The polynomials on $w_1$, $w_2$ and $w_3$ in the universal Wu classes

Dedicated to Professor Teiichi Kobayashi on his 60th birthday

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ABSTRACT. The cohomology ring $H^*(BO; \mathbb{Z}_2)$ is the polynomial algebra $\mathbb{Z}_2[w_1, w_2, w_3, \cdots]$, where $w_i$ is the $i$-dimensional universal Stiefel-Whitney class. The $i$-dimensional universal Wu class $v_i$ is defined inductively as follows: $v_0 = w_0 = 1$ and $v_i = v_i + \sum_{j=1}^i Sq^j v_{i-j} (i \geq 1)$, where $Sq^j$ is the Steenrod squaring operation. We can describe explicitly the polynomials on $w_1$, $w_2$ and $w_3$ in $v_i$.

1. Introduction

Let $BO$ be the space which classifies stable real vector bundles. Then its mod 2 cohomology $H^*(BO; \mathbb{Z}_2)$ is the polynomial algebra over $\mathbb{Z}_2$ on the universal Stiefel-Whitney classes $w_i \in H^i(BO; \mathbb{Z}_2)$ for $i > 1$ (cf. [4], [10]).

The $i$-dimensional universal Wu class $v_i$ ($i \geq 0$) is the element of $H^i(BO; \mathbb{Z}_2)$, and this is defined inductively by using the Steenrod squaring operations $Sq^j$ in the following way (cf. [3], [6], [7], [8]):

\begin{equation}
0 = v_0 = 1 \text{ and } v_i = v_i + Sq^1 v_{i-1} + \cdots + Sq^i v_0 \quad \text{if } i \geq 1.
\end{equation}

The $i$-dimensional Wu class $v_i(M)$ of a closed $n$-dimensional manifold $M$ is the unique element of $H^i(M; \mathbb{Z}_2)$ such that

\begin{equation}
Sq^j x = xv_i(M) \quad \text{for all } x \in H^{n-i}(M; \mathbb{Z}_2),
\end{equation}

and the following relations between the Stiefel-Whitney classes and the Wu classes of $M$ hold (cf. [4], [9]):

\begin{equation}
v_0(M) = 1 \text{ and } w_i(M) = v_i(M) + Sq^1 v_{i-1}(M) + \cdots + Sq^i v_0(M) \quad \text{if } i \geq 1.
\end{equation}

So if $f$ denotes the classifying map for the stable tangent bundle of $M$, then

\begin{equation}
f^*w_i = w_i(M) \quad \text{and } f^*v_i = v_i(M) \quad \text{if } i \geq 0.
\end{equation}

Let $J$ be the ideal of $H^*(BO; \mathbb{Z}_2)$ generated by the squares $w_1^2$, $w_2^2$, $w_3^2$, \cdots.
Then the total universal Wu class \( v = 1 + v_1 + v_2 + \cdots \mod J \) is as follows (cf. [13]):

\[
v \equiv 1 + \sum w_i \cdots w_i \mod J,
\]

where \( \Sigma \) is taken over all sequences \( 1 \leq i_1 < \cdots < i_l \) \((l \geq 1)\) satisfying \( \{i_1, \cdots, i_l\} = \{a_1, b_1, \cdots, a_m, b_m, \gamma_1, \cdots, \gamma_n\} \) \((l = 2m + n, m \geq 0, n \geq 0)\) such that \( a_j + b_j \) and \( \gamma_j \) are all powers of 2.

So the final goal is to describe all monomials in \( v \) which belong to \( J \). It is known that \( w_j^2 \) appears in \( v_j \) \((i = 2^j > 0)\) if and only if \( a(i) = 1 \) or 2, and also \( w_j w_{i-j}(i > j > 2) \) appears in \( v_i \) if and only if \( a(i) = 1 \) and \( i/2 < j < i \), or \( a(i) = 2 \) and \( 2^b < j \leq 2^a \) with \( i = 2^a + 2^b \) \((a > b \geq 0)\), where \( a(i) \) denotes the number of \( \Gamma \)s in the dyadic expansion of \( i \) (cf. [2]).

Now applying the Wu formula (cf. [5], [11])

\[
(1.3) \quad Sq^{t} w_i = \sum_{t=0}^{i} \binom{i-j-1+t}{t} w_{j-t} w_{i+t} \quad (0 \leq j < i),
\]

we can suppose the following (cf. Theorem 4.9, [12]):

\[
(1.4) \quad v_j \equiv \begin{cases} 
(w_2 + w_2^2)^{l/2} \mod I_3 & \text{if } j = 2^a \ (a \geq 1), \\
(S_{a-b-1} w_2^2 w_1^{l/2-2^{a+b+1}})^{2^b} \mod I_3 & \text{if } j = 2^a + 2^b \ (a > b \geq 0), \\
0 \mod I_3 & \text{if } a(j) \geq 3,
\end{cases}
\]

where \( I_3 \), generally \( I_k \) denotes the ideal of \( H^*(BO; Z_2) \) generated by \( w_k, w_{k+1}, \ldots \).

But the Wu classes modulo \( I_4 \) seem very complicated. So we study these classes in this paper.

Let \( P_t \) be the element of \( H^{2^{t+1}+1}(BO; Z_2) \) defined by

\[
P_t = Sq^{2^t} Sq^{2^{t-1}} \cdots Sq^{1} w_2^2 \quad \text{if } t \geq 0; \text{ and } P_t = 0 \quad \text{if } t = -1.
\]

Then the Wu classes modulo \( I_4 \) are given by the following theorems.

**Theorem 1.5.**

(i) \( v_j \equiv (v_{a})^{l/4} \equiv (w_3 w_1 + w_2^2 + w_1^4)^{l/4} \mod I_4 \) \( \text{if } j = 2^{a} \ (a \geq 2). \)

(ii) \( v_i \equiv P_{a-2} w_1^{l-1/2} + w_2^{l-1/2} \mod I_4 \) \( \text{if } i = 2^{a} + 1 \ (a \geq 1). \)

(iii) \( v_i \equiv P_{a-1} w_1 + (P_{a-2} + P_{a-3} w_2^{l-2}) w_1^2 + w_2 \mod I_4 \) \( \text{if } i = 2^{a} + 2 \ (a \geq 2); \text{ and } v_j \equiv (v_{a})^{l/2} \mod I_4 \) \( \text{if } j = 2^{a} + 2^b \ (a > b \geq 1) \) and \( i = j/2^{b-1}. \)

(iv) \( v_i \equiv P_{a-1} w_1^{2^b} + P_{a-b-1} w_1 + P_{b-1} w_1^{2^a} \mod I_4 \) \( \text{if } i = 2^{a} + 2^b + 1 \ (a > b \geq 1); \text{ and } v_j \equiv (v_{a})^{l/2} \mod I_4 \) \( \text{if } j = 2^{a} + 2^b + 2^c \ (a > b > c \geq 0) \) and \( i = j/2^c. \)

(v) \( v_j \equiv 0 \mod I_4 \) \( \text{if } a(j) \geq 4. \)
The above theorem will be proved in §3.
Let \( F_{m,n} \) be the element of \( H^{2m-2n} (BO; \mathbb{Z}_2) \) defined by
\[
F_{m,n} = \sum_{i=n}^{m-1} w_2^{i-n} w_1^{2m-2i+1} \quad \text{if } m > n \geq 0; \quad \text{and } F_{m,n} = 0 \quad \text{if } n \geq m \geq 0.
\]
And if \( p = 2^{p_1} + 2^{p_2} + \cdots + 2^{p_s} \) with \( s \geq 1 \) and \( p_1 > p_2 > \cdots > p_s \geq 0 \), then set
\[
G_p = \begin{cases} 
F_{p_1, p_2+1} F_{p_2, p_3+1} \cdots F_{p_{s-1}, p_s+1} & \text{if } s \geq 2, \\
1 & \text{if } s = 1.
\end{cases}
\]
and also set \( h(p) = p_1 \) and \( l(p) = p_s \).
Let \( P_t(p) \) be the sum consisting of all monomials on \( w_3, w_2 \) and \( w_1 \) in \( P_t \) such that each power of \( w_3 \) for such monomials is \( p \). If there are no such monomials, then also set \( P_t(p) = 0 \) in \( H^{2t+1} (BO; \mathbb{Z}_2) \).
Then we have the following, which will be proved in §4.

