \) is represented by the Wu classes as the following Wu formula:
\[
(3.1) \quad (6, \text{Th. 11. 14}) \quad w_k = \sum_{l=0}^{k} Sq^{l} v_{k-l}.
\]

Conversely, the Wu class is represented by the Stiefel-Whitney classes as follows:

**Proposition 3.2.**
\[ v_n = \sum_{i=1}^{n} \theta^{n-i} w_i, \]
where \( \theta^{n-i} \in \mathfrak{so}(2) \) is the element given by (2.1).

**Proof.** By (3.1), \( w_1 = v_1 + Sq^1 v_0 = v_1 \). Suppose inductively that the
equality holds for \( n < k \). Then, by (3.1) and (2.1)', we see that
\[
v_k = w_k + \sum_{j=1}^{k-1} S^j v_{k-j} = w_k + \sum_{j=1}^{k-1} S^j (\sum_{i=1}^{j-1} \theta^{k-j} v_i)
\]
\[
= w_k + \sum_{j=1}^{k-1} (\sum_{i=1}^{j-1} S^j (\theta^{k-j} w_i)) = \sum_{j=1}^{k} \theta^{k-j} w_j,
\]
as desired. q. e. d.

To prove Theorems 3.9 and 3.10 which are the main results in this section, we prepare several lemmas, where we use the notations \( t' = 2^t - 1 \) for any positive integer \( t \), and \( l(I) = l \) and \( ||I|| = l'_1 + \cdots + l'_t \) for any sequence \( I = (i_1, \ldots, i_t) \) of positive integers.

**Lemma 3.3.** (i) If \( l = l'_1 + \cdots + l'_k = ||L|| \) for \( L = (l_1, \ldots, l_k) \) with \( l_1 > \cdots > l_k \geq 1 \), then
\[
\sum_{I} w_1^2 ||I|| \theta^{m-2} ||I|| w_n = \sum_{I+m} ||I+m|| \theta^{m-2} ||I+m|| w_n,
\]
where \( J + L = (j_1 + l_1, \ldots, j_k + l_k) \) for \( J = (j_1, \ldots, j_k) \) and \( \theta^j = 0 \) if \( j < 0 \).

(ii) If \( l = 2^s - 1 \geq 1 \), then
\[
\sum_{I} w_1^2 ||I|| \theta^{m-2} ||I|| w_n = \sum_{l \geq 2} w_1^{(i+1)'} \theta^{m-(i+1)'} w_n.
\]

(iii) If \( l = 2^s - 1 \geq 1 \), then
\[
\sum_{I} w_1^2 ||I|| \theta^{m-2} ||I|| w_n = \sum_{l \geq 2} w_1^{(i+1)'} \theta^{m-(i+1)'} w_n + \sum_{l \geq 2} w_1^{\varphi(s,k,i)} \theta^{m-\varphi(s,k,i)} w_n,
\]
where \( \varphi(s,k,i) = (s + i)' + (s + 2 - k)' - 2 \).

**Proof.** (i) In the left hand side of the equality, the sum of the terms for \( I = (i_1, i_2, i_3, \ldots, i_t) \) and \( I' = (i_2, i_1, i_3, \ldots, i_t) \) with \( i_1 \neq i_2 \) is 0, and the term for \( I = (i_1, i_1, i_3, \ldots, i_t) \) is equal to
\[
w_1^{2l} \theta^{m-2} ||I|| w_n \quad \text{with} \quad I'' = (i_1 + 1, i_3, \ldots, i_t).
\]
Let \( k = 1 \), i.e., \( l = l'_1 \). Then, by using these facts repeatedly, we see easily that the left hand side of the equality is equal to
\[
\sum_{I} w_1^{l/2} \theta^{m-4} ||I|| w_n,
\]
and hence to \( \sum_{l \geq 1} w_1^{(l+1)'} \theta^{m-(l+1)'} w_n \), which is the right hand side of the equality. In the same way, we can prove (i) for \( k > 1 \).

(ii) The equality is proved in the above proof.

(iii) Since \( l = 2^s - 1 = ||S|| \) where \( S = (s, s-1, \ldots, 1) \), (i) implies that the left hand side of the equality in (iii) is equal to
\[
\sum_{I \geq s} w_1^{l+s} \theta^{m-l-s} ||I|| w_n.
\]
In this summation, let \( \sigma_k (1 \leq k \leq s) \) be the partial sum on
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Then, (*) is equal to

\[ w_{2t}^2 \theta^{m-2t} w_n + \sum_{k=1}^t \sigma_k, \]

since the term in (*) for \( J = (1, \ldots, 1) \) is equal to the first term.

Now, by the same consideration as in the proof of (i), \( \sigma_s \) is equal to the partial sum on \( J \) with \( j_{s-1} + 1 = j_s \geq 2 \), and hence to that on \( J \) with \( j_{s-2} + 2 = j_{s-1} + 2 = j_s + 1 \geq 3 \), and so on. Hence, \( \sigma_s \) is equal to the partial sum on \( J \) with \( j_1 = j_2 = \cdots = j_{s-1} = j_s - 1 \geq 1 \), which is clearly equal to \( \sum_{i \geq 2} w_1^{(i+s)^t} \theta^{m-(i+s)} w_n \).

Similarly, we see that

\[ \sigma_k = \sum_{i \geq 2} w_1^{(i; k, i)} \theta^{m-\varphi(i; k, i)} w_n. \]

Thus we have proved (iii).

**Lemma 3.4.** (i) For \( t = 2^{r-1} \geq 2 \),

\[ \sum_{q=t-1}^2 \sum_{l(t)=q} w_1^{2l} \theta^{m-2l} \| w_n = w_1^{2r} \theta^{m-2r} w_n. \]

(ii) \( \sum_{q \geq 1} \sum_{l(t)=q} w_1^{2l} \theta^{m-2l} \| w_n = \sum_{t \geq 3} w_1^{2t} \theta^{m-2t} w_n. \)

**Proof.** (i) For \( t = 2 \), the above lemma implies the desired equality as follows:

\[ \sum_{q=1}^2 \sum_{l(t)=q} w_1^{2l} \theta^{m-2l} \| w_n = w_1^2 \theta^{m-2} w_n + \sum_{i \geq 2} w_1^{(1; 1, i)} \theta^{m-\varphi(1; 1, i)} w_n + \sum_{i \geq 2} w_1^{(i+1)} \theta^{m-(i+1)} w_n \]

\[ = w_1^2 \theta^{m-2} w_n. \]

We prove (i) by the induction on \( t \). In the left hand side of the equality, we see easily by (i) of the above lemma that the sum on \( q = t' + p \) with \( 1 \leq p \leq t - 2 \) is equal to

\[ \sum_{i \geq 2} w_1^{(i')} \{ \sum_{p=1}^{t-2} \sum_{l(t)=p} w_1^{2l} \theta^{m-i'-2} \| w_n \}. \]

By the inductive assumption, this is equal to

\[ \sum_{i \geq 2} w_1^{(i')} \{ \sum_{p=1}^{t-2} w_1^{2p} \theta^{m-i'-2} \| w_n \} = \sum_{i \geq 2} w_1^{(i')} \theta^{m-i'-2} w_n + \sum_{i \geq 2} w_1^{(i')} \theta^{m-i'} w_n. \]

On the other hand, the terms for \( q = t' - 1 \) and \( t' \) are given by (iii) and (ii) of the above lemma for \( s = t - 1 \), respectively. Thus we see (i).

(ii) (ii) follows immediately from (i).

**Lemma 3.5.**

(i) \( \theta^{2t}(w_1 w_2 m) = \sum_{i \geq 2} w_1^{(i-1)} \theta^{2i+2-2} w_2 m + \sum_{i \geq 3} w_1^{(i-2)} \theta^{2i+2-2} w_2 m + 1 \).
(ii) \( \theta^{2i}(w_1w_{2m+1}) = \sum_{t \geq 2} w_1^{t-1} \theta^{2i+2-t'}w_{2m+1} \).

