On the Homotopy Type of Manifolds

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BY J. H. C. WHITEHEAD

(Received July 15, 1939)

1. The object of this note is to call attention to certain theorems, which follow very easily from some results due to E. Stiefel, ¹ H. Seifert,² Hassler Whitney,³ and myself.⁴ They refer to a class of manifolds which we call the class II, and are intended to throw light on the question, raised by W. Hurewicz,⁵ whether two closed manifolds of the same homotopy type are necessarily homeomorphic. The theorems depend both on M. H. A. Newman's⁶ theory of combinatorial equivalence, as re-developed by J. W. Alexander⁷ and carried further in S. S., and on theorems concerning differentiable manifolds. Therefore it is necessary to give a precise meaning to the term 'manifold'.

By an $n$-dimensional manifold, $M^n$, we shall mean a class of combinatorially equivalent, simplicial complexes covering the same space, each complex being a formal manifold, meaning that the complement of each vertex is combinatorially equivalent to $A^n$ or to $A^{n-1}$, according as the vertex in question is inside $M^n$ or on $M^n$, where $A^k$ stands for a closed $k$-simplex and $M^n$ is the boundary of $M^n$. These covering complexes will be called proper triangulations of $M^n$ (of course any simplicial complex covering $M^n$ is a proper triangulation if the 'Hauptvermutung' is true). The proper triangulations of an unbounded manifold of class $C^1$, or smooth manifold, are to be $C^1$-triangulations.⁸ By a smooth, bounded, $n$-dimensional manifold we shall mean the manifold of which a sub-complex $K^n_0 \subseteq K^n$ is a proper triangulation, where $K^n$ is a $C^1$ triangulation of a smooth, unbounded $n$-dimensional manifold and $K^n_0$ is a formal manifold. By the topological product $M^n \times A^k$ we shall mean the manifold having a normal subdivision of the cell-complex $K^n \times A^k$ as a proper triangulation, where $K^n$ is a proper triangulation of $M^n$. We shall use $\equiv$ to indicate combinatorial equivalence, and $M^n_1 \equiv M^n_2$ will mean that $K^n_1 \equiv K^n_2$, where $K^n_1$ is a proper triangulation of $M^n_1$.

⁴ J. H. C. Whitehead, Proc. London Math. Soc., 45 (1939), 243. This paper will be referred to as S. S.
⁵ W. Hurewicz, Akad. Wet. Amsterdam, 29 (1936), 125.
⁸ J. H. C. Whitehead, Annals of Math. this number, 809-824. This paper will be referred to as C. C. Relevant to the present paper are theorems 4, 5, 7 and 8 of C. C.
We now state some of our theorems, postponing the proof of theorem 1 and
the definition of the class \( \Pi \) till §2. It is to be understood that the manifolds
referred to in these theorems are connected and covered by finite complexes.

**Theorem 1.** If \( M^n_i \in \Pi (i = 1, 2) \) and \( M^n_1 \) and \( M^n_2 \) have the same nucleus,\(^1\) then

\[
M^n_1 \times A^k = M^n_2 \times A^k
\]

for sufficiently large values of \( k \).

It is shown in S. S. that, provided their fundamental group satisfies a certain
condition,\(^2\) two (finite) complexes have the same nucleus if they are of the
same homotopy type. For manifolds with such a group, theorem 1 can there-
fore be restated with ‘have the same nucleus’ replaced by ‘are of the same
homotopy type’.