**Theorem 1.6.** Let \( p \geq 2 \) be an even integer. Then
\begin{enumerate}
  \item \( P_t(1) = w_3 F_{t+1, 0} \).
  \item \( P_t(p) = \ell F_{t+1, h(p)+1} G_p F_{l(p), 0} w_2 \).
  \item \( P_t(p+1) = \ell^{p+1} F_{t+1, h(p)+1} G_p F_{l(p), 0} \).
\end{enumerate}

If we apply Theorem 1.6 to Theorem 1.5, then common monomials will appear and they will cancel each other. Explicit descriptions by the distinct monomials of the Wu classes modulo \( I_4 \) will be obtained in §5.

### 2. Iterated Steenrod operations on the Stiefel–Whitney classes

Let \( \theta^i \) be the elements of the mod 2 Steenrod algebra defined inductively by
\[
\begin{align*}
\theta^0 &= Sq^0 = 1, \quad \theta^1 = Sq^1, \\
\theta^i &= Sq^i + Sq^{i-1} \theta^1 + Sq^{i-2} \theta^2 + \cdots + Sq^1 \theta^{i-1} \quad \text{if } i \geq 2 \quad \text{(cf. [12])}. \quad \tag{2.1}
\end{align*}
\]
Then \( \theta^i = \sum Sq^{j_1} \cdots Sq^{j_s} \), where \( \Sigma \) is taken over all sequences \( (j_1, \cdots, j_s) \) consisting of positive integers such that \( j_1 + \cdots + j_s = i \) for \( i \geq 1 \); and this implies the following equality:
\[
\theta^i = Sq^i + \theta^1 Sq^{i-1} + \theta^2 Sq^{i-2} + \cdots + \theta^{i-1} Sq^1 \quad \text{if } i \geq 2. \quad \tag{2.2}
\]

From (2.1) and (2.2), the following equalities hold:
\[
(Sq^0 + Sq^1 + Sq^2 + \cdots)(\theta^0 + \theta^1 + \theta^2 + \cdots) = 1, \tag{2.3}
\]
\[
(\theta^0 + \theta^1 + \theta^2 + \cdots)(Sq^0 + Sq^1 + Sq^2 + \cdots) = 1.
\]
Thus the inverse \( Sq^{-1} \) of \( Sq = Sq^0 + Sq^1 + Sq^2 + \cdots \) is given by

\[
Sq^{-1} = \theta^0 + \theta^1 + \theta^2 + \cdots
\]

**Proposition 2.5.** Let \( i \geq 0 \). Then

\[
v_i = \theta^i w_0 + \theta^{i-1} w_1 + \cdots + \theta^0 w_i.
\]

**Proof.** Set \( w = w_0 + w_1 + w_2 + \cdots \) and \( v = v_0 + v_1 + v_2 + \cdots \). Then using (1.1) we see \( w = Sq v \), and so \( v = Sq^{-1} w \). Thus (2.4) implies the conclusion. \( \square \)

The following lemma is well-known (cf. [12]).

**Lemma 2.6.** (i) Let \( x \) be a one dimensional cohomology class. Then

\[
\theta^i x = \begin{cases} 
  x^{i+1} & \text{if } i + 1 \text{ is a power of } 2, \\
  0 & \text{otherwise}.
\end{cases}
\]

(ii) Let \( x \) and \( y \) be cohomology classes. Then

\[
\theta^i(xy) = \sum_{j+k=i} (\theta^j x)(\theta^k y).
\]

**Proof.** (i) For a sequence \((j_1, j_2, \ldots, j_s)\) consisting of positive integers, we see

\[
Sq^{j_1}Sq^{j_2} \cdots Sq^{j_s} = \begin{cases} 
  x^{2^s} & \text{if } (j_1, j_2, \ldots, j_s) = (2^{s-1}, 2^{s-2}, \ldots, 1), \\
  0 & \text{otherwise}.
\end{cases}
\]

Thus we obtain (i).

(ii) It holds that \( Sq\{Sq^{-1}(xy)\} = Sq\{(Sq^{-1}x)(Sq^{-1}y)\} \) since the left side is \((SqSq^{-1})(xy) = xy\) and the right side is \((SqSq^{-1}x)(SqSq^{-1}y) = xy\). Applying \( Sq^{-1} \) on both sides of this equality, we see \( Sq^{-1}(xy) = (Sq^{-1}x)(Sq^{-1}y) \). Thus (ii) follows from (2.4). \( \square \)

**Proposition 2.7.** Let \( i \) and \( j \) be positive integers such that \( i(j) > j \). Then

\[
\theta^{i-j} w_j = 0.
\]

**Proof.** Let \( BO(n) \) be the space which classifies real \( n \)-plane bundles. Then \( H^*(BO(n), Z_2) \) is the polynomial algebra over \( Z_2 \) on the Stiefel–Whitney classes \( w_m(\gamma^n) \in H^m(BO(n); Z_2) \) (\( 1 \leq m \leq n \)) of the universal bundle \( \gamma^n \) over \( BO(n) \). And if \( g: BO(n) \to BO \) denotes the natural inclusion map, then \( g^* w_m = w_m(\gamma^n) \) and

\[
g^*: H^k(BO; Z_2) \to H^k(BO(n); Z_2)
\]

is an isomorphism for all \( k \leq n \) (cf. [4], [10]).
The polynomials on $w_1$, $w_2$ and $w_3$ in the universal Wu classes

Now by the splitting principle, there exists a space $X$ and a map $f: X \to BO(n)$ such that the induced bundle $f^*y^n$ is isomorphic to the Whitney sum $\xi_1 \oplus \cdots \oplus \xi_n$ of suitable real line bundles $\xi_s$ ($1 \leq s \leq n$) over $X$, and also

$$f^*: H^k(BO(n); \mathbb{Z}_2) \to H^k(X; \mathbb{Z}_2)$$

is a monomorphism for all $k$ (cf. [5]).

Let $x_s = w_1(\xi_s)$ ($1 \leq s \leq n$). Then

$$(f^*g^*)w_j = f^*(\gamma^n) = w_j(f^*\gamma^n) = w_j(\xi_1 \oplus \cdots \oplus \xi_n) = \sum x_{t_1} \cdots x_{t_j},$$

where $\Sigma$ is taken over all sequences $(t_1, \cdots, t_j)$ such that $1 \leq t_1 < \cdots < t_j \leq n$. So we see

$$(f^*g^*)\theta^{i-j}w_j = \theta^{i-j}(f^*g^*)w_j = \sum \theta^{i-j}(x_{t_1} \cdots x_{t_j}).$$

From Lemma 2.6 we have

$$\theta^{i-j}(x_{t_1} \cdots x_{t_j}) = \sum x_{p_{t_1}} \cdots x_{p_{t_j}},$$

where $\Sigma$ is taken over all sequences $(p_{t_1}, \cdots, p_{t_j})$ consisting of powers of 2 such that $p_{t_1} + \cdots + p_{t_j} = i$. But such a sequence does not exist since $\alpha(p_{t_1} + \cdots + p_{t_j}) \leq j < \alpha(i)$. Thus $f^*g^*w_j = 0$, and so $\theta^{i-j}w_j = 0$ by choosing $n$ such that $i \leq n$. □

Next we consider the case $\alpha(i) \leq j$, and obtain the following.

**Proposition 2.8.** Let $i$ and $j$ be positive integers such that $\alpha(i) \leq j \leq \alpha(i) + l(i)$. Then

$$\theta^{i-j}w_j = (\theta^{i/2m-l}w_j)^{2m},$$

where $m = \alpha(i) + l(i) - j$.

