On Smooth Surgery

SHMUEL WEINBERGER
University of Chicago and Courant Institute

In this note I would like to fill a small lacuna in the theory of surgery on smooth manifolds. We prove and apply the following:

**Theorem.** If $M$ is a smooth manifold, then the map of structure sets $S^{\text{Diff}}(M) \to S^{\text{Top}}(M)$ is finite to one with image containing a subgroup of finite index.

Recall that the structure set is: $\{f: W \to M$ a simple homotopy equivalence, $W$ a Cat manifold\}/Cat-isomorphism. In the topological case, one can turn $S^{\text{Top}}(M)$ into an abelian group (and in fact, $F/\text{Top}$ has an infinite loop space structure, so that the surgery exact sequence is a long exact sequence of abelian groups). For this see [2]. While no such result is yet known for the smooth case, it is reasonable to believe that one can do as much rationally (suitably interpreted) since $\text{Top}/\text{O}$ has finite homotopy groups. The above is our version. The difficulty\(^1\) is, of course, that the map

$$s: F/\text{Top} \to B(\text{Top}/\text{O}).$$

need not be an $H$ map. We show that, for each $k$, there is an $N(k)$ such that, for any $k$-dimensional space and map $f: X \to F/\text{Cat}$, $s(N(k)f)$ is nullhomotopic. The result then follows from a diagram chase with the surgery exact sequence.

Clearly, we can work one prime at a time since $\text{Top}/\text{O}$ has finite homotopy groups. Consider the infinite mapping cylinder of $p: F/\text{Top} \to F/\text{Top}$ (i.e., the infinite cyclic cover of the mapping torus of this map). It is homotopy equivalent to $F/\text{Top}[1/p]$ so that any map to $B(\text{Top}/\text{O})_{(p)}$ is nullhomotopic. We restrict to a skeleton $S$ of $F/\text{Top}$. Because, again, the homotopy groups of $B(\text{Top}/\text{O})$ are finite, lim\(^1\)'s vanish, so that

$$\lim [S; B(\text{Top}/\text{O})(p)] = [S[1/p]; B(\text{Top}/\text{O})(p)] = 0,$$

and thus some number of multiplications by $p$ will kill the map from $S$. The result now follows for arbitrary $X$.

\(^1\)This only affects the existence of smooth manifolds, not the finiteness of the number of smooth structures on any such manifold, a well-known consequence of smoothing theory.

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Remark. The exponent of the image can be bounded just in terms of the dimension, although not the index. We do not know whether the image is a subgroup.

Remark. This theory has some obvious applications to questions regarding the finiteness of numbers of structures of various manifolds and the like. More interesting are the equivariant applications. The surgery theory of [1] is quite formal, and while, hedged by many restrictions, notably transverse isovariance, in that setting it is quite computable. Thus for transverse isovariant topological structures one gets an excellent (if conceptually unnatural) surgery exact sequence.\(^2\) The formality of the above arguments then shows that one can obtain finite index smooth \(G\)-manifolds from these. In particular, the result of [4] that all finite groups act effectively on fake complex projective spaces can be improved as far as the dimension estimates necessary for producing the actions. (They are halved.) It seems conceivable that these are now the best bounds, and that a \(G\)-signature argument could prove this.

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Bibliography


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\(^2\)There is now a nontransverse but isovariant equivariant topological surgery exact sequence which is rather more involved; see [3].