C. T. C. Wall's contributions to the topology of manifolds

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Note. Numbered references in this survey refer to the Research Papers in Wall's Publication List.

1 A quick overview

C. T. C. Wall[1] spent the first half of his career, roughly from 1959 to 1977, working in topology and related areas of algebra. In this period, he produced more than 90 research papers and two books, covering

- cobordism groups,
- the Steenrod algebra,
- homological algebra,
- manifolds of dimensions 3, 4, ≥ 5,
- quadratic forms,
- finiteness obstructions,
- embeddings,
- bundles,
- Poincaré complexes,
- surgery obstruction theory,

* Partially supported by NSF Grant # DMS-96-26817.

† This is an expanded version of a lecture given at the Liverpool Singularities Meeting in honor of Wall in August 1996.

‡ Partially supported by NSF Grant # DMS-96-25336.

[1] Charles Terence Clegg Wall, known in his papers by his initials C. T. C. and to his friends as Terry.

- homology of groups,
- 2-dimensional complexes,
- the topological space form problem,
- computations of K- and L-groups,
- and more.

One quick measure of Wall's influence is that there are two headings in the *Mathematics Subject Classification* that bear his name:

- 57Q12 (Wall finiteness obstruction for CW complexes).
- 57R67 (Surgery obstructions, Wall groups).

Above all, Wall was responsible for major advances in the topology of manifolds. Our aim in this survey is to give an overview of how his work has advanced our understanding of classification methods. Wall's approaches to manifold theory may conveniently be divided into three phases, according to the scheme:

1. All manifolds at once, up to cobordism (1959–1961).
2. One manifold at a time, up to diffeomorphism (1962–1966).
3. All manifolds within a homotopy type (1967–1977).

2 Cobordism

Two closed n-dimensional manifolds M_1^n and M_2^n are called *cobordant* if there is a compact manifold with boundary, say W^{n+1}, whose boundary is the disjoint union of M_1 and M_2. Cobordism classes can be added via the disjoint union of manifolds, and multiplied via the Cartesian product of manifolds. Thom (early 1950's) computed the cobordism ring $\mathfrak{N}_*$ of unoriented smooth manifolds, and began the calculation of the cobordism ring Ω_* of oriented smooth manifolds.

After Milnor showed in the late 1950's that Ω_* contains no odd torsion, Wall [1, 3] completed the calculation of Ω_*. This was the ultimate achievement of the pioneering phase of cobordism theory. One version of Wall's main result is easy to state:

Theorem 2.1 (Wall [3]) *All torsion in Ω_* is of order 2. The oriented cobordism class of an oriented closed manifold is determined by its Stiefel-Whitney and Pontrjagin numbers.*

For a fairly detailed discussion of Wall's method of proof and of its remarkable corollaries, see [Ros].

3 Structure of manifolds

What is the internal structure of a cobordism? Morse theory has as one of its main consequences (as pointed out by Milnor) that any cobordism between smooth manifolds can be built out of a sequence of *handle attachments.*

Definition 3.1 Given an m-dimensional manifold M and an embedding $S^r \times D^{m-r} \hookrightarrow M$, there is an associated *elementary cobordism* $(W; M, N)$ obtained by *attaching an* $(r+1)$*-handle* to $M \times I$. The cobordism W is the union

$$W = M \times I \cup_{\left(S^r \times D^{m-r}\right) \times \{1\}} D^{r+1} \times D^{m-r},$$

and N is obtained from M by deleting $S^r \times D^{m-r}$ and gluing in $S^r \times D^{m-r}$ in its place:

$$N = \left(M \smallsetminus \left(S^r \times D^{m-r}\right)\right) \cup_{S^r \times S^{m-r-1}} D^{r+1} \times S^{m-r-1}.$$

The process of constructing N from M is called *surgery on an* r*-sphere*, or *surgery in dimension* r *or in codimension* $m-r$. Here $r = -1$ is allowed, and amounts to letting N be the *disjoint* union of M and S^m.