**Proof.** We use the same notations as the proof of Proposition 2.7. Then

$$(f^*g^*)\theta^{i-j}w_j = \sum \sum x_{p_{t_1}} \cdots x_{p_{t_j}}.$$ 

Here we have $p_k \geq 2^m$ for all $k$ ($1 \leq k \leq j$). To show this equality, if $j = 1$, then $\alpha(i) = 1$, and so $p_1 = i = 2^{l(i)} = 2^m$. Next let $j \geq 2$. If $p_1 = 2^a < 2^m$ for example, then $\alpha(p_2 + \cdots + p_j) = \alpha(i - p_1) = (\alpha(i) - 1) + (l(i) - a) > (\alpha(i) - 1) + (j - \alpha(i)) = j - 1$, which is incompatible with $\alpha(p_2 + \cdots + p_j) \leq j - 1$. Thus we obtain

$$(f^*g^*)\theta^{i-j}w_j = (\sum \sum x_{q_{t_1}} \cdots x_{q_{t_j}})^{2^m},$$

where $\Sigma$ is taken over all sequences $(t_1, \cdots, t_j)$ and $(q_1, \cdots, q_j)$ such that $1 \leq t_1 < \cdots < t_j \leq n$ and $q_1 + \cdots + q_j = i/2^m$ with $q_s$ ($1 \leq s \leq j$) powers of 2.
Thus \((f^*g^*)(\theta^{i-j} w_j) = (f^*g^*)(\theta^{(i/2^m-j)} w_j)^{2^m}\), and so \(\theta^{i-j} w_j = (\theta^{(i/2^m-j)} w_j)^{2^m}\) by choosing \(n\) such that \(i \leq n\). □

We obtain the following by using the above propositions.

**Theorem 2.9.** Let \(j\) and \(k\) be positive integers such that \(\alpha(j) + 1 \leq k \leq \alpha(j) + l(j) + 1\). Then

\[ v_j \equiv (v_i)^{2^m} \mod I_k, \]

where \(i = j/2^m\) and \(m = \alpha(j) + l(j) + 1 - k\).

**Proof.** Since \(l(j) - m = k - \alpha(j) - 1 \geq 0\), \(i\) is a positive integer, and also \(i \geq k - 1\) holds because of \(i \geq 2k^{-2} + 2k^{-3} + \cdots + 2^{k-\alpha(j)-1} \geq 2^{k-2} \geq k - 1\). Now from Proposition 2.7 and (1.3), we see \(\theta^{i-s} w_s = 0\) \((1 < s < \alpha(j))\) and \(\theta^{i-s} w_s \equiv 0 \mod I_k\) \((k < s < j)\). Thus by Proposition 2.5 with \(\theta^j w_0 = 0\) \((j \geq 1)\), we have

\[ v_j \equiv \sum_{s=\alpha(j)}^{k-1} \theta^{j-s} w_s \mod I_k. \]

Using Proposition 2.8 because of \(\alpha(j) \leq s \leq k - 1 \leq \alpha(j) + l(j)\), we see

\[ \theta^{j-s} w_s = (\theta^{j/2^p-s} w_s)^{2^p} \text{ for } p = \alpha(j) + l(j) - s. \]

Similarly noting \(\alpha(i) = \alpha(j) \leq k - 1\), we have

\[ v_i \equiv \sum_{s=\alpha(i)}^{k-1} \theta^{i-s} w_s \mod I_k. \]

Since \(\alpha(i) + l(i) = \alpha(j) + l(j) - m = k - 1\), we see

\[ \theta^{i-s} w_s = (\theta^{i/2^q-s} w_s)^{2^q} \text{ for } q = \alpha(i) + l(i) - s. \]

Since \(i/2^q = j/2^{m+q} = j/2^p\), we obtain

\[ (\theta^{i-s} w_s)^{2^m} = (\theta^{i/2^q-s} w_s)^{2^{m+q}} = (\theta^{j/2^p-s} w_j)^{2^p}. \]

Therefore \(v_j \equiv (v_i)^{2^m} \mod I_k\). □

**Remark 2.10.** Let \(j\) and \(k\) be positive integers such that \(\alpha(j) \geq k\). Then in the proof of Theorem 2.9, it is shown that \(v_j \equiv 0 \mod I_k\).

3. The Wu classes modulo \(I_4\)

In this section we will study the Wu classes modulo \(I_4\) and prove Theorem 1.5.
PROPOSITION 3.1.
(i) \( v_j \equiv (v_4)^{ij4} \mod I_4 \) if \( \alpha(j) = 1 \) \( (j \geq 4) \).
(ii) \( v_j \equiv (v_i)^{ijj} \mod I_4 \) if \( \alpha(j) = 2 \) and \( i = j/2^{l(j) - 1}, l(j) \geq 1 \).
(iii) \( v_j \equiv (v_i)^{ijij} \mod I_4 \) if \( \alpha(j) = 3 \) and \( i = j/2^{l(j)} \).
(iv) \( v_j \equiv 0 \mod I_4 \) if \( \alpha(j) \geq 4 \).

PROOF. Let \( 1 \leq \alpha(j) \leq 3 \). Then Theorem 2.9 implies
\[
v_j \equiv (v_4)^{2m} \mod I_4 \text{ if } l(j) \geq 3 - \alpha(j),
\]
where \( i = j/2^m \) and \( m = \alpha(j) + l(j) - 3 \). Thus (i), (ii) and (iii) follow from this. Also (iv) follows from Remark 2.10. \( \square \)

Thus we have the following remark from the above proposition.

REMARK 3.2. To describe the Wu classes \( v_i \) modulo \( I_4 \), it is sufficient only to describe \( v_i \) modulo \( I_4 \), where \( i = 4, 2^a + 1 \) \( (a \geq 1) \), \( 2^a + 2 \) \( (a \geq 2) \) and \( 2^a + 2^b + 1 \) \( (a > b \geq 1) \).

The following lemma is known (cf. [1], [12]).

LEMMA 3.3. Let \( i \) be a power of 2. Then
(i) \( \theta^{i-1} = Sq^{i/2}Sq^{i/4} \cdots Sq^1 \) if \( i \geq 2 \).
(ii) \( \theta^{i-1-j} = Sq^{i/2-s}\theta^{i/2-1-(j-\delta)} + Sq^{i/2}\theta^{i/2-1-j} \) if \( 1 \leq j < i/2 \) and \( s = 2^{h(j)} \).
(iii) \( \theta^{i-j} = Sq^{i/2}Sq^{i/4} \cdots Sq^s\theta^{s-j} \) if \( 1 \leq j \leq h(i) \) and \( s = 2^{i-1} \).
(iv) \( \theta^{2k+1} = \theta^{2k}Sq^1 \) if \( k \geq 0 \).

The Wu classes modulo \( I_4 \) of the dimensions in Remark 3.2 are as follows:

PROPOSITION 3.4. Let \( i \) be \( 2^a + 1 \) \( (a \geq 1) \), \( 2^a + 2 \) \( (a \geq 2) \), \( 2^a + 2^b + 1 \) \( (a > b \geq 1) \). Then
\[
v_i \equiv \begin{cases} (\theta^{i/2-2}w_2)^2 + \theta^{i-3}(w_2w_1) & \mod I_4 \text{ if } i = 2^a + 2, \\ \theta^{i-3}(w_2w_1) & \mod I_4 \text{ otherwise }.
\end{cases}
\]

PROOF. We use Propositions 2.5 and 2.7. For \( i \geq 1 \), \( \theta^iw_0 = 0 \); and using (1.3), \( \theta^{i-s}w_0 \equiv 0 \mod I_4 \) for \( 4 \leq s \leq i \).