A bounded manifold \( M^n \), which is an absolute retract (i.e. is of the same
homotopy type as a single point) belongs to the class \( \Pi \) if it is combinatorially
equivalent to a smooth manifold. If \( M^n \) is smooth we may assume that
\( M^n \subset M^n_i \subset R^{n+k} \) for any \( k > 0 \), where \( M^n_i \) is an unbounded analytic manifold
and \( R^n \) is Euclidean \( n \)-space. Since \( M^n \) has the same homology and cohomology
groups as a cell its normal sphere-space \(^3\) in \( R^{n+k} \) is simple. Taking \( k = 5 \)
we have, from theorem 5, below, and S. S., theorem 25, corollary 3:

**Theorem 2.** \( I^{5p} \pi_1(M^n) = 1 \), \( \beta(M^n) = 0 \) (\( r = 1, \ldots, n \)) and \( M^n \) is smooth,
then

\[
M^n \times A^{n+5} = A^{2n+5}.
\]

It will be seen that any (bounded) polyhedral \( M^n \subset R^n \) belongs to \( \Pi \). Therefore \( M^n \times A^{n+5} = A^{2n+5} \) if \( M^n \) is the finite region bounded by a polyhedral
\((n - 1)\)-sphere in \( R^n \), or even if \( M^n \) is of the same homotopy type as \( A^n \).

The Poincaré hypothesis, in its combinatorial form and as generalized by
Hurewicz\(^4\) from \( n = 3 \) to any \( n \), is equivalent to the hypothesis.

If \( M^n \) is an \((n - 1)\)-sphere and if \( M^n \) is an absolute retract, then \( M^n = A^n \).

Discarding the condition that \( M^n \) is an \((n - 1)\)-sphere, we have what may be
called the extended Poincaré hypothesis, namely:

A bounded, \( n \)-dimensional manifold, which is an absolute retract is an \( n \)-element.

From theorem 2, since a \( k \)-element is the topological product of \( k \) linear seg-
ments, we have:

**Theorem 3.** The extended Poincaré hypothesis, for smooth manifolds at least,
is equivalent to the hypothesis:

If \( M^n \times A^k = A^{n+1} \), then \( M^n = A^n \).

This theorem raises various questions, one of which can be answered very

\(^1\) See S. S. p. 287. See also a paper by G. Higman to be published shortly by the London
Math. Soc.


\(^3\) Appendix, Theorem 2, corollary.

\(^4\) \( \pi_1(M^n) \) denotes the (multiplicative) fundamental group and \( \beta(M^n) \) the (additive)
\( r^\text{th} \) homology group of \( M^n \).
simply, namely: are there manifolds \( M^n_i \neq M^n_2 \) such that \( M^n_i \times A^l = M^n_2 \times A^l \)? The answer is in the affirmative. For let \( M^n_i = M^n_1 \times A_1 \), where \( M^n_1 \) is a torus with one hole and \( M^n_2 \) is a 2-sphere with three holes. Then \( M^n_1 \neq M^n_2 \). On the other hand, taking \( M^n_2 \subset R^3 \), it is easily verified that \( M^n_1 = M^n_2 \), since \( M^n_1 \subset R^3 \) is obviously a regular neighborhood (S. S., p. 293) of two simple circuits with a single point in common. As another, and perhaps more interesting example, let \( M^n_i \) (i = 1, 2) be a lens space of type \( (p, q_i) \), from which the interior of a 3-simplex \( A^3_1 \) has been removed, where \( q_2 \neq \pm 1 \) mod \( p \). Then \( M^n_i \) contracts (S. S., pp. 248 and 258) into the 2-cell, bounded by a circuit taken \( p \) times, which, taken twice, bounds a lens model of \( M^n_i \). Therefore \( M^n_1 \) and \( M^n_2 \) have the same nucleus. It will be seen that \( M^n_i \in \Pi \), whence, by theorem 1, \( M^n_i \times A^k = M^n_2 \times A^k \) for large values of \( k \) (actually for \( k \geq 6 \)). But \( M^n_1 \) and \( M^n_2 \) are not combinatorially equivalent. For if they were, the lens spaces \( M^n_1 + A^1 \) and \( M^n_2 + A^1 \) would be combinatorially equivalent, which they are not since \( q_2 \neq \pm 1 \) mod \( p \).