**Proof.** We notice that the equalities

\[
Sq^jw_{2m} = w_1w_{2m} + w_{2m+1}, \quad Sq^jw_{2m+1} = w_1w_{2m+1}
\]

hold as special cases of Wu's formula

(3.7) ([10], [3]) \( Sq^jw_i = \sum_{t=0}^{l} \binom{i-j+t-1}{t} w_{j-t}w_{t+i} \) for \( j \leq i \).

By Proposition 2.18, (2.8), Corollary 2.14 and the first equality in (3.6), we see that

\[
\theta^{2i}(w_1w_{2m}) = \sum_{j \geq 0} \left( \theta^j w_1 \right) \left( \theta^{2i-j} w_{2m} \right)
\]

\[= w_1 \theta^{2i}w_{2m} + \sum_{i \geq 2} w_1^{i-1} \theta^{2i-t'} Sq^jw_{2m} \]

\[= w_1 \theta^{2i}w_{2m} + \sum_{i \geq 2} w_1^{i-1} \theta^{2i-t'}w_{2m+1} + \sum_{i \geq 2} w_1^{i-1} \theta^{2i-t'}(w_1w_{2m}).
\]

Consider the equality (3.8.1), and substitute (3.8.1 - i'/2) for its last term \( \theta^{2i-t'}(w_1w_{2m}) \) (\( i' = 2t-1 \geq 2 \)) if \( 2t-i' \geq 0 \), and so on. Then, we see easily that

\[
\theta^{2i}(w_1w_{2m}) = w_1 \theta^{2i}w_{2m} + \sum_{j=1}^{\infty} \sum_{(I) = q} w_1^{I+2} \| \theta^{2i-1} || w_{2m} + \sum_{j=1}^{\infty} \sum_{(I) = q} w_1^{I+2} \| \theta^{2i-2} || w_{2m+1}.
\]

Thus, (i) is seen by (ii) of the above lemma.

We can prove (ii) similarly by using the second equality in (3.6). q.e.d.

By the above lemmas, we have the following results.

**Theorem 3.9.** The equality in Proposition 3.2 can be rewritten as follows, where \( a \geq 1 \) and \( i' = 2t-1 \) for any positive integer \( t \):

(i) \( v_{2a} = \sum_{p \geq 1} \theta^{2a-2p}w_{2p} + \sum_{i \geq 0, t \geq 2} w_1^{i-1} \theta^{2a-2p-t'}w_{2p+1} \).

(ii) \( v_{2a+1} = \sum_{p \geq 1, t \geq 2} w_1^{i-1} \theta^{2a+2-2p-t'}w_{2p} + \sum_{p \geq 1, t \geq 3} w_1^{i-1} \theta^{2a+2-2p-t'}w_{2p+1} \).

**Proof.** By Proposition 3.2, Corollary 2.14 and the first equality in (3.6), we see that

\[
v_{2a+1} = \sum_{p \geq 1} \left( \theta^{2a+1-2p}w_{2p} + \theta^{2a-2p}w_{2p+1} \right)
\]

\[= \sum_{p \geq 1} \{ \theta^{2a-2p}(w_1w_{2p} + w_{2p+1}) + \theta^{2a-2p}w_{2p+1} \}
\]

\[= \sum_{p \geq 1} \theta^{2a-2p}(w_1w_{2p}).
\]

Thus, we have (ii) by (i) of the above lemma.

(i) is shown in the same way. q.e.d.
THEOREM 3.10. The odd dimensional Wu class $v_{2a+1}$ of a closed manifold can be represented by the lower and even dimensional Wu classes and the first Stiefel-Whitney class $w_1$ by the equality

$$v_{2a+1} = \sum_{l \geq 2} w_1^{l'-1} v_{2a+2-l'}, \quad (l' = 2^{l'-1}).$$

PROOF. The equality for $a=0$ is clear.

Let $a$ be positive. Then, by the above theorem,

$$\sum_{l \geq 2} w_1^{l'-1} v_{2a+2-l'} = \sum_{l \geq 2, p \geq 1} w_1^{l'-1} \theta^{2a+2-2p-l'} w_{2p} + \sum_{l \geq 2, p \geq 0} w_1^{l'+1-2p-2p-l'-t'} w_{2p+1}$$

$$+ \begin{cases} w_1^{2a+1} & (a = 2^l - 1) \\ 0 & (a \neq 2^l - 1). \end{cases}$$

Here, in the same way as in the proof of Lemma 3.3(ii), we see that the second term is equal to

$$\sum_{l \geq 2, p \geq 0} w_1^{2l'-2} \theta^{2a+2-2p-2l'} w_{2p+1} = \sum_{l \geq 2} w_1^{2l'-2} \theta^{2a+2-2l'} w_1 + \sum_{l \geq 2, p \geq 1} w_1^{2l'-2p+1} \theta^{2a+2-2p-2l'} w_{2p+1},$$

whose first sum is equal to

$$w_1^{2a+1} \quad (a = 2^l - 1), \quad 0 \quad (a \neq 2^l - 1),$$

by (2.8). Thus we obtain the desired equality by (ii) of the above theorem. q.e.d.

As an application of the above theorem, we obtain the following known result:

COROLLARY 3.11 ([5, Lemma 3]). If a closed manifold $M$ is orientable, then the odd-dimensional Wu classes of $M$ vanish.

PROOF. By [4, p. 244, Th. 12.1], the assumption is equivalent to $w_1 = 0$. Thus the corollary follows immediately from the above theorem. q.e.d.

§ 4. Wu classes of certain manifolds

In the rest of this paper, we only consider a closed manifold $M$ whose $i$th Stiefel-Whitney class $w_i$ satisfies

$$w_i = 0 \quad \text{if } i \text{ is not a power of } 2;$$

i.e., we assume that the total Stiefel-Whitney class $wM$ is given by

$$wM = 1 + \sum_{i \geq 1} w_i, \quad w_i \in H^i(M; \mathbb{Z}_2),$$
where we use at all times the notation
\[ b' = 2^{b-1} \text{ for any positive integer } b. \]

Under the above assumption, we have the following

**Proposition 4.2.** If \( c \geq b+2 \), then \( w_b w_c = 0 \).

**Proof.** \( w_{c'-b'} = 0 \) by the assumption and (4.1). Therefore
\[
0 = Sq^{2b'}w_{c'-b'} = \sum_{t=0}^{2b'} \binom{c'-b' - 2b' + t - 1}{t} w_{2b'-t}w_{c'-b'+t}
\]
by (3.7), and the last sum is equal to \( w_b w_c \) by (2.12) and (4.1). q.e.d.

By Proposition 3.2, (3.7) and this proposition, we see that the Wu class \( v_t \) can be written as a sum of cohomology classes \( (w_b)^j(w_{2b})^k \). More precisely, the purpose of this section is to prove the following

**Theorem 4.3.** The \( i \)th Wu class \( v_i \) of a closed manifold \( M \) satisfying the condition (4.1) can be represented by the Stiefel-Whitney classes \( w_b \) of \( M \) as follows, where

\[ i = a_1' + a_2' + \cdots + a_k' \text{ with } a_1 > a_2 > \cdots > a_k \geq 1: \]

(i) If \( k = 1 \), i.e., if \( i = a' \) with \( a \geq 1 \), then
\[
v_i = \sum_{b=1}^{a} (w_b)^{(a-b+1)'}.
\]

(ii) If \( k = 2 \), i.e., if \( i = a_1' + a_2' \) with \( a_1 > a_2 \geq 1 \), then
\[
v_i = \sum_{b=1}^{a_1} \sum_{j=a_2+1}^{a_2} (w_b)^{(j-1')/b'} (w_{2b})^{(j-b)'}.
\]

(iii) If \( k \geq 3 \), then \( v_i = 0 \).

To prove this theorem, we prepare several lemmas.

**Lemma 4.4.** For any cohomology class \( y \) and \( t' = 2^{i-1} \),
\[
Sq^{ty} = \begin{cases} (Sq^{ty})' & \text{if } i \text{ is a multiple of } t', \\ 0 & \text{otherwise}. \end{cases}
\]

**Proof.** We see easily by the Cartan formula that
\[
Sq^{2^a z^2} = (Sq^a z)^2, \quad Sq^{2^a+1 z^2} = 0.
\]
These imply immediately the lemma. q.e.d.