Let \( i = 2^a + 1 \) \( (a \geq 1) \). Then since \( w_3 = Sq^1w_2 + w_2w_1 \) by (1.3), and \( \theta^{i-3}Sq^1 = \theta^{i-2} \) by Lemma 3.3 (iv), we see
\[
v_i \equiv \theta^{i-2}w_2 + \theta^{i-3}w_3 = \theta^{i-2}w_2 + \theta^{i-3}(Sq^1w_2 + w_2w_1) = \theta^{i-3}(w_2w_1) \mod I_4.
\]

Let \( i = 2^a + 2^b + 1 \) \( (a > b \geq 1) \). Then similarly we see
\[
v_i \equiv \theta^{i-3}w_3 = \theta^{i-3}(Sq^1w_2 + w_2w_1) = \theta^{i-2}w_2 + \theta^{i-3}(w_2w_1) \mod I_4
\]
since \( \theta^{i-2}w_2 = 0 \) also by Proposition 2.7.
Let \( i = 2^a + 2 \) \((a \geq 2)\). Then

\[
v_i \equiv \theta^{i-3}w_2 + \theta^{i-3}w_3 = \theta^{i-2}w_2 + \theta^{i-3}Sq^1w_2 + \theta^{i-3}(w_2w_1) \mod I_4.
\]

Here \( \theta^{i-2}w_2 = (\theta^{i-2}w_2)^2 \) by Proposition 2.8, and \( \theta^{i-3}Sq^1 = \theta^{i-4}Sq^1Sq^1 = 0 \) by Lemma 3.3 (iv) and the Adem relation \( Sq^1Sq^1 = 0 \). \( \square \)

In the following proposition, we consider \( \theta^{i-3}(w_2w_1) \) in the above proposition.

**PROPOSITION 3.5.** Let \( i \) be \( 2^a + 1 \) \((a \geq 2)\), \( 2^a + 2 \) \((a \geq 2)\), \( 2^a + 2^b + 1 \) \((a > b \geq 1)\). Then

\[
\theta^{i-3}(w_2w_1)
\]

\[
\begin{cases}
w_2^m w_1 + P_{a-2} w_1^m & \text{if } i = 2^a + 1 \text{ and } m = 2^{a-1}, \\
P_{a-1} w_1 + w_2^m w_1^2 + P_{a-3} w_1^m + w_2 w_1^{2m} & \text{if } i = 2^a + 2 \text{ and } m = 2^{a-1}, \\
P_{a-b-1} w_1 + P_{a-1} w_1^a + P_{b-1} w_1^{2m} & \text{if } i = 2^a + 2^b + 1 \text{ and } m = 2^{a-1}, n = 2^b.
\end{cases}
\]

**PROOF.** Let \( i = 2^a + 1 \) \((a \geq 2)\) and \( m = 2^{a-1} \). Then using Lemma 2.6 and Proposition 2.7, we see

\[
\theta^{i-3}(w_2w_1) = (\theta^{2m-2}w_2)w_1 + \sum_{p=1}^{a-1} \{ \theta^{(2m-2p+1)-2}w_2 \} (\theta^{2p-1}w_1)
\]

since \( \alpha(2m - 2p + 1) = a - p + 1 > 2 \) for \( 1 \leq p < a - 1 \).

Here from Proposition 2.8 and Lemma 3.3 (i), we see

\[
\theta^{2m-2}w_2 = (\theta^0 w_2)^m = w_2^m \quad \text{and} \quad \theta^{m-1}w_2 = Sq^m Sq^{m+1} \cdots Sq^1 w_2 = P_{a-2}.
\]

Let \( i = 2^a + 2 \) \((a \geq 2)\) and \( m = 2^{a-1} \). Then similarly we see

\[
\theta^{i-3}(w_2w_1) = (\theta^{2m-1}w_2)w_1 + (\theta^{2m-2}w_2)w_1^2 + \sum_{p=2}^{a-1} \{ \theta^{(2m-2p+2)-2}w_2 \} w_1^p + w_2 w_1^{2m}
\]

\[
= (\theta^{2m-1}w_2)w_1 + (\theta^{2m-2}w_2)w_1^2 + (\theta^m w_2)w_1^m + w_2 w_1^{2m}
\]

since \( \alpha(2m - 2p + 2) = a - p + 1 > 2 \) for \( 2 \leq p < a - 1 \).

Here \( \theta^{2m-1}w_2 = P_{a-1} \), \( \theta^{2m-2}w_2 = w_2^m \) and \( \theta^m w_2 = (\theta^{m-2}w_2)^2 = P_{a-3}^2 \) for \( a \geq 3 \).

Thus noting \( P_{a-3} = 0 \) for \( a = 2 \), we obtain the conclusion.

Let \( i = 2^a + 2^b + 1 \) \((a > b \geq 1)\) and \( m = 2^{a-1}, n = 2^b \). Then similarly we see
\[ \theta^{i-3}(w_2 w_1) = (\theta^{2m+n-2}w_2)w_1 + \sum_{p=1}^{b-1} \{ \theta^{(2m+n-2p+1)-2}w_2 \} w_1^{2p} + (\theta^{2m-1}w_2)w_1^{n} \]
\[ + \sum_{p=b+1}^{a-1} \{ \theta^{(2m-2p+n+1)-2}w_2 \} w_1^{2p} + (\theta^{n-1}w_2)w_1^{2m} \]
\[ = (\theta^{2m+n-2}w_2)w_1 + (\theta^{2m-1}w_2)w_1^{n} + (\theta^{n-1}w_2)w_1^{2m} \]
since \( \alpha(2m + n - 2p + 1) = b - p + 2 > 2 \) for \( 1 \leq p \leq b - 1 \), and also \( \alpha(2m - 2p + n + 1) = a - p + 2 > 2 \) for \( b + 1 \leq p \leq a - 1 \).

Here \( \theta^{2m+n-2}w_2 = (\theta^{2m/n-1}w_2)^n = P_{a-b-1}^n, \theta^{2m-1}w_2 = P_{a-1} \) and \( \theta^{n-1}w_2 = P_{b-1} \).

We are now in a position to prove Theorem 1.5.

**PROOF OF THEOREM 1.5.** (i) \( v_4 = \theta^3 w_1 + \theta^2 w_2 + \theta w_3 \) mod \( I_4 \). Here by Proposition 2.8 and (1.3), we see \( \theta^3 w_1 = (\theta^0 w_1)^4 = w_1^4 \), \( \theta^2 w_2 = (\theta^0 w_2)^2 = w_2^2 \)
and \( \theta^1 w_3 = Sq^1 w_3 = w_3 w_1 \). Hence (i) follows from Proposition 3.1 (i).

(ii) \( v_3 = w_2 w_1 \) by Proposition 3.4. So (ii) holds for \( a = 1 \) since \( P_{-1} = 0 \) by the definition. Next let \( i = 2^a + 1 \) (\( a \geq 2 \)). Then Propositions 3.4 and 3.5 imply (ii).

(iii) Using Propositions 3.4, 3.5 and letting \( m = 2^{a-1} \), we see
\[ v_i = (\theta^{m-1}w_2)^2 + \theta^{i-3}(w_2 w_1) \]
\[ = P_{a-2}^2 + P_{a-1} w_1 + w_2 w_1^2 + P_{a-3}^2 w_1^m + w_2 w_1^{2m} \]
\[ = P_{a-1} w_1 + (P_{a-2} + P_{a-3} w_1^{m/2} + w_2 w_1^{2m})^2 + w_2 w_1^{2m} \) mod \( I_4 \),
which is the result for \( v_i \). And the one for \( v_j \) follows from Proposition 3.1 (ii).

(iv) Propositions 3.4 and 3.5 imply the result for \( v_i \). And the one for \( v_j \) follows from Proposition 3.1 (iii).

(v) This follows from Proposition 3.1 (iv). \( \square \)

4. \( Sq^i Sq^{2^i-1} \cdots Sq^1 w_2 \) modulo \( I_4 \)

In this section we will study the terms \( P_i = Sq^2 Sq^{2^i} \cdots Sq^1 w_2 \) which remain to be known in Theorem 1.5.

The following lemma will be used often.