Any cobordism may be decomposed into such elementary cobordisms. In particular, any closed smooth manifold may be viewed as a cobordism between empty manifolds, and may thus be decomposed into handles.

Definition 3.2 A cobordism W^{n+1} between manifolds M^n and N^n is called an *h-cobordism* if the inclusions $M \hookrightarrow W$ and $N \hookrightarrow W$ are homotopy equivalences.

The importance of this notion stems from the *h-cobordism theorem* of Smale (ca. 1960), which showed that if M and N are simply connected and of dimension ≥ 5, then every h-cobordism between M and N is a cylinder $M \times I$. The crux of the proof involves handle cancellations as well as Whitney's trick for removing double points of immersions in dimension > 4. In particular, if M^n and N^n are simply connected and h-cobordant, and if $n > 4$, then M and N are diffeomorphic (or PL-homeomorphic, depending on whether one is working in the smooth or the PL category).

For manifolds which are not simply connected, the situation is more complicated and involves the fundamental group. But Smale's theorem was extended a few years later by Barden,[2] Mazur, and Stallings to give the *s-cobordism theorem*, which (under the same dimension restrictions) showed that the possible h-cobordisms between M and N are in natural bijection with the elements of the *Whitehead group* $\mathrm{Wh}\,\pi_1(M)$. The bijection sends

[2] One of Wall's students!

an h-cobordism W to the *Whitehead torsion* of the associated homotopy equivalence from M to W, an invariant from algebraic K-theory that arises from the combinatorics of handle rearrangements. One consequence of this is that if M and N are h-cobordant and the Whitehead torsion of the h-cobordism vanishes (and in particular, if $\mathrm{Wh}\,\pi_1(M) = 0$, which is the case for many π_1's of practical interest), then M and N are again diffeomorphic (assuming $n > 4$).

The use of the Whitney trick and the analysis of handle rearrangements, crucial to the proof of the h-cobordism and s-cobordism theorems, became the foundation of Wall's work on manifold classification.

4 4-Manifolds

Milnor, following J. H. C. Whitehead, observed in 1956 that a simply connected 4-dimensional manifold M is classified up to homotopy equivalence by its *intersection form*, the non-degenerate symmetric bilinear from on $H_2(M;\mathbb{Z})$ given by intersection of cycles, or in the dual picture, by the cup-product

$$H^2(M;\mathbb{Z}) \times H^2(M;\mathbb{Z}) \to H^4(M;\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}.$$

Note that the isomorphism $H^4(M;\mathbb{Z}) \to \mathbb{Z}$, and thus the form, depends on the orientation.

Classification of 4-dimensional manifolds up to homeomorphism or diffeomorphism, however, has remained to this day one of the hardest problems in topology, because of the failure of the Whitney trick in this dimension. Wall succeeded in 1964 to get around this difficulty at the expense of "stabilizing." He used handlebody theory to obtain a stabilized version of the h-cobordism theorem for 4-dimensional manifolds:

Theorem 4.1 (Wall [19]) *For two simply connected smooth closed oriented 4-manifolds M_1 and M_2, the following are equivalent:*

1. *they are h-cobordant;*
2. *they are homotopy equivalent (in a way preserving orientation);*
3. *they have the same intersection form on middle homology.*

If these conditions hold, then $M_1 \# k(S^2 \times S^2)$ and $M_2 \# k(S^2 \times S^2)$ are diffeomorphic (in a way preserving orientation) for k sufficiently large (depending on M_1 and M_2).

Note incidentally that the converse of the above theorem is not quite true: $M_1 \# k(S^2 \times S^2)$ and $M_2 \# k(S^2 \times S^2)$ are diffeomorphic (in a way preserving orientation) for k sufficiently large if and only if the intersection forms of M_1 and M_2 are *stably* isomorphic (where stability refers to addition of the hyperbolic form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$).

From the 1960's until Donaldson's work in the 1980's, Theorem 4.1 was basically the *only* significant result on the diffeomorphism classification of simply-connected 4-dimensional manifolds. Thanks to Donaldson's work, we now know that the stabilization in the theorem (with respect to addition of copies of $S^2 \times S^2$) is unavoidable, in that without it, nothing like Theorem 4.1 could be true.