2. Let a proper triangulation, \( K^n \), of a given manifold, \( M^n \), be represented as a rect-linear complex in \( R^{n+k} \), and let \( U(K^n, R^{n+k}) \) be a regular neighborhood11 of \( K^n \). Then our definition of \( \Pi \) is: \( M^n \in \Pi \) if, and only if,

\[
U(K^n, R^{n+k}) = K^n \times A^k
\]

for large values of \( k \). Provided \( k \geq n + 3 \) it follows from S. S., theorems 23 and 24, that this definition is independent of the choice of the proper triangulation \( K^n \), of the choice of the regular neighborhood \( U(K^n, R^{n+k}) \) and of the way in which \( K^n \) is imbedded in \( R^{n+k} \). If \( K^n \subset R^{n+k} \subset R^{n+k+l} = R^{n+k} \times R^l \) \((l > 0)\) we may take

\[
U(K^n, R^{n+k+l}) = U(K^n, R^{n+k}) \times A^1 \times \cdots \times A^1.
\]

For the latter is a manifold and, by an obvious induction on \( l \), it contracts into \( U(K^n, R^{n+k}) \), and hence into \( K^n \). Therefore, if the condition (2.1) is satisfied by some \( K^n \subset R^{n+k} \), it is satisfied for every \( k_1 > k \) and a suitable \( K^n \subset R^{n+k_1} \).

Theorem 1, above, is now seen to be an immediate consequence of S. S., theorem 25.

It follows from an argument in S. S. (p. 298) that an \( n \)-sphere belongs to \( \Pi \) for each value of \( n \). Moreover, if \( M^n \in \Pi \) and \( M^n_0 \subset M^n \), then \( M^n_0 \in \Pi \). For let \( t \) be a semi-linear homeomorphism of \( K^n \times A^k \) on \( U(K^n, R^{n+k}) \), where \( K^n \) is a proper triangulation of \( M^n \) which contains a sub-complex, \( K^n_0 \), covering \( M^n_0 \). Then \( t(K^n_0 \times A^k) \) is a manifold and contracts geometrically into

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15 S. S., p. 298. Observe that regular neighbourhoods are not necessarily neighbourhoods in the sense of topology.
\( i(K^n \times p) \), for any point \( p \in A^k \), and (2.1) is satisfied by \( i(K^n \times p) \subset R^{n+k} \). Therefore \( i(K^n \times A^k) \) is a regular neighborhood of \( K^n \). More generally, let \( M'_0 \subset M^n \) and let a proper triangulation \( K^n \), of \( M'_0 \), be a sub-complex of \( K^n \).

If a regular neighborhood \( U^*_0 = U(K^n, K^n) = K^n \times A^n \) we shall say that \( M'_0 \) is in regular position at \( M^n \). This is always the case if \( r = n \), for then we may take \( U^*_0 \) to be \( K^n \) itself.

**Theorem 4.** If \( M^n \subset M^n \) is in regular position in \( M^n \) and \( M^n \in \Pi \), then \( M'_0 \in \Pi \).

For, with the above notation, \( U^*_0 \) is an \( n \)-dimensional manifold in \( M^n \) and we have shown that if \( M^n \in \Pi \), then \( U^*_0 \in \Pi \). That is to say

\[
U^{n+k} = U(U^*_0, R^{n+k}) = U^*_0 \times A^k
\]

for some value of \( k \) and some rectilinear \( U^*_0 \subset K^n \subset R^{n+k} \). But \( U^{n+k} \) contracts into \( U^*_0 \) and the latter contracts into \( K^n \). Therefore \( U^{n+k} \) is also a regular neighborhood of \( K^n \), and if \( U^*_0 \equiv K^n \times A^{n-r} \) we have

\[
U^{n+k} = U^*_0 \times A^k \equiv K^n \times A^{n-r} \times A^k \equiv K^n \times A^{n-r+k},
\]

and the theorem is established.