**LEMMA 4.1 ([6]).** Let \( a = \Sigma_i a_i 2^i \) and \( b = \Sigma_i b_i 2^i \) (\( 0 \leq a_i, b_i \leq 1 \)). Then
\[ \left( \frac{a}{b} \right) = \prod_i \left( \frac{a_i}{b_i} \right) \mod 2. \]

We have the following formula on \( P_i \).
Proposition 4.2. Let \( x = w_3 + w_2 w_1 \) and \( y = w_2 + w_1^2 \). Then

\[
P_t = \sum_{0 \leq 3i < n} \binom{3i}{i} x^{2i+1} y^{n-3i-1} \mod I_4,
\]

where \( n = 2^t \) \((t \geq 0)\).

Proof. We notice the following (4.4):

\[
(4.4) \quad S^1 x = 0, \quad S^2 x \equiv xy \mod I_4, \quad S^3 x = x^2, \quad S^1 y = x, \quad S^2 y = y^2.
\]

In fact using (1.3) and the Cartan formula, we see

\[
S^1 x = w_3 w_1 + w_2 w_1^2 + (w_3 + w_2 w_1) w_1 = 0,
\]

\[
S^2 x = w_5 + w_4 w_1 + w_3 w_2 + (w_3 + w_2 w_1) w_1^2 + w_2 w_1^2 \equiv xy \mod I_4,
\]

\[
S^1 y = S^2 w_1 = w_3 + w_2 w_1 = x.
\]

And since \( \dim x = 3 \) and \( \dim y = 2 \), \( S^3 x = x^2 \) and \( S^2 y = y^2 \) hold.

We prove (4.3) by induction on \( t \). Since \( P_0 = S^1 w_2 = x \) and \( P_1 = S^2 P_0 = S^2 x \equiv xy \mod I_4, \) (4.3) holds for \( t = 0, 1 \). Assume that (4.3) holds for \( t - 1 \) \((t \geq 2)\). Then since \( S^t I_4 \subseteq I_4 \) and \( n = 2^t = \dim P_{t-1} - 1 \), the left side of (4.3) for \( t \) is as follows:

\[
P_t = S^n P_{t-1} = S^n \sum_{0 \leq 3i < n/2} \binom{3i}{i} x^{2i+1} y^{n/2-3i-1}
\]

\[
= \sum_{0 \leq 3i < n/2} \binom{3i}{i} \{(2i + 1)(S^2 x)x^{4i}y^{n-6i-2}
\]

\[
+ (n/2 - 3i - 1)x^{4i+2}(S^1 y)y^{n-6i-4}\}
\]

\[
= \sum_{0 \leq 6j < n/2} \binom{6j}{2j} \{(S^2 x)x^{8j}y^{n-12j-2} + x^{8j+2}(S^1 y)y^{n-12j-4}\}
\]

\[
+ \sum_{0 \leq 6j+3 < n/2} \binom{6j+3}{2j+1} \{(S^2 x)x^{8j+4}y^{n-12j-8}\}
\]

\[
= \sum_{0 \leq 6j < n/2} \binom{6j}{2j} \{x^{8j+1}y^{n-12j-1} + x^{8j+3}y^{n-12j-4}\}
\]

\[
+ \sum_{0 \leq 6j+3 < n/2} \binom{6j+3}{2j+1} \{x^{8j+5}y^{n-12j-7}\} \mod I_4.
\]

On the other hand, using Lemma 4.1, we see

\[
\sum_{0 \leq 3i < n} \binom{3i}{i} x^{2i+1} y^{n-3i-1} = S_1 + S_2 + S_3 + S_4,
\]
The polynomials on $w_1$, $w_2$ and $w_3$ in the universal Wu classes

where

\[
S_1 = \sum_{0 \leq 12j < n} \binom{12j}{4j} x^{8j+1} y^{n-12j-1} = \sum_{0 \leq 6j < n/2} \binom{6j}{2j} x^{8j+1} y^{n-12j-1},
\]

\[
S_2 = \sum_{0 \leq 12j+3 < n} \binom{12j+3}{4j+1} x^{8j+3} y^{n-12j-4} = \sum_{0 \leq 6j < n/2} \binom{6j}{2j} x^{8j+3} y^{n-12j-4},
\]

\[
S_3 = \sum_{0 \leq 12j+6 < n} \binom{12j+6}{4j+2} x^{8j+5} y^{n-12j-7} = \sum_{0 \leq 6j+3 < n/2} \binom{6j+3}{2j+1} x^{8j+5} y^{n-12j-7},
\]

and since \( \binom{12j+9}{4j+3} \) is even

\[
S_4 = \sum_{0 \leq 12j+9 < n} \binom{12j+9}{4j+3} x^{8j+7} y^{n-12j-10} = 0.
\]

Therefore (4.3) for \( t \) holds. \( \square \)

Next we have another inductive formula on \( P_t \).

**Proposition 4.5.** Let \( x = w_3 + w_2 w_1 \) and \( y = w_2 + w_1^2 \). Then

\[
P_t \equiv y^n P_{t-1} + x^n P_{t-2} \mod I_4,
\]

where \( n = 2^t - 1 \) (\( t \geq 1 \)).

**Proof.** We prove (4.6) by induction on \( t \). Since \( P_{t-1} = 0 \), \( P_0 = x \), \( P_1 \equiv xy \mod I_4 \), \( P_2 = S^2 q^3 P_1 \equiv S^3 (xy) = (S^2 x) y^2 + x^2 (S^1 y) \equiv xy^3 + x^3 \equiv y^2 P_1 + x^2 P_0 \mod I_4 \), (4.6) holds for \( t = 1, 2 \). Assume that (4.6) holds for \( t - 1 \) (\( t \geq 3 \)). Then the left side of (4.6) for \( t \) is as follows:

\[
P_t = S^2 q^n P_{t-1} \equiv S^2 q^n (y^{n/2} P_{t-2} + x^{n/2} P_{t-3})
\]

\[
= (S^{n-1} y^{n/2}) P_{t-2} + y^n (S^2 P_{t-2})
\]

\[
+ (S^{n+n/2-1} x^{n/2}) P_{t-3} + x^n (S^{n/2} P_{t-3}) \mod I_4.
\]

Here \( S^{n-1} y^{n/2} = 0 \) and \( S^{n+n/2-1} x^{n/2} = 0 \) since \( n - 1 \) and \( n + n/2 - 1 \) are both odd and \( n/2 \) is even; \( S^2 P_{t-2} = P_{t-1} \) and \( S^{n/2} P_{t-3} = P_{t-2} \). Thus \( P_t \equiv y^n P_{t-1} + x^n P_{t-2} \mod I_4 \), which completes induction on \( t \). \( \square \)

**Corollary 4.7.** Let \( t \geq 0 \). Then

\[
P_t \equiv \sum_{i=0}^{t} w_2^i w_1^{m-2i+1} \mod I_3,
\]

where \( m = 2t+1 + 1 \).
PROOF. We prove (4.8) by induction on \( t \). Since \( P_0 = Sq^1 w_2 \equiv w_2 w_1 \mod I_3 \) and \( P_1 = Sq^2 P_0 = Sq^2(w_2 w_1) = (Sq^1 w_2)w_1^2 + w_2^2 w_1 \equiv w_2 w_1^2 + w_2^2 w_1 \mod I_3 \), (4.8) holds for \( t = 0, 1 \). Assume that (4.8) holds for less than \( t - 1 \) \((t \geq 2)\). Then the left side of (4.8) for \( t \) is as follows by using Proposition 4.5 and letting \( n = 2^{t-1} \), \( m = 2^{t+1} + 1 \):

\[
P_t \equiv (w_2 + w_1^2)^{t-1} P_{t-1} + (w_2 w_1)^{t-1} P_{t-2}
\]

\[
\equiv (w_2 + w_1^2) \sum_{i=0}^{t-1} w_2^i w_1^{2i+1} + (w_2 w_1) \sum_{i=0}^{t-2} w_2^i w_1^{4i+2} + w_2^i w_1^{2i+3} + w_2^i w_1^{2i+3} \mod I_3,
\]

which completes induction on \( t \). \( \square \)

We obtain the following theorem which is (1.4) in §1 (cf. [12]).