**Lemma 4.5.** (i) For \( b' = 2^{b-1} \geq 1 \), \( t' = 2^{t-1} \geq 1 \) and \( i \geq 1 \),
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\[
Sq^i(w_{2b'})^t = \begin{cases} 
  (w_{2b'})^{t'} & \text{if } i = b't', \\
  (w_{2b'})^{2t'} & \text{if } i = 2b't', \\
  0 & \text{otherwise}.
\end{cases}
\]

(ii) \( Sq^i w_{2b'} = 0 \) if \(|l|\) is not a multiple of \( b' \).

**Proof.** (i) By (3.7), (4.1) and Proposition 4.2, we see that

\[
Sq^i w_{2b'} = w_i w_{2b'} = \begin{cases} 
  w_{2b'}^i & \text{if } i = b', \\
  0 & \text{otherwise},
\end{cases}
\]

for \( 0 < i < 2b' \). Thus we see the equality for \( t' = t = 1 \).

The lemma for \( t > 1 \) follows immediately from that for \( t = 1 \) and the above lemma.

(ii) (ii) is clear by (i), Proposition 4.2 and the Cartan formula. q.e.d.

**Lemma 4.6.** For \( q \geq p + 1 \geq 3 \),

\[
Sq^{2b'+q'(q'-p')}(w_{2b'})^{q'-p'+1} = \begin{cases} 
  (w_{2b'})^{2q'-2p'+1} & \text{if } p > 2, \\
  (w_{2b'})^{2q'-3} + (w_{2b'})^{2q'-4} & \text{if } p = 2.
\end{cases}
\]

**Proof.** We prove the lemma by the induction on \( q = p + 1, p + 2, \ldots \).

If \( q = p + 1 \), then \( q' = p' \) and we see that

\[
Sq^{2b'+p'}(w_{2b'})^{p'+1} = Sq^{2b'+p'}((w_{2b'})^p w_{2b'})
\]

\[
= (w_{2b'})^p Sq^{2b'+p'} w_{2b'} + (w_{2b'})^{p'} Sq^{b'+p'} w_{2b'} + (w_{2b'})^{2p'} w_{2b'}
\]

by the Cartan formula and the above lemma for \( t = p \). Furthermore,

\[
Sq^{2b'+p'} w_{2b'} = 0, \quad Sq^{b'+p'} w_{2b'} = \begin{cases} 
  0 & \text{if } p > 2, \\
  (w_{2b'})^2 & \text{if } p = 2,
\end{cases}
\]

by the above lemma for \( t = 1 \). Thus we see the equality for \( q = p + 1 \geq 3 \).

By the Cartan formula, (i) of the above lemma and the dimensional reason that \( Sq^i x = 0 \) for \( i \geq \dim x \), we see easily that

\[
Sq^{2b'+(2q'-p')}(w_{2b'})^{2q'-p'+1} = Sq^{2b'+(2q'-p')}(w_{2b'})^{q'-p'+1}
\]

\[
= (w_{2b'})^{2q'} Sq^{2b'+(q'-p')}(w_{2b'})^{q'-p'+1}.
\]

Thus, we see the equality by the induction on \( q \). q.e.d.

**Lemma 4.7.** For \( q \geq p + 1 \geq 3 \),

\[
Sq^{2b'+(q'-p'-1)}(w_{2b'})^{q'-p'+1}
\]
\[
\begin{align*}
&= \begin{cases} 
0 & \text{if } p \geq 4, \\
(w_b)^4(w_{2b})^2q^{r-10} & \text{if } p = 3, \\
(w_{2b})^{2q^{r-4}} + (w_b)^2(w_{2b})^{2q^{r-5}} + & \begin{cases} 
0 & \text{if } p = 2, q = 3, \\
(w_b)^4(w_{2b})^{2q^{r-6}} & \text{if } p = 2, q \geq 4.
\end{cases}
\end{cases}
\end{align*}
\]

**Proof.** If \( q = p + 1 \) or \( p + 2 \), then the left hand side of the equality is equal to
\[
Sq^{2b}p^{r-2b'}((w_{2b})^{p'}(w_{2b})) \quad \text{or} \quad Sq^{6b}p^{r-2b'}((w_{2b})^{2p'}(w_{2b}))^{p'+1},
\]
respectively. Thus, we see the equality for \( q = p + 1 \) or \( p + 2 \) by the Cartan formula and Lemma 4.5 (i).

If \( q \geq p + 2 \), then we see easily that
\[
Sq^{2b}((2q^{r-p'+1}) (w_{2b}))^{q-p'+1} = (w_{2b})^{2q'} Sq^{2b'}(q'-p'-1) (w_{2b})^{q'-p'+1}
\]
by a way similar to the inductive proof of the above lemma. Thus, we see the equality by the induction on \( q \). q.e.d.

**Lemma 4.8.** For \( l \geq 0 \) and \( b' = 2b-1 \geq 1 \),

(i) \( Sq^{l(b+1; b)}w_{2b'} = \sum_{l=0}^{\infty} (w_b)^{4l'-4l'+1}(w_{2b})^{2l'} \),

(ii) \( Sq^{l(b+1; b)'}w_{2b'} = 0 \),

where \( J(k; b) = (k', (k-1)', \ldots, b') \).

**Proof.** (i) The equality holds for \( l = 0 \) by Lemma 4.5 (i). Assume inductively the equality for \( l \). Then
\[
Sq^{l(b+1; b)}w_{2b'} = Sq^k Sq^{l(b+1; b)}w_{2b'} \quad (k = b + l + 1)
\]
\[
= Sq^k \{ \sum_{l=0}^{\infty} (w_b)^{4l'-4l'+1}(w_{2b})^{2l'} \}
\]
\[
= \sum_{l=0}^{\infty} \sum_{e=0}^{2} \{ Sq^{k-2eb'}(w_b)^{4l'-4l'+1} \} \{ Sq^{2eb'}(w_{2b})^{2l'} \}
\]
\[
= (w_b)^{8l'-1}w_{2b'} \quad + \sum_{l=0}^{\infty} \{ Sq^{k-4eb'}(w_b)^{4l'-4l'+1}(w_{2b})^{4l'} \}
\]
\[
= \sum_{l=0}^{\infty} (w_b)^{8l'-4l'+1}(w_{2b})^{2l'},
\]
as desired, by Lemmas 4.5 (i), 4.6 and Proposition 4.2.

(ii) The equality holds for \( l = 0 \) by Lemma 4.5 (i). Assume \( l \geq 1 \). By (i), it is sufficient to show that
\[
\sum_{l=0}^{\infty} Sq^{k-2b'}((w_b)^{4l'-4l'+1}(w_{2b})^{2l'}) = 0 \quad (k = b + l + 1).
\]
The left hand side is equal to
\[
\sum_{l=0}^{\infty} \sum_{e=0}^{2} \{ Sq^{k-2eb'}(w_b)^{4l'-4l'+1} \} \{ Sq^{2eb'}(w_{2b})^{2l'} \}
\]
\[
= (w_b)^{8l'-2}w_{2b'} \quad + \{ Sq^{k-2b'}(w_b)^{4l'-1} \} w_b w_{2b'} \quad + (w_b)^{8l'-6}(w_{2b})^2
\]
\[
+ \sum_{l=0}^{\infty} \{ Sq^{k-4eb'}(w_b)^{4l'-4l'+1}(w_{2b})^{4l'} \}
\]
\[
= 0,
\]
as desired, by Lemmas 4.5(i), 4.6, Proposition 4.2 and Lemma 4.7. q.e.d.

Now, by using the above results and Theorem 2.4, we can prove the following lemma which implies (i) and (ii) of Theorem 4.3.