5 Highly connected manifolds

The investigation of simply connected 4-dimensional manifolds suggested the more general problem of classifying $(n-1)$-connected $2n$-dimensional manifolds, for all n. The intersection form on middle homology again appears as a fundamental algebraic invariant of oriented homotopy type. In fact this invariant also makes sense for an $(n-1)$-connected $2n$-dimensional manifold M with boundary a homology sphere $\partial M = \Sigma^{2n-1}$. If ∂M is a homotopy sphere, it has a potentially *exotic* differentiable structure for $n \geq 4$.[3]

Theorem 5.1 (Wall [10]) *For $n \geq 3$ the diffeomorphism classes of differentiable $(n-1)$-connected $2n$-dimensional manifolds with boundary a homotopy sphere are in natural bijection with the isomorphism classes of $\mathbb{Z}$-valued non-degenerate $(-1)^n$-symmetric forms with a quadratic refinement in $\pi_n(BSO(n))$.*

(The form associated to a manifold M is of course the intersection form on the middle homology $H_n(M;\mathbb{Z})$. This group is isomorphic to $\pi_n(M)$, by the Hurewicz theorem, so every element is represented by a map $S^n \to M^{2n}$. By the Whitney trick, this can be deformed to an embedding, with normal bundle classified by an element of $\pi_n(BSO(n))$. The quadratic refinement is defined by this homotopy class.)

The sequence of papers [15, 16, 22, 23, 37, 42] extended this diffeomorphism classification to other types of highly-connected manifolds, using a combination of homotopy theory and the algebra of quadratic forms. These papers showed how far one could go in the classification of manifolds without surgery theory.

[3] It was the study of the classification of 3-connected 8-dimensional manifolds with boundary which led Milnor to discover the existence of exotic spheres in the first place. [Mil]

6 Finiteness obstruction

Recall that if X is a space and $f : S^r \to X$ is a map, the space obtained from X by *attaching an* $(r+1)$*-cell* is $X \cup_f D^{r+1}$. A *CW complex* is a space obtained from $\emptyset$ by attaching cells. It is called *finite* if only finitely many cells are used. One of the most natural questions in topology is:

> When is a space homotopy equivalent to a *finite CW* complex?

A space X is called *finitely dominated* if it is a *homotopy* retract of a finite CW complex K, i.e., if there exist maps $f : X \to K$, $g : K \to X$ and a homotopy $gf \simeq 1 : X \to X$. This is clearly a necessary condition for X to be of the homotopy type of a finite CW complex. Furthermore, for spaces of geometric interest, finite domination is much easier to verify than finiteness. For example, already in 1932 Borsuk had proved that every compact ANR, such as a compact topological manifold, is finitely dominated. So another question arises:

> Is a finitely dominated space homotopy equivalent to a finite CW complex?

This question also has roots in the study of the free actions of finite groups on spheres. A group with such an action necessarily has periodic cohomology. In the early 1960's Swan had proved that a finite group π with cohomology of period q acts freely on an infinite CW complex Y homotopy equivalent to S^{q-1}, with Y/π finitely dominated, and that π acts freely on a *finite* complex homotopy equivalent to S^{q-1} if and only if an algebraic K-theory invariant vanishes. Swan's theorem was in fact a special case of the following general result.

Theorem 6.1 (Wall [26, 43]) *A finitely dominated space X has an associated obstruction $[X] \in \widetilde{K}_0(\mathbb{Z}[\pi_1(X)])$. The space X is homotopy equivalent to a finite CW complex if and only if this obstruction vanishes.*

The obstruction defined in this theorem, now universally called the *Wall finiteness obstruction*, is a fundamental algebraic invariant of non-compact topology. It arises as follows. If K is a finite CW complex dominating X, then the cellular chain complex of K, with local coefficients in the group ring $\mathbb{Z}[\pi_1(X)]$, is a finite complex of finitely generated free modules. The domination of X by K thus determines a direct summand subcomplex of a finite chain complex, attached to X. Since a direct summand in a free module is projective, this chain complex attached to X consists of finitely generated projective modules. The Wall obstruction is a kind of "Euler characteristic" measuring whether or not this chain complex is chain equivalent to a finite complex of finitely generated free modules.