**Theorem 4.9.**

(i) \( v_j \equiv (v_j)^{2^m} \mod I_3 \) if \( j = 2^a (a \geq 1) \).

(ii) \( v_i \equiv \sum_{j=0}^{a-1} w_2^2 w_1^{2j+2} \mod I_3 \) if \( i = 2^a + 1 (a \geq 1) \); and \( v_j \equiv (v_j)^{2^m} \mod I_3 \) if \( j = 2^a + 2^b (a > b \geq 0) \) and \( i = j/2^b \).

(iii) \( v_j \equiv 0 \mod I_3 \) if \( \alpha(j) \geq 3 \).

**Proof.** (iii) follows from Remark 2.10. So assume \( \alpha(j) \leq 2 \). Then by Theorem 2.9, we have

\[
v_j \equiv (v_j)^{2^m} \mod I_3 \text{ if } l(j) \geq 2 - \alpha(j),
\]

where \( i = j/2^m \) and \( m = \alpha(j) + l(j) - 2 \).

(i) If \( j = 2^a (a \geq 1) \), then \( v_j \equiv (v_2)^{2^m} \mod I_3 \). Here we see that \( v_2 = \theta^i w_1 + \theta^0 w_2 = w_1^2 + w_2 \).

(ii) If \( j = 2^a + 2^b (a > b \geq 0) \), then \( v_j \equiv (v_2)^{2^m} \mod I_3 \) where \( i = j/2^b \). Also if \( i = 2^a + 1 (a \geq 1) \), then using Corollary 4.7, we see

\[
v_i \equiv \theta^{i-2} w_2 = P_{a-1} \equiv \sum_{s=0}^{a-1} w_2^s w_1^{-2s} \mod I_3 . \square
\]

**Remark 4.10.** Using Theorem 1.5, we can also prove Theorem 4.9 in the following way:

(i) Let \( j = 2^a (a \geq 2) \). Then Theorem 1.5 (i) implies that \( v_j \equiv (w_2^2 + w_1^4)^{2^m} \mod I_3 \).
(ii) Let \( i = 2^a + 1 \) \((a \geq 1)\) and \( m = 2^{a-1} + 1 \). Then Theorem 1.5(ii) and Corollary 4.7 imply

\[
v_i \equiv \left( \sum_{s=0}^{a-2} w_2^2 w_1^{m-2s+1} \right) w_1^{m-1} + w_2^{m-1} w_1 = \sum_{s=0}^{a-1} w_2^2 w_1^{2s+1} \mod I_3.
\]

Let \( j = 2^a + 2 \) \((a \geq 2)\), \( i = j/2 \) and \( m = 2^a + 1 \). Then Theorem 1.5(iii), Corollary 4.7 and the above result for \( i = j/2 \) imply

\[
v_j \equiv \left( \sum_{s=0}^{a-1} w_2^2 w_1^{m-2s+1} \right) w_1 \]

\[
+ \left( \sum_{s=0}^{a-2} w_2^2 w_1^{m-1+2-2s+1} + \left( \sum_{s=0}^{a-3} w_2^2 w_1^{m-1+4-2s+1} \right) w_1^{m-1/4} + w_2^{m-1/4} w_1 \right)^2
\]

\[
+ w_2 w_1^{m-1} = \sum_{s=0}^{a-1} w_2^2 w_1^{2s+1} + w_2 w_1^{m-1} = \sum_{s=0}^{a-1} w_2^2 w_1^{2s+1}
\]

\[
= \left( \sum_{s=0}^{a-1} w_2^2 w_1^{2s+1} \right)^2 \equiv (v_i)^{ii} \mod I_3.
\]

Let \( j = 2^a + 2^b \) \((a > b \geq 1)\) and \( i = j/2^b \). Then Theorem 1.5(iii) and the above result for \( v_{2i} \) imply that \( v_j \equiv (v_{2i})^{j/2} \equiv (v_i)^{ii} \mod I_3 \).

(iii) Let \( i = 2^a + 2^b + 1 \) \((a > b \geq 1)\), \( m = 2^a + 1 \) and \( n = 2^b + 1 \). Then Theorem 1.5(iv) and Corollary 4.7 imply

\[
v_i \equiv \left( \sum_{s=0}^{a-1} w_2^2 w_1^{m-2s+1} \right) w_1^{m-1} + \left( \sum_{s=0}^{a-b-1} w_2^2 w_1^{m-1(n-1)+1-2s+1} \right) w_1^{n-1}
\]

\[
+ \left( \sum_{s=0}^{b-1} w_2^2 w_1^{m-2s+1} \right) w_1^{m-1}
\]

\[
= \sum_{s=0}^{a-1} w_2^2 w_1^{2s+1} + \sum_{s=b}^{a-1} w_2^2 w_1^{2s+1} + \sum_{s=0}^{b-1} w_2^2 w_1^{2s+1} = 0 \mod I_3.
\]

Let \( j = 2^a + 2^b + 2^c \) \((a > b > c \geq 0)\) and \( i = j/2^c \). Then Theorem 1.5(iv) and the above result for \( j/2^c \) imply that \( v_j \equiv (v_j)^{j/2} \equiv 0 \mod I_3 \).

Let \( \alpha(j) \geq 4 \). Then \( v_j \equiv 0 \mod I_3 \) by Theorem 1.5(v).

We are now in a position to prove Theorem 1.6.

**Proof of Theorem 1.6 (i).** By the definition of \( F_{r+1,0} \), (i) means the following:

\[
P_t(1) = w_3 \sum_{i=0}^{t} w_2^{i-1} w_1^{m-2i+1},
\]

where \( m = 2^{t+1} \).
We prove (4.11) by induction on \( \tau \). Since \( P_0 = w_3 + w_2 w_1 \) and \( P_1 = (w_3 + w_2 w_1) (w_2 + w_2^2) \mod I_4 \), we see that \( P_0(1) = w_3 \) and \( P_1(1) = w_4 (w_2 + w_2^2) \). So (4.11) holds for \( \tau = 0, 1 \). Assume that (4.11) holds for less than \( \tau - 1 \) \((\tau \geq 2)\). Then the left side of (4.11) for \( \tau \) is as follows by using Proposition 4.5 and letting \( m = 2^{\tau+1} \):

\[
P_\tau(1) = (w_2 + w_2^2)^{m/4} P_{\tau-1}(1) + (w_2 w_1)^{m/4} P_{\tau-2}(1)
= (w_2 + w_1^2)^{m/4} w_3 \sum_{i=0}^{\tau-1} w_2^{2i-1} w_1^{m/2 - 2^{\tau+1}} + (w_2 w_1)^{m/4} w_3 \sum_{i=0}^{\tau-2} w_2^{2i-1} w_1^{m/4 - 2^{\tau+1}}
= w_3 \sum_{i=0}^{\tau} w_2^{2i-1} w_1^{m - 2^{\tau+1}},
\]

which completes induction on \( \tau \). \( \square \)

**Proof of Theorem 1.6 (ii)**. We prove (ii) by induction on \( \tau \). We see that \( P_0(p) = 0 \) and \( P_1(p) = 0 \) since \( \dim P_0 = 3 < 3p \) and \( \dim P_1 = 5 < 3p \). On the other hand, \( F_{1, h(p)+1} = 0 \) and \( F_{2, h(p)+1} = 0 \) since \( h(p) + 1 \geq 2 \). Thus (ii) holds for \( \tau = 0, 1 \). Also using Propositions 4.2 or 4.5, we see that \( P_2 \equiv xy^3 + x^3 \mod I_4 \), where \( x = w_3 + w_2 w_1 \) and \( y = w_2 + w_1^2 \). So \( P_2(2) = w_3^2 w_2 w_1 \) and \( P_2(p) = 0 \) for \( p \geq 4 \). On the other hand, \( w_3^2 F_{3,2} F_{1,0} w_2 w_1 = w_3^2 w_2 w_1 \) and \( F_{3, h(p)+1} = 0 \) for \( p \geq 4 \) since \( h(p) \geq 2 \) for \( p \geq 4 \). Hence (ii) also holds for \( \tau = 2 \). Assume that (ii) holds for less than \( \tau - 1 \) \((\tau \geq 3)\). Then the left side of (ii) for \( \tau \) is as follows by using Proposition 4.5 and letting \( n = 2^{\tau-1} \):