**Lemma 4.9.**

(i) For $a \geq b \geq 1$,

$$\theta^{a-b'} w_{b'} = (w_{b'})^{(a-b+1)'}.$$

(ii) If $i = a_1 + a_2$ for $a_1 > a_2 \geq 1$ and $a_1 > b \geq 1$, then

$$\theta^{i-2b'} w_{2b'} = \begin{cases} \sum_{j=a_2+1}^{\infty} (w_{b'})^{(i-j')/b'} (w_{2b'})^{(j-b')'} & \text{if } b \leq a_2, \\ 0 & \text{otherwise.} \end{cases}$$

**Proof.**

(i) If $b=1$, then the equality is clear by (2.8). Also, the equality for $a=b$ is trivial.

Let $a > b > 1$. Then $a'-b' = a'-1-(b'-1) = a'-1-[J(b-1)]$, where $J(b-1) = ((b-1)', (b-2)', \ldots, 1)$, and Theorem 2.4(ii) shows that

$$\theta^{a-b'} w_{b'} = (\text{Sq}^j(a-1) - (J(b-1))) w_{b'} = \sum_j \text{Sq}^j w_{b'},$$

where $J=(j_1, \ldots, j_a, 1)$ is given by

$$j_p = (a-p')' \quad (s=1, \ldots, b-1), \quad j_p = (a-p)' \quad (p \neq p_1, \ldots, p_{b-1})$$

for $1 \leq p_1 < \cdots < p_{b-1} \leq a-1$. Since $\text{Sq}^j w_{b'} = 0$ for $0 < j < b'/2$ by Lemma 4.5(i),

$$\text{Sq}^j w_{b'} = 0 \quad \text{if } (p_2, \ldots, p_{b-1}) \neq (a-b+2, \ldots, a-1),$$

and hence we see that

$$\theta^{a-b'} w_{b'} = \text{Sq}^j(a-1; b) w_{b'} + \sum_{l=1}^{a-1} \text{Sq}^j(a-1; l+1) \text{Sq}^l(-b' , 2) \text{Sq}^l(-1; b-1) w_{b'},$$

This is equal to $(w_{b'})^{(a-b+1)'}$ by Lemmas 4.5(i) and 4.8(ii), and (i) is proved.

(ii) Let $a_2 > b \geq 1$. Then, $i-2b' = 2a_1' - 1 - (a_1' - a_2') - (2b' - 1)$ and

$$\theta^{i-2b'} = \text{Sq}^{(a_1-1)'} \theta^{(a_1-1)'} + a_2 - 2b' + \text{Sq}^{a_1} \theta^{a_2 - 2b'}$$

by Corollary 2.5. Therefore, by the dimensional reason that $\text{Sq}^j x = 0$ if $j > \dim x$,

$$\theta^{i-2b'} w_{2b'} = \text{Sq}^{(a_1-1)'} \theta^{(a_1-1)'} + a_2 - 2b' \theta^{a_2 - 2b'} w_{2b'}.$$

By repeating this process, we see that

$$\theta^{i-2b'} w_{2b'} = \theta^{a_1} \theta^{a_2 - 2b'} w_{2b'}.$$

By (i) and Lemma 4.4, the last is equal to
which is equal to the right hand side of the equality in (ii) by Lemma 4.8 (i).

Let $a_2 = b$. Then, $i - 2b' = a_1' - 1 - (b' - 1)$ and we see that

$$
\theta^{i - 2b'}w_{2b'} = \{Sq^{J(a_1 - 1)} - (J(b - 1))\}w_{2b'} = Sq^{J(a_1 - 1;b)}w_{2b'}
$$

in the same way as in the proof of (i). Thus, we see (ii) for $a_2 = b$ by Lemma 4.8 (i).

Let $b > a_2$. Then $i - 2b' = a_1' + a_2' - 2b'$ is not a multiple of $b'$. Thus

$$
\theta^{i - 2b'}w_{2b'} = 0
$$

by Lemma 4.5 (ii) and (2.1). q.e.d.

**Proof of (i) and (ii) of Theorem 4.3.** The desired results follow immediately from Proposition 3.2, the assumption (4.1), (2.8) and the above lemma. q.e.d.

To prove Theorem 4.3 (iii), we use the following two lemmas which are valid without assuming (4.1).

**Lemma 4.10.** Let $i = a_1 + \ldots + a_k$ with $a_1 > \ldots > a_k \geq 1$ and $k \geq 3$. If $b < a_k$, then

$$
\theta^{i - 2b'}w_{2b'} = Sq^{A_2} \ldots Sq^{A_k} Sq^{a_k(k - 1) - 2b'}w_{2b'},
$$

where $A_s = ((a_s - 1)'(s' - 1), (a_s - 2)'(s' - 1), \ldots, (a_s + 1)'(s' - 1))$ and $Sq^b = 1$.

**Proof.** Set $i_s = a_{s+1} + \ldots + a_k$ for $1 \leq s \leq k$. Then, we can prove that

$$
\theta^{i - 2b'}w_{2b'} = Sq^{A_2} \ldots Sq^{A_s} \theta^{\varphi_s - 2b'}w_{2b'},
$$

by the induction on $s$ ($2 \leq s \leq k$) as follows.

If $a_1 = a_2 + 1$, then (4.11) for $s = 2$ is trivial. If $a_1 > a_2 + 1$, then

$$
i - 2b' = a_1' + i_1 - 2b' = 2a_1' - 1 - (a_1' - i_1) - (2b' - 1) \quad \text{with} \quad a_1' - i_1 \geq (a_1 - 1)',
$$

and we see in the same way as in the first part of the proof of Lemma 4.9 (ii) that

$$
\theta^{i - 2b'}w_{2b'} = Sq^{(a_1 - 1)'}(a_1 - 1)' + i_1 - 2b'w_{2b'} = \ldots = Sq^{(a_1 - 1)'}(a_1 - 1)' + i_1 - 2b'w_{2b'},
$$

by using Corollary 2.5. Thus we see (4.11) for $s = 2$.

Assume inductively (4.11) for $s < k$. If $a_s = a_{s+1} + 1$, then $\varphi_{s+1} = \varphi_s$ and (4.11) for $s + 1$ is trivial. Let $a_s > a_{s+1} + 1$. Then

$$
\varphi_s - 2b' = (a_s + s)' - a_s' + i_s - 2b' = 2(a_s + s - 1)' - 1 - (a_s' - i_s) - (2b' - 1)
$$

with $a_s' - i_s \geq (a_s - 1)'$, and in the same way, we see (4.11) for $t = s + 1$ by

$$
\theta^{\varphi_s - 2b'}w_{2b'} = Sq^{(a_s + s - 1)'}(a_s - 1)' + i_s - 2b'w_{2b'} = \ldots = Sq^{A_t} \theta^{(a_t + 1)'}(a_t + 1)' + i_t - 2b'w_{2b'} = Sq^{A_t} \theta^{(a_t + 1)'}(a_t + 1)' + i_t - 2b'w_{2b'},
$$
Thus, we see (4.11). Furthermore, since \( \varphi_k - 2b = (a_k + k)' - a_k' - 2b' = 2(a_k + k - 1)' - 1 - a_k' - (2b' - 1) \), we see in the same way that

\[ \theta^{\varphi_k - 2b'} w_{2b'} = Sq^{(a_k + k - 1)'} - a_k \theta^{(a_k + k - 1)'} - 2b' w_{2b'} \]

This equality and (4.11) for \( s = k \) imply the lemma.

**q.e.d.**

**Lemma 4.12.** For \( i \) in the above lemma and \( b = a_k \),

\[ \theta^{l - 2b'} w_{2b'} = Sq^{d_2} \cdots Sq^{d_{i-1}} Sq^{a_i(l-1)} \theta^{a_i + l - 1)' - b'} w_{2b'} \quad (l = k - 1). \]

**Proof.** We see that (4.11) is also valid in the case \( b = a_k \) for \( 2 \leq s \leq k - 1 = l \). Furthermore, since

\[ \varphi_l - 2b' = (a_l + l)' - a_l' - b' = 2(a_l + l - 1)' - 1 - a_l' - (b' - 1), \]

we see in the same way as in the above proof that

\[ \theta^{\varphi_l - 2b'} w_{2b'} = Sq^{a_l + l - 1)' - a_l \theta^{a_l + l - 1)' - b'} w_{2b'} \]

Thus, we see the lemma.