With the help of theorem 4 we can dispose of the case \( n = 2 \). No non-orientable manifold can belong to \( \Pi \). For its regular neighborhood in \( R^{n+k} \), being an \((n+k)\)-dimensional manifold in \( R^{n+k} \), is orientable, while its topological product with a cell is not. On the other hand any orientable surface may be represented as a polyhedron in \( R^3 \) and is necessarily in regular position. Therefore it belongs to \( \Pi \). Also any orientable, polyhedral surface in \( R^m \) is in regular position if \( m \geq 7 \). Of course theorem 1 is trivial for any closed surface, whether orientable or not. Also it follows from special arguments, as in the remarks following theorem 3, that theorem 1, with \( k = 1 \), is true of bounded, orientable surfaces.

Now let \( M^n \subset R^{n+k} \) be a smooth manifold which, without loss of generality, we may assume to be analytic.

**Theorem 5.** \( M^n \in \Pi \) if its normal sphere-space in \( R^{n+k} \) is simple.

Since \( M^n \) is compact there is a positive \( \delta \) such that the flat \( k \)-spaces normal to \( M^n \) at two different points do not meet at a distance less than \( 2\delta \) from \( M^n \). Therefore no two of the \( k \)-cells \( E^k(p) \) meet each other, where \( E^k(p) \) is the interior and boundary of a \((k-1)\)-sphere with centre \( p \) and radius \( \delta \) in the normal flat \( k \)-space at \( p \). To say that the normal sphere-space is simple is to say that \( k \) mutually orthogonal, unit vectors \( e_1(p), \ldots, e_k(p) \) are defined in the normal flat \( k \)-space at each point \( p \in M^n \), and that \( e_j(p) \) varies continuously with \( p \).


\[\text{[14] Though this lower limit for } m \text{ can probably be reduced from 7 to 5 it cannot be discarded. For if } K \text{ is a knotted circuit in a 3-sphere, } S^3, \text{ it may be verified that the 2-sphere } (a + b)K \text{ is not in regular position in the 4-sphere } (a + b)S^4, \text{ where } a \text{ and } b \text{ are vertices not in } S^4. \text{ (Cf. E. Artin, Abh. Math. Sem. Hamburg., 4 (1925), 174-7.)}\]
After a process of approximation, projection in $\mathbb{E}^k(p)$, and a final normalization, we may assume that $e_i(p)$ varies analytically with $p$. The bounded manifold $M^{n+k}$, which is swept out by $E^i(p)$ as $p$ describes $M^n$, is then seen to be the image of $M^n \times E^k(p_0) \equiv M^n \times A^k$ in an analytic transformation which maps $M^n$ on itself. Therefore a suitable triangulation of $M^n \times A^k$ determines a $C^1$-triangulation, $P^{n+k}$ of $M^{n+k}$, which contains a proper triangulation of $M^n$ as a sub-complex. Let $K^{n+k}$ be a rectilinear model of $P^{n+k}$ and let $K^n \subset K^{n+k}$ be the subcomplex representing $M^n$. By C. C., theorem 4, there is a semi-linear, topological map $F(K^{n+k}) \subset R^{n+k}$. Then $F(K^{n+k}) (= M^n \times A^k)$ is a regular neighborhood of $F(K^n) (= M^n)$, and the theorem is established.

It follows from this theorem, and the results referred to at the beginning of §1, that $M^n \times I$, where $M^n$ is a smooth, orientable manifold, if any one of the following conditions is satisfied:

1. $M^n$ is closed and admits an internal parallelism, as is always the case if $n = 3$, or for example, if $M^n$ is a Lie group.
2. $M^n$ is closed and can be represented as a manifold of class $C^2$ in $R^{n+1}$ or in $R^{n+1}$ (Seifert's).
3. $M^n$ is bounded and all its cohomology groups vanish with integral, and hence with all coefficients. It can be shown that this follows from the general theory of sphere-spaces.