The Wall finiteness obstruction has turned out to have many applications to the topology of manifolds, most notably the Siebenmann end obstruction for closing tame ends of open manifolds.

7 Surgery theory and the Wall groups

The most significant of all of Wall's contributions to topology was undoubtedly his development of the general theory of non-simply-connected surgery. As defined above, surgery can be viewed as a means of creating new manifolds out of old ones. One measure of Wall's great influence was that when other workers (too numerous to list here) made use of surgery, they almost invariably drew upon Wall's contributions.

As a methodology for classifying manifolds, surgery was first developed in the 1961 work of Kervaire and Milnor [KM] classifying homotopy spheres in dimensions $n \geq 6$, up to h-cobordism (and hence, by Smale's theorem, up to diffeomorphism). If W^n is a parallelizable manifold with homotopy sphere boundary $\partial W = \Sigma^{n-1}$, then it is possible to kill the homotopy groups of W by surgeries if and only if an obstruction

$$\sigma(W) \in P_n = \begin{cases} \mathbb{Z} & \text{if } n \equiv 0 \pmod 4, \\ \mathbb{Z}/2 & \text{if } n \equiv 2 \pmod 4, \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

vanishes. Here

$$\sigma(W) = \begin{cases} \text{signature}(W) \in \mathbb{Z} & \text{if } n \equiv 0 \pmod 4, \\ \text{Arf invariant}(W) \in \mathbb{Z}/2 & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

In 1962 Browder [Br] used the surgery method to prove that, for $n \geq 5$, a simply-connected finite CW complex X with n-dimensional Poincaré duality

$$H^{n-*}(X) \cong H_*(X)$$

is homotopy equivalent to a closed n-dimensional differentiable manifold if and only if there exists a vector bundle η with spherical Thom class such that an associated invariant $\sigma \in P_n$ (the simply connected surgery obstruction) is 0. The result was proved by applying Thom transversality to η to obtain a suitable degree-one map $M \to X$ from a manifold, and then killing the kernel of the induced map on homology. For $n = 4k$ the invariant $\sigma \in P_{4k} = \mathbb{Z}$ is one eighth of the difference between signature(X) and the $4k$-dimensional component of the $\mathcal{L}$-genus of $-\eta$. (The minus sign comes from the fact that the tangent and normal bundles are stably the negatives of one another.) In this case, the result is a converse of the Hirzebruch signature theorem. In other words, X is homotopy-equivalent to a differentiable manifold if and only if the formula of the theorem holds with η playing the role of the stable normal bundle. The hardest step was to find enough embedded spheres with trivial normal bundle in the middle dimension, using the Whitney embedding theorem for embeddings $S^m \subset M^{2m}$ — this requires $\pi_1(M) = \{1\}$ and $m \geq 3$. Also in 1962, Novikov initiated the use of surgery in the study of the uniqueness of differentiable manifold structures in the homotopy type of a manifold, in the simply-connected case.

From about 1965 until 1970, Wall developed a comprehensive surgery obstruction theory, which also dealt with the non-simply-connected case. The extension to the non-simply-connected case involved many innovations, starting with the correct generalization of the notion of Poincaré duality. A connected finite *CW* complex X is called a *Poincaré complex* [44] of dimension n if there exists a *fundamental homology class* $[X] \in H_n(X;\mathbb{Z})$ such that cap product with $[X]$ induces isomorphisms from cohomology to homology with local coefficients,

$$H^{n-*}(X;\mathbb{Z}[\pi_1(X)]) \xrightarrow{\cong} H_*(X;\mathbb{Z}[\pi_1(X)]).$$

This is obviously a *necessary* condition for X to have the homotopy type of a closed n-dimensional manifold. A *normal map* or *surgery problem*

$$(f,b)\ :\ M^n \to X$$

is a degree-one map $f : M \to X$ from a closed n-dimensional manifold to an n-dimensional Poincaré complex X, together with a bundle map $b : \nu_M \to \eta$ so that

$$\begin{array}{ccc} \nu_M & \xrightarrow{b} & \eta \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & X \end{array}$$

commutes.