**q.e.d.**

Now, we use the assumption (4.1) in the following

**Lemma 4.13.** Let \( q > p > b \geq 1 \). Then,

\[ Sq^{q'-p'} \{ \sum_{i=p}^{q'} (w_{b'})^{2(q-b)' - 2l'+1} (w_{2b'})^{l'} \} = 0. \]

**Proof.** Put

\[ x_i = (w_{b'})^{2(q-b)' - 2l'+1} \quad \text{for} \quad 1 \leq i \leq q - b. \]

Then, by Lemma 4.5(i) and the Cartan formula, we see that

\[ Sq^{q'-p'} \{ \sum_{i=p}^{q'} x_i (w_{2b'})^{l'} \} = \sum_1 + \sum_2 + \sum_3, \]

\[ \sum_1 = \sum_{i=p}^{q'} (Sq^{q'-p'} x_i)(w_{2b'})^{l'}, \]

\[ \sum_2 = \sum_{i=p}^{q'} (Sq^{q'-p'-2b'} x_i)(w_{b'} w_{2b'})^{l'}, \]

\[ \sum_3 = \sum_{i=p}^{q'} (Sq^{q'-p'-2b'} x_i)(w_{2b'})^{2l'}. \]

Thus we can prove the lemma by showing

\[ \sum_1 = \sum_{i=p}^{q'} (w_{b'})^{4(q-b)' - 2(p-b)' - 2l'+1} (w_{2b'})^{l'}, \]

\[ \sum_2 = (w_{b'})^{4(q-b)' - 6(p-b)' + 1} (w_{2b'})^{2(p-b)'} + (w_{b'})^{4(q-b)' - 2(p-b)' - 1} w_{2b'}, \]

\[ \sum_3 = \sum_{i=p}^{q'} (w_{b'})^{4(q-b)' - 2(p-b)' - 4l'+1} (w_{2b'})^{2l'}. \]
Proof of (4.15). If \( i \geq p - b + 1 \), then \( \dim x_i \leq q' - 2p' + b' < q' - p' \) and so \( Sq'^{p'-b'}x_i = 0 \). For \( 1 \leq i \leq p - b \), by the Cartan formula, Lemma 4.5 (i) and Proposition 4.2, we see that

\[
(Sq'^{p'-b'}x_i)(w_{2b})^{l'} = (Sq'^{p'-b'}((w_{b'})^{(q-b)'}(w_{b})(q-b')^{2l'+1}) (w_{2b})^{l'}
= (w_{b'})^{2(q-b)'}[Sq'^{q-1}-p'(w_{b'})^{(q-b')^{2l'+1}}] (w_{2b})^{l'}
= \cdots (w_{b'})^{2(q-b)'} + \cdots + (w_{b'})^{2(p-b)'} - 2l' + 1) (w_{2b})^{l'}
= (w_{b'})^{4(q-b')^{2l' - 2l' + 1}(w_{2b})^{l'}}.
\]

Thus, we see (4.15).

Proof of (4.16). In the same way as in the above proof, we see (4.16) by the following

\[
\begin{align*}
(Sq'^{p'-b'-b'}x_i)(w_{2b})^{l'}
&= (w_{b'})^{2(q-b)'}[Sq'^{q-2p'-b'}-b'(w_{b})(q-b')^{2l'+1}] (w_{2b})^{l'}
&= \cdots (w_{b'})^{4(q-b)'} - 8(p-b)'[Sq'^{p'-b'-b'}(w_{b'})^{4(p-b')^{2l'+1}}] (w_{2b})^{l'}
&= (w_{b'})^{4(q-b)^{2l' - 2l' + 1)} (w_{2b})^{l'} (w_{2b})^{2(p-b)'}
&\begin{cases}
(w_{b'})^{2(p-b)'} + 1 (w_{2b})^{2(p-b)'} & \text{if } i = p - b + 1 \\
(w_{b'})^{6(p-b)'} - 1 w_{2b} & \text{if } i = 1 \\
0 & \text{otherwise}.
\end{cases}
\end{align*}
\]

Proof of (4.17). Let \( i \leq p - b - 1 \). Then, in a way similar to the above proof, we see that

\[
(Sq'^{p'-2b'}x_i)(w_{2b})^{2l'} = (w_{b'})^{4(q-b)'} - 4l' + 1 (w_{2b})^{2l'}.
\]

Here, by using Lemma 4.6, we see that

\[
\begin{align*}
(Sq'^{p'-2b'}x_i)(w_{2b})^{2l'}
&= (w_{b'})^{4(q-b)'} - 4l' + 1 (w_{2b})^{2l'}
\end{align*}
\]

Let \( i = p - b \). Then in the same way as above, we see that

\[
(Sq'^{p'-2b'}x_i)(w_{2b})^{2l'} = (w_{b'})^{4(q-b)'} - 6(p-b)'+ 1 (w_{2b})^{2(p-b)'}.
\]

Let \( p - b < i \leq q - b \). Then, in the same way,

\[
(Sq'^{p'-2b'}x_i)(w_{2b})^{2l'}
= (w_{b'})^{4(q-b)'} - 6l' (w_{2b})^{2l'} = 0,
\]

\[
\begin{align*}
(Sq'^{p'-2b'}x_i)(w_{2b})^{2l'}
&= (w_{b'})^{4(q-b)'} - 6l' (w_{2b})^{2l'}
&= 0.
\end{align*}
\]
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because \( Sq^{2b'}w_{b'} = 0 \) by the dimensional reason. Thus, we see (4.17); and the proof of the lemma is complete. q. e. d.

**Lemma 4.18.** Let \( i = a'_1 + \cdots + a'_k \) with \( a'_1 > \cdots > a'_k \geq 1 \) and \( k \geq 3 \). Then

\[
\theta^{i - b'}w_{b'} = 0 \quad \text{for} \quad 1 \leq b \leq a_1.
\]

**Proof.** If \( b = 1 \), then the equality holds by (2.8) and the assumption. Let \( 1 \leq b < a_1 \). Then, by Lemmas 4.10 and 4.9 (i), we see that

\[
\theta^{i - 2b'}w_{2b'} = Sq^{A_1} \cdots Sq^{A_k} Sq^a(w_{2b'})^q,
\]

where \( \alpha = (a_k + k - 1)' - a'_k \) and \( \beta = (a_k + k - b - 1)' \). Since \( k \geq 3 \) by the assumption, \( \alpha \) is not a multiple of \( \beta b' = (a_k + k - 2)' \). Therefore \( Sq^a(w_{2b'})^q = 0 \) by Lemmas 4.4 and 4.5 (i). Thus \( \theta^{i - 2b'}w_{2b'} = 0 \).

Let \( b = a_k \). Then, by Lemma 4.12,

\[
\theta^{i - 2b'}w_{2b'} = Sq^{A_2} \cdots Sq^{A_k + 1} Sq^{q - p'} \theta^{q - b'}w_{2b'} \quad (q = a_{k-1} + k - 2, \ p = a_{k-1}).
\]

Furthermore, by Lemmas 4.9 (ii) and 4.13, we see that

\[
Sq^{q - p'} \theta^{q - b'}w_{2b'} = Sq^{q - p'} \{ \sum_{j=b+1}^q (w_{b'})^{(q'+b'-j')/b'} (w_{2b'})^{(j-b)'} \} = 0.
\]

Thus \( \theta^{i - 2b'}w_{2b'} = 0 \).

Let \( a_k < b \). Then, \( i - 2b' \) is not a multiple of \( b' \) by the assumption, and we see \( \theta^{i - 2b'}w_{2b'} = 0 \) by Lemma 4.5 (ii) and (2.1). q. e. d.

**Proof of (iii) of Theorem 4.3.** The desired result follows immediately from Proposition 3.2, the assumption (4.1) and the above lemma. q. e. d.

Thus, we have proved Theorem 4.3 completely. In the rest of this section, we consider some examples of closed manifolds which satisfy (4.1).