The sufficiency of the first condition follows from a theorem similar to theorem 23 on pp. 43 and 44 of Steifel's paper. For let $M^n \subset R^{n+k}$, where $k \geq n + 1$, and let $K^n$ be a triangulation of $M^n$. Then we successively set up outer parallelisms (i.e. parallelisms in the normal flat $k$-spaces) over $K^0, K^1, \ldots, K^n$, where $K^r$ is the $r$-dimensional skeleton of $K^n$. An outer parallelism over $K^r$ ($0 \leq r < n$) determines an $(r + 1)$-dimensional cocycle in $K^{r+1}$, whose coefficients are elements of $\pi_r(G_k)$, where $G_k$ is the group of rotations in $R^k$. The parallelism over $K^r$ may be extended throughout $K^{r+1}$ if this cocycle is zero. If it is not zero, but cohomologous to zero, then the parallelism over $K^r$ may be replaced by one for which the corresponding cocycle is zero. Thus $K^{r+1}$ admits an outer parallelism if the cocycle determined by the outer parallelism over $K^r$ is cohomologous to zero. Since $r + 1 < k$ it follows from the analysis of $G_k (= V_{k,k-1})$ in §1 of Steifel's paper, that a map $f(S') \subset G_k \subset G_{n+k}$, which is homotopic to a point in $G_{n+k}$, is homotopic to a point in $G_k$; also that any $f(S') \subset G_{n+k}$ can be deformed into a map in $G_k$. Therefore a lemma, analogous to the one in Steifel's theorem 23, follows from arguments similar to those in his §3. Therefore the $(r + 1)$-dimensional cocycle in $K^{r+1}$, which is determined by an outer parallelism over $K^r$ is cohomologous to zero. Finally, Steifel's assumption that some triangulation of $M^n$ is a sub-complex of a triangulation of $R^{n+k}$ need not, in this case, be taken as an additional axiom. For we may assume $M^n$ to be analytic and sub-divide it and a recti-linear triangulation of

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$R^{n+k}$ by the van der Waerden-Lefschetz method. The result will not, in
general, be a $C^1$-triangulation, but it will suffice in setting up the outer parallelism. Alternatively we may replace $M^n$ by a homeomorphic polyhedral $F(K^n)$, as in the proof of theorem 5, and attach a flat $n$-space and a flat $k$-space to each point of $F(K^n)$, which are respectively parallel to the tangent and normal flat spaces at the corresponding point of $M^n$. Then an inner parallelism in $M^n$ determines a parallelism in the $n$-spaces attached to the points of $F(K^n)$, and a parallelism in the $k$-spaces at points of $F(K^n)$ will determine an outer parallelism for $M^n$.

APPENDIX

(Extract from a letter of the author to Hassler Whitney under date of Jan. 26, 1940.—
The Editors.)

* * * I omitted to prove that $G_{n+1}$, the group of rotations in Euclidean metric space $R^{n+1}$, is $r$-simple for each $r \geq 1$, as the term is used by S. Eilenberg. This condition may be expressed as follows. Let $X$ be any arcwise connected topological space, let $\tilde{X}$ be its universal covering space and let $\Gamma$ be the group of covering transformations of $\tilde{X}$ (i.e. the group of homeomorphisms $\gamma_1(\tilde{X}) = \tilde{X}$, such that $u_\gamma = u$, where $u(\tilde{X}) = X$ is a locally (1-1) map of $\tilde{X}$ on $X$). Then $X$ is said to be 1-simple if $\pi_1(X)$ is Abelian, and $r$-simple $(r > 1)$ if, and only if, any spherical map $f(S^r) \subset \tilde{X}$ is homotopic in $\tilde{X}$ to the map $\gamma f(S^r)$, for each $\gamma \in \Gamma$. Let us assume that $\Gamma$ is a sub-group of some arcwise connected, topological group $\Gamma$, of homeomorphisms $\gamma(\tilde{X}) = \tilde{X}$, whose topology agrees with that of $\tilde{X}$, meaning that $\gamma(x)$ varies continuously with $x \in \tilde{X}$ and $\gamma \in \Gamma$. Then the identity in $\Gamma$, say $\gamma_0$ is joined to a given $\gamma \in \Gamma$ by a segment $\gamma_t \in \Gamma$ $(0 \leq t < 1)$. Therefore $\gamma f_0(S^r) = f_1(S^r)$, say, is the image of a given map, $f_1(S^r) \subset \tilde{X}$, in the deformation $f_t = \gamma f_0$, whence $X$ is $r$-simple for any $r > 1$. Therefore, and since $\Gamma$ is isomorphic to $\pi_1(X)$, we have the theorem:

**Theorem 1.** If $\Gamma_1$ satisfies the above condition and is also Abelian, then $X$ is $r$-simple for each $r \geq 1$.

Let $X$ be an arcwise connected topological group and let $\tilde{X}$ be its universal covering group. Then $\Gamma_1$ is Abelian, and is also a sub-group of the 'left translations' $\xi = \gamma \xi$ (also of the 'right translations' $\xi = \xi \gamma$, since $\Gamma_1$ is not only Abelian but, if the translation $\xi = \gamma \xi$ is identified with the element $\gamma \in X$, then $\Gamma_1$ belongs to the centre of $X$). Since $X$ is arcwise connected, so is $\tilde{X} = \Gamma$, and we have the corollary:

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Corollary. Any arcwise connected topological group is r-simple for each r \geq 1.

The consequence of this condition which interests us here is that, if X is r-simple, then a unique element of \( \Pi_r(X) \) is determined by a "free" map \( f(S) \subset X \), meaning a map which is independent of the base point for \( \Pi_r(X) \).

Now let an orientable sphere-space \( S(K^n) \) be given, where \( K^n \) is a simplicial complex and the associated spheres are r-dimensional, and let \( S(K^n) \) be simple in the r-dimensional skeleton, \( K_r^n \), of \( K^n \) (0 < r < n). We shall assume that \( S(K^n) \) is not only orientable but oriented, meaning that the associated spheres \( S^r(p) (p \in K^n) \) and the base sphere \( S_0^r \) are oriented, and that the defining maps \( \xi(p, S^r(p)) = S_0^r \) are all direct. Thus the (orthogonal) transformations of \( S_0^r \) into itself by which 'transformations of coordinates' are determined will be rotations. Let \( A_i^{r+1} \) (i = 1, 2, \ldots) be the (oriented) \((r + 1)\)-simplexes in \( K_r^{r+1} \) and, using the rotation, let \( q \to \xi_i(p, q) \in S_0^r \ (p \in A_i^{r+1}, q \in S^r(p)) \) be a local coordinate system for \( A_i^{r+1} \). Since \( S(K^n) \) is simple there is a map \( q \to \eta(p, q) \in S_0^r \) defined for each \( p \in K^n \), \( q \in S^r(p), \) such that the rotation

\[
q_0 \to \Phi_p(q_0) = \xi_i[p, \eta^{-1}(p, q_0)] \quad (p \in A_i^{r+1}, q_0 \in S_0^r)
\]

varies continuously with \( p \). In other words, \( p \to \Phi_p \) is a continuous map of \( A_i^{r+1} \) in \( G_{r+1} \), and since \( G_{r+1} \) is r-simple \( p \to \Phi_p \) defines a unique element \( \alpha_i \in \pi_r(G_{r+1}) \). The element \( \alpha_i \) is independent of the coordinate system \( \xi_i \).