Wall defined ([41], [W1]) the *surgery obstruction groups* $L_*(A)$ for any ring with involution A, using quadratic forms over A and their automorphisms. They are more elaborate versions of the Witt groups of fields studied by algebraists.

Theorem 7.1 (Wall, [41], [W1]) *A normal map* $(f,b) : M \to X$ *has a* surgery obstruction

$$\sigma_*(f,b) \in L_n(\mathbb{Z}[\pi_1(X)]),$$

and (f,b) *is normally bordant to a homotopy equivalence if (and for* $n \geq 5$ *only if)* $\sigma_*(f,b) = 0$.

One of Wall's accomplishments in this theorem, quite new at the time, was to find a way to treat both even-dimensional and odd-dimensional manifolds in the same general framework. Another important accomplishment was the recognition that surgery obstructions live in groups *depending on the fundamental group*, but not on any other aspect of X (except for the dimension modulo 4 and the orientation character w_1. Here we have concentrated on the oriented case, $w_1 = 0$). In general, the groups $L_n(\mathbb{Z}[\pi_1(X)])$ are not so easy to compute (more about this below and elsewhere in this volume!), but in the simply-connected case, $\pi_1(X) = \{1\}$, they are just the Kervaire-Milnor groups, $L_n(\mathbb{Z}[\{1\}]) = P_n$.

Wall formulated various relative version of Theorem 7.1 for manifolds with boundary, and manifold n-ads. An important special case is often quoted, which formalizes the idea that the "surgery obstruction groups only depend on the fundamental group." This is the celebrated:

Theorem 7.2 "π-π Theorem" ([W1], 3.3) *Suppose one is given a surgery problem $(f, b) : M^n \to X$, where M and X each have connected non-empty boundary, and suppose $\pi_1(\partial X) \to \pi_1(X)$ is an isomorphism. Also assume that $n \geq 6$. Then (f, b) is normally cobordant to a homotopy equivalence of pairs.*

The most important consequence of Wall's theory, which appeared for the first time in Chapter 10 of [W1], is that it provides a "classification" of the manifolds in a fixed homotopy type (in dimensions ≥ 5 in the absolute case, ≥ 6 in the relative case). The formulation, in terms of the "surgery exact sequence," was based on the earlier work of Browder, Novikov, and Sullivan in the simply connected case. The basic object of study is the *structure set* $\mathcal{S}(X)$ of a Poincaré complex X. This is the set of all homotopy equivalences (or perhaps simple homotopy equivalences, depending on the way one wants to formulate the theory) $M \xrightarrow{f} X$, where M is a manifold, modulo a certain equivalence relation: $M \xrightarrow{f} X$ and $M' \xrightarrow{f'} X$ are considered equivalent if there is a diffeomorphism $\phi : M \to M'$ such that $f' \circ \phi$ is homotopic to f. One should think of $\mathcal{S}(X)$ as "classifying all manifold structures on the homotopy type of X."

Theorem 7.3 (Wall [W1], Theorem 10.3 *et seq.*) *Under the above dimension restrictions, the structure set $\mathcal{S}(X)$ of a Poincaré complex X is non-empty if and only if there exists a normal map $(f, b) : M \to X$ with surgery obstruction $\sigma_*(f, b) = 0 \in L_n(\mathbb{Z}[\pi_1(X)])$. If non-empty, $\mathcal{S}(X)$ fits into an exact sequence (of sets)*

$$L_{n+1}(\mathbb{Z}[\pi_1(X)]) \to \mathcal{S}(X) \to \mathcal{T}(X) \to L_n(\mathbb{Z}[\pi_1(X)]),$$

where $\mathcal{T}(X) = [X, G/O]$ classifies "tangential data."