**Example 4.19.** Let \( RP^n \) be the real projective \( n \)-space. Then

\[
wRP^n = 1 + u^{b'} + u^a' \quad \text{if} \quad n = a' + b' - 1 \quad \text{with} \quad a > b \geq 1,
\]

where \( u \in H^1(RP^n; Z_2) = Z_2 \) is the generator.

**Proof.** We see the desired result by the fact that

\[
wRP^n = (1 + u)^n + 1
\]

([6, Th. 4.5]) and (2.12). q. e. d.

For a (differentiable real) \( k \)-plane bundle \( \zeta \to V \) over a closed \( d \)-manifold \( V \), we denote by
the associated projective space bundle with fiber $RP^{k-1}$. Then, $RP(\zeta)$ is a closed $(d+k-1)$-manifold.

Let $\zeta_n$ be the canonical line bundle over $RP^n$, and $m\zeta_n$ be the $m$-fold Whitney sum of $\zeta_n$. Consider the natural projection

$$p_i: RP^n \times RP^n \to RP^n$$

of the product manifold $RP^n \times RP^n$ onto the $i$th factor, the induced bundle $p^*_i m\zeta_n$ of $m\zeta_n$ by $p_i$, and the Whitney sum

$$\xi(n, m) = p^*_i m\zeta_n \oplus p^*_i m\zeta_n,$$

which is a $2m$-plane bundle over $RP^n \times RP^n$. Then, we have the associated projective space bundle

$$p: (\xi(n, m)) \to RP^n \times RP^n$$

with fiber $RP^{2m-1}$.

**Example 4.20.** If $n = a' + b' - 1$ and $m = a'$ with $a > b \geq 1$,

then the total Stiefel-Whitney class of the $(2n+2m-1) = 4a' + 2b' - 3)$-manifold $RP(\xi(n, m))$ is given by

$$w_{RP(\xi(n, m))} = 1 + p^*\{(u_1^a + u_2^a) + (u_1 u_2)^b + (u_1 u_2)^a\},$$

where $u_i = p^*_i u \in H^1(RP^n \times RP^n; \mathbb{Z}_2)$ and $u \in H^1(RP^n; \mathbb{Z}_2)$ is the generator.

**Proof.** For the projective space bundle $p: RP(\zeta) \to V$ of a $k$-plane bundle $\zeta$ over a closed manifold $V$, it is proved in [1, (23.3)] that

$$H^*(RP(\zeta); Z_2)$$

is the free $H^*(V; Z_2)$-module with basis $1, c, \ldots, c^{k-1}$, with the relation

$$c^k = \sum_{i=1}^{k} p^*(w_i \zeta)c^{k-i},$$

where $c$ is the first Stiefel-Whitney class of the canonical line bundle over $RP(\zeta)$ and $w_i \zeta$ is the $i$th Stiefel-Whitney class of $\zeta$. Furthermore, the total Stiefel-Whitney class of $RP(\zeta)$ is given by

$$w_{RP(\zeta)} = p^*(wV)\sum_{l=0}^{k} p^*(w_l \zeta)(1+c)^{k-l}.$$
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\[ (m = a') \]

and the first equality in (4.21) is

\[ c^{2a'} = \{ p^*(u_1^{a'} + u_2^{a'}) \} c^{a'} + p^*(u_1 u_2)^{a'} \]

Therefore

\[
\sum_{i=0}^{2m} p^* (w_i \xi(n, m)) (1 + c)^{2m-i} = (1 + c)^{2a'} + \{ p^*(u_1^{a'} + u_2^{a'}) \} (1 + c)^{a'} + p^*(u_1 u_2)^{a'} = 1 + p^*(u_1^{a'} + u_2^{a'}). \]

Thus, by the last equality in (4.21) and Example 4.19, we see that

\[
w_{RP}(\xi(n, m)) = 1 + p^*(u_1^{a'} + u_2^{a'}) + p^*(u_1^{a'} + u_2^{a'}) (1 + c)^{a'} + p^*(u_1^{a'} + u_2^{a'}) \]

as desired. q.e.d.

Similarly, we have the following

**Example 4.22.** If \( n = b' - 1 \) and \( m = a' \) with \( b > a \geq 1 \), then

\[
w_{RP}(\xi(n, m)) = 1 + p^*(u_1^{a'} + u_2^{a'}). \]

**Remark 4.23.** In Proposition 4.2, the assumption is necessary. In fact,

\[
(w_b)^2 \neq 0, \quad w_b w_{2b'} \neq 0
\]

in Example 4.20, where \( w_b = p^*(u_1^{b'} + u_2^{b'}) \) and \( w_{2b'} = p^*(u_1 u_2)^{b'} \).

Finally, in connection with the condition (4.1), we notice the following

**Remark 4.24.** Let \( M \) be a closed manifold.

(i) If \( w_b = 0 \) for some \( b \geq 1 \), then \( w_i = 0 \) for \( b' \leq i < 2b' \).

(ii) If \( wM = 1 + w_i + w_i (i > 1) \) or \( wM = 1 + w_i (i \geq 1) \), and \( i \) is not a power of \( 2 \) in addition, then \( w_i = 0 \).

In fact, we can show (i) by using the equality

\[
Sq^{i-b'} w_{b'} = w_i + \sum_{j=1}^{i-1} \left( \binom{b'+j-i-1}{j-b'} \right) w_{i-j} w_j \quad (b' < i < 2b')
\]

of (3.7) and by the induction on \( i \). (ii) is an immediate consequence of (i).

§ 5. Unoriented bordism classes of certain manifolds

The purpose in this section is to prove the following

**Theorem 5.1.** Assume that a closed manifold \( M \) satisfies (4.1), i.e., the
total Stiefel-Whitney class $wM$ is given by

\[(5.2) \quad wM = 1 + \sum_{b \geq 1} w_b, \quad w_b \in H^{b'}(M; \mathbb{Z}_2) \quad (b' = 2^{b-1}),\]

and let

\[(5.3) \quad \dim M = p_1 + \cdots + p_k \quad \text{with} \quad p_1 > \cdots > p_k \geq 1 \quad (p' = 2^{p-1}).\]

(i) If $k \geq 4$ in (5.3) and

\[(w_b)^{\dim M/b'} = 0 \quad \text{for} \quad 2 \leq b \leq p_k,\]

then the unoriented bordism class $[M]$ of $M$ is 0.

(ii) If $\dim M$ is odd and $k \geq 3$ in (5.3), then $[M] = 0$.

**Theorem 5.4.**

(i) If $wM$ is given by

\[wM = 1 + w_b + w_c \quad \text{for some} \quad c > b \geq 1,\]

and $k \geq 2$ in (5.3), then $[M] = 0$.

(ii) If $wM = 1 + w_1 + w_i$ where $i > 1$ is not a power of 2, then $w_i = 0$ and $[M] = 0$.

(iii) If $wM = 1 + w_i$ for some $i \geq 1$, then $[M] = 0$.

To prove these theorems, we study the Stiefel-Whitney numbers of $M$, which is assumed throughout this section to satisfy (5.2) and $k \geq 2$ in (5.3), as follows.

By the assumption $k \geq 2$ in (5.3), we put

\[(5.3)' \quad \dim M = p' + q' + m \quad \text{with} \quad p > q \quad \text{and} \quad q' > m \geq 0,\]

and consider the following cohomology classes in $H^*(M; \mathbb{Z}_2)$:

\[A_i(b) = \sum_{i=1}^{p' + q' + m} (w_b)^{(p' - 1)^{s} t' - j'}/b' (w_{2b})^{(j - b)'}, \quad (q \leq t \leq p - 2, 1 \leq b \leq t),\]

\[B(b) = (w_b)^{(p' - b + 1)'} \quad (1 \leq b \leq p),\]

\[B_s(b) = \sum_{s=1}^{p' + q' + m} (w_b)^{(p' - s') t' - j'}/b' (w_{2b})^{(j - b)'}, \quad (1 \leq s \leq q, 1 \leq b \leq s).\]

Then, we have the following

**Lemma 5.6.** \[\sum_{b=1}^{t} A_i(b) = 0 \quad (q \leq t \leq p - 2),\]

\[\sum_{b=1}^{s} B(b) = 0, \quad \sum_{b=1}^{s} B_s(b) = 0 \quad (1 \leq s \leq q).\]

**Proof.** By Theorem 4.3 (i)-(ii), the $i$th Wu class $v_i$ is equal to

\[\sum_{b=1}^{t} A_i(b) \quad \text{if} \quad i = (p - 1)^{s} + t', \quad \sum_{b=1}^{s} B(b) \quad \text{if} \quad i = p',\]

and $\sum_{b=1}^{s} B_s(b)$ if $i = p' + s'$, respectively. On the other hand, $v_i = 0$ if $2i > \dim M$ by the definition of the Wu classes. Thus we see the lemma.