For if \( \xi_i(p, q) \) is a second coordinate system for \( A_i^{r+1} \), then \( p \to \xi_i^{-1}(p, q) = \psi_0(p) \), say, is a map of \( A_i^{r+1} \) in \( G_{r+1} \). Since \( A_i^{r+1} \) can be shrunk into a point there is a deformation \( \psi_t \) \((0 \leq t \leq 1), \) of \( \psi_0(p) \) into the map given by \( \psi_1(p) = 1 \), the identity in \( G_{r+1} \). Therefore the coordinate system \( \xi_i = \xi_i^0 \) may be deformed into \( \xi = \xi_i^0 \); Thus

\[
\xi_i(p, q) = \psi_t(p) \xi_i(p, q),
\]

remembering that \( \psi_t(p) \) is a rotation of \( S_0^r \) into itself. Therefore the map of \( A_i^{r+1} \) in \( G_{r+1} \), which is defined by \( \xi_i^0 \) and \( \eta_i \), is homotopic to the above map \( p \to \Phi_p \), and hence determines the same element \( \alpha_i \in \pi_r(G_{r+1}) \). Let \( B_i^r \) be an oriented r-simplex, which is common to \( A_i^{r+1} \) and to \( A_i^{r+1} \) and let \( \eta_i^0(p, q) \) be any coordinate system for \( B_i^r \), which coincides with \( \eta_i(p, q) \) in \( B_i^r \). Then the map \( \xi_i \eta_i^{-1} \), of \( B_i^r \) in \( G_{r+1} \), together with the map \( \xi_i \eta_i^{-1} \), in which the orientation of \( B_i^r \) is reversed, determine an element \( \beta_i \in \pi_r(G_{r+1}) \). It follows from a similar argument to the one just given that the same element, \( \beta_i \), is determined by \( \eta_i, \eta_i^0 \) and a coordinate system \( \xi_i \) for \( A_i^{r+1} \). Thus, if \( \eta_i \) is constructed in such a way that \( \eta_i, \eta_i^0 \) and \( \xi_i \) determine a given element \( \beta_i \), then \( \eta_i, \eta_i^0 \) and \( \xi_i \) lead back to the same element \( \beta_i \).

After these preliminaries it follows from arguments which are similar to some of those used by E. Stiefel and by Eilenberg that

1. \( C^{r+1} = \sum_i \alpha_i A_i^{r+1} \)

is a co-cycle, with coefficients in \( \pi_r(G_{r+1}) \).

2. If \( C^{r+1} \sim 0 \), then \( S(K^n) \) is simple in \( K^{r+1} \). For this is obviously so if
\( C^{r+1} = 0 \). If \( C^{r+1} \neq 0 \) but \( \sim 0 \), then the coordinate system \( n \) may be replaced by one for which the corresponding co-cycle vanishes.

Since \( S(K^n) \) is orientable by hypothesis, it follows that it is simple in \( K^1 \), and we have the theorem:

**Theorem 2.** If the \((r + 1)\)-dimensional co-homology group of \( K^n \) vanishes for each \( r = 1, \ldots, n - 1 \), with coefficients in \( \pi_r(G_{r+1}) \), then any orientable sphere-space \( S(K^n) \), in which the associated spheres are \( v \)-dimensional \((v > 0)\), is simple.

If the 1-dimensional co-homology group of \( K^n \) vanishes with integral coefficients, reduced mod 2, then any sphere-space \( S(K^n) \) is orientable. Also the co-homology groups vanish with all coefficients if they all vanish with integral coefficients. Hence we have the corollary:

**Corollary.** If all the co-homology groups of \( K^n \) vanish, with integral coefficients, then any sphere space \( S(K^n) \) is simple.

Notice, on the other hand, that no condition is imposed on the \((r + 1)\)-dimensional cohomology groups for those values of \( r \) such that \( \pi_r(G_{r+1}) = 0 \). Do you know if there are any, beyond \( r = 2 \), for any \( v > 1 \)?

We also have, for the reasons indicated in my paper on homotopy types:

**Theorem 3.** If a differentiable \( n \)-dimensional manifold admits an absolute parallelism, then its normal sphere-space in \( R^{2n+k} \) \((k > 0)\) is simple.

In the paper on homotopy types I was interested only in finite (i.e. closed or bounded) manifolds. But this theorem is obviously true in general, provided one requires the manifold to be a closed, but not necessarily compact, sub-set of \( R^{2n+k} \).

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