**Case 1.** \( p < n \).

In this case \( \tau \geq h(p) + 2 \) holds, and we see

\[
P_\tau(p) = (w_2 + w_1^2)^n P_{\tau-1}(p) + (w_2 w_1)^n P_{\tau-2}(p)
= w_3^2 G_p F_{1, h(p)+1} + w_1^n F_{1, h(p)+1} \}
\]

Here by letting \( m = h(p) + 1 \), we see

\[
(w_2^n + w_{1}^2) F_{t,m} + w_2 w_1^n F_{t-1,m} = \sum_{i=m}^{t} w_2^{2i-2m} w_1^{4n-2^{\tau+1}} = F_{t+1,m}.
\]

Thus \( P_\tau(p) = w_3^2 F_{t+1, h(p)+1} G_p F_{1, h(p)+1} w_2 w_1 \), which completes induction on \( \tau \).

**Case 2.** \( p = n \).

In this case, using Corollary 4.7 and noting \( P_{\tau-1}(p) = 0 \), \( P_{\tau-2}(p) = 0 \), we see

\[
P_\tau(p) = w_3^2 \sum_{i=0}^{\tau-2} w_2^{2i} w_1^{n - 2^{\tau+1}} + 1.
\]

On the other hand, since \( h(p) = l(p) = t - 1 \), \( F_{t+1,t} = 1 \) and \( G_p = 1 \), we see
The polynomials on $w_1$, $w_2$ and $w_3$ in the universal Wu classes

$$w_3^p F_{t+1,h(p)+1} G_p F_{l(p)+1,0} w_2 w_1 = w_3^p F_{t-1,0} w_2 w_1 = w_3^p \sum_{i=0}^{t-2} w_2^i w_1^{n-2i} + 1,$$

which completes induction on $t$.

Case 3. $n < p < 2n$.
In this case $h(p) = t - 1$ and $p - n (> 0)$ is even. So noting $P_{t-1}(p) = 0$, $P_{t-2}(p) = 0$, $F_{t-1,h(p-n)+1} G_{p-n} = G_p$ and $l(p-n) = l(p)$, we see

$$P_t(p) = w_3^p P_{t-2}(p-n)$$

$$= w_3^p w_3^{n-p} F_{t-1,h(p-n)+1} G_{p-n} F_{l(p-n),0} w_2 w_1 = w_3^p G_p F_{l(p)+1,0} w_2 w_1.$$

On the other hand, since $F_{t+1,h(p)+1} = F_{t+1,t} = 1$, we see

$$w_3^p F_{t+1,h(p)+1} G_p F_{l(p),0} w_2 w_1 = w_3^p G_p F_{l(p),0} w_2 w_1,$$

which completes induction on $t$.

Case 4. $p \geq 2n$.
Since $\dim P_t = 4n + 1$ and $3p \geq 6n > 4n + 1$, we see $P_t(p) = 0$. On the other hand, since $h(p) + 1 \geq t + 1$, we see $F_{t,1,h(p)+1} = 0$, which completes induction on $t$. $\square$

**Proof of Theorem 1.6 (iii).** We prove (iii) in the same way as the proof of (ii). Then we see that (iii) holds for $t = 0, 1$. And since $P_2(3) = w_3^3$ and $P_2(p+1) = 0$ for $p \geq 4$, (iii) also holds for $t = 2$. Assume that (iii) holds for less than $t - 1$ ($t \geq 3$). Then the left side of (iii) for $t$ is as follows by using Proposition 4.5 and letting $n = 2^{t-1}$:

Case 1. $p + 1 < n$.
In this case $t \geq h(p+1) + 2 = h(p) + 2$ holds, and we see

$$P_t(p+1) = (w_2 + w_2^2)^p P_{t-1}(p+1) + (w_2 w_1)^p P_{t-2}(p+1)$$

$$= w_3^{t+1} G_p F_{l(p)+1,0} F_{t+1,h(p)+1}.$$

Case 2. $p + 1 = n + 1$.
In this case, using Theorem 1.6(i) and noting $P_{t-1}(p+1) = 0$, $P_{t-2}(p+1) = 0$, $h(p) = l(p) = t - 1$, $F_{t+1,h(p)+1} = F_{t+1,t} = 1$ and $G_p = 1$, we see

$$P_t(p+1) = w_3^p P_{t-2}(1) = w_3^{n+1} F_{t-1,0} = w_3^{n+1} F_{t+1,h(p)+1} G_p F_{l(p)+1,0}.$$

Case 3. $n + 1 < p + 1 < 2n$.
In the same way as the Case 3 in the proof of Theorem 1.6(ii), we see

$$P_t(p+1) = w_3^p P_{t-2}(p-n+1) = w_3^{n+1} F_{t-1,h(p-n)+1} G_{p-n} F_{l(p-n),0}$$

$$= w_3^{n+1} F_{t+1,h(p)+1} G_p F_{l(p)+1,0}.$$
Case 4. \( p + 1 > 2n \).
Since \( 3(p + 1) > \dim P \), we have \( P_{t(p + 1)} = 0 \). Also since \( h(p) + 1 \geq t + 1 \), we have \( F_{t+1,h(p)+1} = 0 \). □

5. Some explicit descriptions of the Wu classes modulo \( I_4 \)

In this section, using Theorems 1.5 and 1.6, we will describe \( v_i \) modulo \( I_4 \) by the distinct monomials on \( w_3, w_2 \) and \( w_1 \). We will do it only for \( i = 2^a + 1 \) \((a \geq 1)\), \( 2^a + 2 \) \((a \geq 2)\) and \( 2^a + 2b + 1 \) \((a > b \geq 1)\), but this is sufficient by Theorem 1.5.

In the following theorems, \( p \equiv q(n) \) denotes \( p \equiv q \mod n \).

**Theorem 5.1.** Let \( i = 2^a + 1 \) \((a \geq 1)\). Then

\[
v_i \equiv F_{a,0}w_2w_1 + w_3F_{a-1,0}w_1^m
+ \sum_{p=0(2)} w_3^pF_{a-1,h(p)+1}G_pF_{l(p),0}(w_3 + w_2w_1)w_1^m \mod I_4 ,
\]

where \( p > 0 \) and \( m = 2^{a-1} \).

**Proof.** By Theorem 4.9(ii) or Remark 4.10(ii), we have

\[
v_i \equiv \sum_{s=0}^{a-1} w_2^s w_1^{i-2s-1} = F_{a,0}w_2w_1 \mod I_3 .
\]

Thus the conclusion follows from Theorems 1.5 and 1.6. □

**Theorem 5.2.** Let \( i = 2^a + 2 \) \((a \geq 2)\). Then

\[
v_i \equiv (F_{a-1,0}w_2w_1)^2 + w_3F_{a,0}w_1 + w_3^2(w_2^{m-2} + F_{a,2}w_2w_1^2)
+ \sum_{p=0(2)} w_3^pF_{a,h(p)+1}G_pF_{l(p),0}w_1
+ \sum_{p=0(4)} w_3^pF_{a,h(p)+1}G_pF_{l(p),1}w_1^{2m+2}
+ \sum_{p=0(4)} w_3^pF_{a,h(p)+1}G_pF_{l(p),2}w_2w_1^2 + w_2^{m-2^{h(p)+1}}G_pF_{l(p),1} \mod I_4 ,
\]

where \( p > 0 \), \( m - 2^{h(p)+1} \geq 0 \) and \( m = 2^{a-1} \).