$q.e.d.$
LEMMA 5.7. For any $b$ with $2 \leq b \leq q$,
\[ \bar{A}_i(b) \equiv w_{b}w_{2b}A_i(b) = 0 \quad (q \leq t \leq p-2), \]
\[ \bar{B}(b) \equiv w_{b}w_{2b}B(b) + w_{b}w_{2b}B(b+1) = 0, \]
\[ \bar{B}_s(b) \equiv w_{b}w_{2b}B_s(b) = 0 \quad (b \leq s \leq q). \]

PROOF. Multiply the equalities in Lemma 5.6 by $w_{b}w_{2b}$. Then, we see the lemma by Proposition 4.2. q.e.d.

LEMMA 5.8. \[ \bar{A}_i = w_1A_i(1) = 0 \quad (q \leq t \leq p-2), \]
\[ \bar{B} = w_1B(1) + w_1B(2) = 0, \]
\[ \bar{B}_s = w_1B_s(1) = 0 \quad (1 \leq s \leq q). \]

PROOF. In the same way, by multiplying the equalities in Lemma 5.6 by $w_1$, we see the lemma. q.e.d.

LEMMA 5.9. For any $b$ with $2 \leq b \leq q$, the equality
\[ (w_{b})^p(w_{2b})^q = 0 \]
holds for $\alpha$ and $\beta$ given as follows:

(1) $\alpha = 1 + (p' + q')/b'$, $\beta = 1$.

(2) $\alpha = 1 + (p' - q' + s')/b'$, $\beta = 2 + (q' - s')/b'$ $\quad (b < s \leq q)$.

(3) $\alpha = 1 + p'/2b'$, $\beta = 1 + q'/b'$.

(4) $\alpha = 1 + t'/b'$, $\beta = 1 + (p' - 2t')/2b'$ $\quad (q \leq t \leq p - 2)$.

(5) $\alpha = 1$, $\beta = 2 + p'/2b'$.

PROOF. (3) By Lemma 5.7 and (5.5), we see that
\[ (w_{b})^p(w_{2b})^q = (w_{b})^{1+p'/2b'}(w_{2b})^{1+q'/b'} + \bar{A}_{q+1}(b) \]
\[ = (w_{b})^{1+p'/b'}\bar{A}_q(b) = 0 \quad \text{if } p \geq q + 2; \]
\[ (w_{b})^p(w_{2b})^q = (w_{b}w_{2b})^{1+p'/2b'} = \bar{B}_q(b) = 0 \quad \text{if } p = q + 1. \]

(4) $(w_{b})^p(w_{2b})^q = (w_{b})^{1+t'/b'}(w_{2b})^{1+(p' - 2t')/2b'}$
\[ + \bar{A}_{p-2}(b) \sum_{j=t+1}^{p-2} (w_{b})^{(p-2) + t' - j'/b'}(w_{2b})^{(j' - 2t')/2b'} \]
\[ = (w_{2b})^{(p-2) - t'/b'}\bar{A}_t(b) = 0. \]

(2) With $s = q$: $(w_{b})^{1+p'/b'}(w_{2b})^2$ is equal to
\[ (w_{b})^{1+p'/b'}(w_{2b})^2 + \bar{B}_{b+1}(b) = w_{b}\bar{B}_q(b) = 0. \]

(5) By the above result, we see that
\[ w_{b}(w_{2b})^{2+p'/2b'} = w_{2b}\bar{B}(b) = 0. \]
(1) By (3) if \( p = q + 1 \) and by (4) with \( t = q \) if \( p \geq q + 2 \), we see that
\[
(w_b)^{1 + q'/b'}(w_{2b})^{1 + p'/2b'} = 0.
\]
Hence
\[
(w_b)^{1 + (p' + q')/b'} w_{2b'} = (w_b)^{q'/b'} B(b) = 0.
\]

(2) with \( b < s < q \): In the equality
\[
\mathcal{B}_s(b) = \sum_{j=s+1}^{p} (w_b)^{1 + (p' + s' - j')/b'} (w_{2b})^{1 + j'/2b'},
\]
\( \sum_{j=q+1}^{p} \) is equal to
\[
(w_b)^{(p' + 2s' - 2q')/2b'} \mathcal{A}_s(b) = 0,
\]
and the term for \( j = p \) multiplied by \( w_{2b'} \) is equal to \((w_b)^{1 + s'/b'}(w_{2b})^{2 + p'/2b'}\), which is 0 by (5). Therefore
\[
\sum_{j=q+1}^{p} (w_b)^{1 + (p' + s' - j')/b'} (w_{2b})^{2 + j'/2b'} = w_{2b'} \mathcal{B}_s(b) = 0.
\]
By taking \( s = q - 1 \) especially, we see that
\[
(w_b)^{1 + (2p' - q')/2b'} (w_{2b})^{2 + q'/2b'} = 0.
\]
Thus, the desired equality is shown as follows:
\[
(w_b)^{1 + (p' - q' + s')/b'} (w_{2b})^{2 + (q' - s')/b'}
= \sum_{j=s+1}^{p} (w_b)^{1 + (p' + s' - j')/b'} (w_{2b})^{2 + (j' + q' - 2s')/2b'}
= (w_{2b})^{1 + (q' - 2s')/2b'} \mathcal{B}_s(b) = 0.
\]
These complete the proof of Lemma 5.9. q. e. d.

We notice that the relations in Lemma 5.8 are obtained from those in Lemma 5.7 for \( b = 1 \) by replacing \((w_1)^x(w_2)^\beta\) by \((w_1)^x(w_2)^{\beta-1}\). Thus, for \( b = 1 \), Lemma 5.9 turns out the following

**Lemma 5.10.** The equality
\[
(w_1)^x(w_2)^{\beta-1} = 0
\]
holds for \( x \) and \( \beta \) which are given by the equalities obtained from (1)-(5) of Lemma 5.9 by setting \( b = 1 \).

To study the Stiefel-Whitney numbers of \( M \), we consider cohomology classes
\[
(w_b)^{k(b, l)}(w_{2b})^{l} \in H^{dim M}(M; Z_2) \quad (b \geq 1, \ l \geq 0),
\]
where the integer \( k(b, l) \) is given by
\[
k(b, l)b' + 2lb' = \dim M = p' + q' + m \quad (p' > q' > m \geq 0).
\]
**Lemma 5.12.** If \((w_b)^\alpha(w_{2b})^\beta = 0\) for some \(\alpha\) and \(\beta\), then

\[
(w_b)^{k(b, l)}(w_{2b})^l = 0 \quad \text{for} \quad \beta \leq l \leq n(\alpha) = (\dim M - \alpha b')/2b'.
\]

**Proof.** The lemma is clear, since \(k(b, l) \geq \alpha\) for the above \(l\) by (5.11).

\[\text{q.e.d.}\]

By using Lemmas 5.9 and 5.12, we see the following

**Lemma 5.13.** In (5.11), assume that

\[(5.14) \quad m = ar' \quad \text{for} \quad r \geq 1 \quad \text{and an odd integer} \quad a \geq 3.
\]

Then, for any \(b\) with \(2 \leq b \leq r\),

\[
(w_b)^{k(b, l)}(w_{2b})^l = 0 \quad \text{if} \quad 1 \leq l \leq (\dim M - b')/2b'.
\]

**Proof.** For \(\alpha\) and \(\beta\) given in Lemma 5.9, we see easily that \(\beta\) and \(n(\alpha)\) in the above lemma are given as follows, where \(n_0 = (m - b')/2b'\):

1. \(\beta = 1, \quad n(\alpha) = n_0.
2. \(\beta = 2 + (q' - s')/b', \quad n(\alpha) = n_0 + (q' - (s - 1))/b' \quad (b < s \leq q).
3. \(\beta = 1 + q'/b', \quad n(\alpha) = n_0 + ((p - 1)' + q')/2b'.
4. \(\beta = 1 + (p' - 2t')/2b', \quad n(\alpha) = n_0 + (p' + q' - t')/2b' \quad (q \leq t \leq p - 2).
5. \(\beta = 2 + p'/2b', \quad n(\alpha) = (\dim M - b')/2b'.

Thus, for these \(\beta\) and \(n(\alpha)\),

\[(5.15) \quad (w_b)^{k(b, l)}(w_{2b})^l = 0 \quad (\beta \leq l \leq n(\alpha)).
\]

Here, we notice that \(n_0 = (ar' - b')/2b' \geq 1\) by the assumptions (5.14) and \(b \leq r\). Therefore, we see immediately that \(n(\alpha)\) in (1) (resp. (2) for \(s = u \geq b + 2\), (2) for \(s = b + 1\), (3), (4) for \(t = v > q\) or (4) for \(t = q\)) is not smaller than \(\beta - 1\) of \(\beta\) in (2) for \(s = q\) (resp. (2) for \(s = u - 1\), (3), (4) for \(t = p - 2\), (4) for \(t = v - 1\) or (5)). Thus, we have the lemma by (5.15).

\[\text{q.e.d.}\]

**Lemma 5.16.** In (5.11), assume that

\[(5.17) \quad m = ar' \quad \text{for} \quad r \geq 1 \quad \text{and an odd integer} \quad a \geq 1.
\]

Then

\[
(w_1)^{k(1, l)}(w_2)^l = 0 \quad \text{if} \quad 0 \leq l \leq (\dim M - 1)/2.
\]

**Proof.** By using Lemma 5.10 instead of Lemma 5.9, we see the lemma in the same way as in the above proof, since we have

\[\text{(5.15)} \quad (w_1)^{k(1, l)}(w_2)^l = 0 \quad (\beta - 1 \leq l \leq n(\alpha)),\]
instead of (5.15), for $\beta$ and $n(\alpha)$ obtained from the above (1)–(5) by setting $b=1$, where $n_0 = (ar'-1)/2 \geq 0$.

Now, we are ready to prove Theorem 5.1.

PROOF OF THEOREM 5.1. (i) By the assumption that $k \geq 4$ in (5.3), we see (5.14) where $r = p_k$. Therefore, by the above two lemmas and Proposition 4.2, we see immediately that all the Stiefel-Whitney numbers of $M$ are 0 except for

$$\langle (w^{(b)}_b)^{k(b,0)}, \mu \rangle \quad (2 \leq b \leq r = p_k).$$

Thus the desired result is an immediate consequence of the theorem of R. Thom (cf. [8, p. 95, Th.]) that

$$[M] = 0 \quad \text{if all the Stiefel-Whitney numbers of } M \text{ are 0.} \quad (5.18)$$

(ii) By the assumption that $\dim M$ is odd and $k \geq 3$, we see (5.17) with $r = 1$. Thus we see that all the Stiefel-Whitney numbers of $M$ are 0 by the above lemma and Proposition 4.2, and that $[M] = 0$ by (5.18).

To prove Theorem 5.4, we notice the following

LEMMA 5.19. Assume that

$$wM = 1 + w_b' + w_{2b'} \quad \text{for some } b \geq 1,$$

and let $k(b, l)$ be the integer given by (5.11). Then

$$(w^{(b)}_b)^{k(b,1)}(w_{2b'})^l = 0 \quad \text{for } 0 \leq l \leq \dim M/2b'.$$

PROOF. By the assumption (*), Lemma 5.7 for $b$ in (*) holds without multiplying $w_b', w_{2b'}$. Thus, we see by the same proof as in Lemma 5.9 that

$$(w^{(b)}_b)^{x-1}(w_{2b'})^{x-1} = 0$$

for $\alpha$ and $\beta$ given by (1)–(5) in Lemma 5.9, and hence we have

$$(w^{(b)}_b)^{k(b,1)}(w_{2b'})^l = 0 \quad (\beta - 1 \leq l \leq n(\alpha - 1))$$

instead of (5.15) by Lemma 5.12. Here, $n(\alpha - 1) = n(\alpha) + 1/2$ and so $n(\alpha - 1)$ is given by the equalities obtained from those of $n(\alpha)$ in (1)–(5) in the proof of Lemma 5.13 by replacing $n_0$ with $n_0 + 1/2 = m/2b' \geq 0$ and $(\dim M - b')/2b'$ with $\dim M/2b'$. Therefore, we have the lemma in the same way as in the proof of Lemma 5.13.

PROOF OF THEOREM 5.4. (i) Let $c = b + 1$. Then, the desired result follows immediately from the above lemma and (5.18).
Let $c > b + 1$. Then, by the second equality in Lemma 5.6,

$$(w_b)^{(p-b+1)'} + (w_c)^{(p-c+1)'} = 0.$$  

By Proposition 4.2, this equality implies that

$$(w_b)^{(p-b+1)'+1} = 0 \text{ and } (w_c)^{(p-c+1)'+1} = 0.$$  

Hence $(w_b)^{(b,0)} = 0 = (w_c)^{(c,0)}$ and all the Stiefel-Whitney numbers of $M$ are 0. Thus, the desired result for $c > b + 1$ follows immediately from (5.18).

(ii), (iii) By Remark 4.24 (ii), it is sufficient to show that

$$(*) \quad \text{if } wM = 1 + w_b \text{ for some } b \geq 1, \text{ then } [M] = 0.$$  

If $k \geq 2$ in (5.3), then $(*)$ is a special case of (i).

Let $k = 1$ in (5.3), i.e., $\dim M = p'$ for some $p \geq 1$. Then, by the assumption of $(*)$, Theorem 4.3 (i) and the dimensional reason, we see that

$$(w_b)^{(p-b+1)'} = v_{p'} = 0.$$  

Thus $[M] = 0$ by (5.18).

**Example 5.20.** The unoriented bordism classes of the $(4a'+2b'-3)$-manifold $RP(ξ(a'+b'-1, a'))$ given in Example 4.20 and the $(2b'+2a'-3)$-manifold $RP(ξ(b'-1, a'))$ given in Example 4.22 are all 0.

Finally, we notice that Theorem 5.4 (i) does not hold if $k = 1$ in (5.3) (i.e., $\dim M$ is a power of 2), as is seen by the following two examples.

**Example 5.21.** Consider the closed $(2n + 2(=2^t))$-manifold $RP(n, n, 0) = RP(p_1^i\xi_n \oplus p_2^i\xi_n \oplus p_3^i\xi_0) \quad (n = t' - 1, t = 2, 3, 4)$, given in [9, Lemma 3.4], where $p_i$ is the projection of $RP^n \times RP^n \times RP^0$ onto the $i$th factor and $ξ_i$ is the canonical line bundle over $RP^i$. Then,

$$[RP(n, n, 0)] \neq 0, \text{ w}_i[RP(n, n, 0)] = 0 \quad \text{for } i \geq 3.$$  

**Proof.** The first assertion is valid, because $[RP(n, n, 0)]$ is indecomposable by [9, Lemma 3.4]. The second assertion is shown by using [11, Lemma 2.9] and [6, p. 39, Prop. 4].

**Example 5.22.** For $RP^{p'}$ with $p > 1$, it holds that

$$[RP^{p'}] \neq 0 \text{ and } wRP^{p'} = 1 + w_1 + w_{p'}.$$  

**Proof.** This is clear by Example 4.19 and $(w_1)^{p'} \neq 0$. 
References


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