**Proof.** By Remark 4.10(ii), we have

\[
v_i \equiv (v_{i/2})^2 \equiv (F_{a-1,0}w_2w_1)^2 \mod I_3 .
\]

Let \( v_j(p) \) be the sum consisting of all monomials on \( w_3, w_2 \) and \( w_1 \) in \( v_i \) such that each power of \( w_3 \) for such monomials is \( p \). If there are no such monomials, then also set \( v_j(p) = 0 \) in \( H^j(BO; Z_2) \).
The polynomials on $w_1$, $w_2$ and $w_3$ in the universal Wu classes 205

Then using Theorem 1.5(iii) and Theorem 1.6(i), (ii) and noting $(F_m)_n^{2^e} = F_{m+n,n+1}$, we see

\[ v_1(1) = P_{a-1}(1)w_1 = w_3 F_{a,0} w_1. \]
\[ v_1(2) = P_{a-1}(2)w_1 + (P_{a-2}(1) + P_{a-3}(1)w_1^{m/2})^2 \]
\[ = w_2^2 F_{a,2} G_2 F_{1,0} w_2 w_1^2 + (w_3 F_{a-1,0} + w_3 F_{a-2,0} w_1^{m/2})^2 \]
\[ = w_2^2 (F_{a,2} w_2 w_1^2 + F_{a,1} + F_{a-1,1} w_1^m) = w_2^2 (F_{a,2} w_2 w_1^2 + w_2^{m-2}). \]

Let $p \equiv 0(2)$ and $p > 0$. Then from Theorems 1.5(iii) and 1.6(iii), we see

\[ v_1(p + 1) = P_{a-1}(p + 1)w_1 = w_2^{p+2} F_{a,h(p)+1} G_p F_{(p),0} w_1. \]

Let $p \equiv 0(4)$ and $p > 0$. Then using Theorems 1.5(iii), 1.6(ii) and noting $(G_p)^2 = G_p$, we see

\[ v_1(p) = P_{a-1}(p)w_1 + (P_{a-2}(p/2) + P_{a-3}(p/2) w_1^{m/2})^2 \]
\[ = w_2^{p+2} \{ F_{a,h(p)+1} G_p F_{(p),0} w_2 w_1^2 + F_{a-1,h(p)+1} G_p F_{(p),1} w_2^2 w_1^{m+2} \}. \]

Here we see that $F_{(p),0} w_2 w_1^2 + F_{(p),1} w_2^2 w_1^2 = w_2^{2h(p)}$.

Let $p \equiv 0(4)$ and $p > 0$. Then using Theorems 1.5(iii) and 1.6(ii), (iii) and noting $G_{p+2} = G_p F_{(p),2}$, we see

\[ v_1(p + 2) = P_{a-1}(p + 2)w_1 + (P_{a-2}(p/2 + 1) + P_{a-3}(p/2 + 1)w_1^{m/2})^2 \]
\[ = w_2^{p+2} \{ F_{a,h(p)+1} G_p F_{(p),2} w_2 w_1^2 + (F_{a,h(p)+1} + F_{a-1,h(p)+1} w_1^m) G_p F_{(p),1} \}. \]

Here we see that $F_{a,h(p)+1} + F_{a-1,h(p)+1} w_1^m = w_2^{m-2h(p)}$ if $h(p) \leq a - 2$; 0 otherwise.

\[ \square \]

**Theorem 5.3.** Let $i = 2^a + 2^b + 1$ ($a > b \geq 1$). Then

\[ v_i \equiv w_3 \left( \sum_{s=b}^{a-1} w_2^{2^s-1} w_1^{i-2^s+1-1} \right) \]
\[ + \sum_{p=0(2), h(p) \leq a-2} w_3^{p+1} \left( \sum_{s=\max(b,h(p)+1)}^{a-1} w_2^{2^s-2^{h(p)+1}} w_1^{i-2^s+1-1} \right) G_p F_{(p),0} \]
\[ + \sum_{p=0(2), p \neq 0(m), h(p) \leq a-2} w_3^{p+2} \left( \sum_{s=\max(b,h(p)+1)}^{a-1} w_2^{2^s-2^{h(p)+1}} w_1^{i-2^s+1-1} \right) G_p F_{(p),0} w_2 w_1 \]
\[ + w_3^m (F_{a,b+1} F_{b,0} w_2 w_1^{m+1} + F_{a,b} w_1) \]
\[ + \sum_{p=0(2m)} w_3^{p+m} F_{a,h(p)+1} G_p (F_{(p),b+1} F_{b,0} w_2 w_1^{m+1} + F_{(p),b} w_1) \]
\[ + \sum_{p=0(2m)} w_3^{p+m} F_{a,h(p)+1} G_p \left( \sum_{s=0}^{b-1} w_2^{2^s} w_1^{2^{h(p)+m-2^s+1}+1} \right) \text{mod } I_4, \]

where $p > 0$ and $m = 2^b$. 
PROOF. We have \( v_i \equiv 0 \mod I_3 \) by Theorem 4.9(iii) or Remark 4.10(iii). We also use the notation \( v_i(p) \) used in the proof of Theorem 5.2.

From Theorems 1.5(iv) and 1.6(i), we see

\[
v_i(1) = P_{a-1}(1)w_1^{2b} + P_{b-1}(1)w_1^{2a} = w_3\left(\sum_{s=b}^{a-1} w_2^{2s-1}w_1^{-2s}\right).
\]

Let \( p \equiv 0(2) \) and \( p > 0 \). Then using Theorems 1.5(iv), 1.6(iii) and letting \( m = 2^b, n = 2^a \), we see

\[
v_i(p + 1) = P_{a-1}(p + 1)w_1^m + P_{b-1}(p + 1)w_1^n
= w_3^{s+1}(F_{a,h(p)+1}w_1^m + F_{b,h(p)+1}w_1^n)G_pF_{(p),0}.
\]

Here we see that \( F_{a,h(p)+1}w_1^m + F_{b,h(p)+1}w_1^n = \sum_{s=\max(b,h(p)+1)}^{a-1} w_2^{2s-2h(p) + 1}w_1^{2s-1} \) if \( h(p) \leq a - 2; \ 0 \) if \( h(p) \geq a - 1 \).

Let \( p \equiv 0(2), p > 0 \) and \( p \not\equiv 0(m) \). Then since \( v_i(p) = P_{a-1}(p)w_1^m + P_{b-1}(p)w_1^n \) \( (m = 2^b, n = 2^a) \), we obtain the conclusion in the same way as the above case.

From Theorems 1.5(iv), 1.6(i) and (ii), we have

\[
v_i(m) = P_{a-1}(m)w_1^m + (P_{a-b-1}(1))^m = w_3(F_{a,b+1}F_{b,0}w_2w_1^{m+1} + F_{a,b}w_1).
\]

Let \( p \equiv 0(2m) \) and \( p > 0 \). Then using Theorems 1.5(iv), 1.6(ii) and (iii), we see

\[
v_i(p + m) = P_{a-1}(p + m)w_1^m + (P_{a-b-1}(p/m + 1))^m w_1
= w_3^{p+m}(F_{a,h(p)+1}G_p(F_{(p),b+1}F_{b,0}w_2w_1^{m+1} + F_{(p),b}w_1)).
\]

Let \( p \equiv 0(2m) \) and \( p > 0 \). Then using Theorems 1.5(iv) and 1.6(ii), we see

\[
v_i(p) = P_{a-1}(p)w_1^m + (P_{a-b-1}(p/m))^m w_1
= w_3^p(F_{a,h(p)+1}G_p(F_{(p),0}w_2w_1^{m+1} + F_{(p),b}w_2w_1^{m+1})).
\]

Here we see that \( F_{(p),0}w_2w_1^{m+1} + F_{(p),b}w_2w_1^{m+1} = \sum_{s=0}^{b-1} w_2^{2s(p+1)/2}w_1^{2s-1} \).

\[
\]

References

The polynomials on $w_1, w_2$ and $w_3$ in the universal Wu classes


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