Much of [W1] and many of Wall's papers in the late 1960's and early 1970's were taken up with calculations and applications. We mention only a few of the applications: a new proof of the theorem of Kervaire characterizing the fundamental groups of high-dimensional knot complements, results on realization of Poincaré (i.e., homotopy-theoretic) embeddings in manifolds by actual embeddings of submanifolds, classification of free actions of various types of discrete groups on manifolds (for example, free involutions on spheres), the classification of "fake projective spaces," "fake lens spaces," "fake tori," and more. The work on the topological space form problem (free actions on spheres) is particularly significant: the CW complex version had already motivated the work of Swan and Wall on the finiteness obstruction discussed above, while the manifold version was one of the impulses for Wall's (and others') extensive calculations of the L-groups of finite groups.

8 PL and topological manifolds

While surgery theory was originally developed in the context of smooth manifolds, it was soon realized that it works equally well in the PL category of combinatorial manifolds. Indeed, the book [W1] was written in the language of PL manifolds. Wall's theory has the same form in both categories, and in fact the surgery obstruction groups are the same, regardless of whether one works in the smooth or in the PL category. The only differences are that for the PL case, vector bundles must be replaced by PL bundles, and in theorem 7.3, G/O should be replaced by G/PL.

Passage from the PL to the topological category was a much trickier step (even though we now know that G/PL more closely resembles G/TOP than G/O). Wall wrote in the introduction to [44]:

> This paper was originally planned when the only known fact about topological manifolds (of dimension > 3) was that they were Poincaré complexes. Novikov's proof of the topological invariance of rational Pontrjagin classes and subsequent work in the same direction has changed this ....

Novikov's work introduced the torus T^n as an essential tool in the study of topological manifolds. A *fake torus* is a manifold which is homotopy equivalent to T^n. The surgery theoretic classification of PL fake tori in dimensions ≥ 5 by Wall and by Hsiang and Shaneson [HS] was an essential tool in the work of Kirby [K] and Kirby-Siebenmann [KS] on the structure theory of topological manifolds. (See also [KSW].) This in turn made it possible to extend surgery theory to the topological category.

9 Invariance properties of the signature

Wall made good use of the signature invariants of quadratic forms and manifolds. We pick out three particular cases:

1. One immediate (but non-trivial) consequence of the Hirzebruch signature theorem is that if $\widetilde{M}$ is a k-fold covering of a closed manifold M, then
$$\text{signature}(\widetilde{M}) \;=\; k\cdot\text{signature}(M)\in\mathbb{Z}\ .$$
It is natural to ask whether this property is special to manifolds, or whether it holds for Poincaré complexes in general. But in [44], Wall constructed examples of 4-dimensional Poincaré complexes X where this fails, and hence such X are not homotopy equivalent to manifolds.

2. For finite groups π, Wall [W1] showed that the surgery obstruction groups $L_*(\mathbb{Z}[\pi])$ are finitely generated abelian groups, and that the torsion-free part of these groups is determined by a collection of signature invariants called the *multisignature*. This work led to a series of deep interactions between algebraic number theory and geometric topology.

3. The Novikov additivity property of the signature is that the signature of the boundary-connected union of manifolds is the sum of the signatures. In [54], Wall showed that this additivity fails for unions of manifolds along parts of boundaries which are not components, and obtained a homological expression for the non-additivity of the signature (which is also known as the Maslov index).

10 Homological and combinatorial group theory

Wall's work on surgery theory led to him to several problems in combinatorial group theory. One of these was to determine what groups can be the fundamental groups of aspherical Poincaré complexes. Such groups are called *Poincaré duality groups*. There are evident connections with the topology of manifolds:

> ([W2], problem G2, p. 391) Is every Poincaré duality group Γ the fundamental group of a closed $K(\Gamma,1)$ manifold? Smooth manifold? Manifold unique up to homeomorphism? (It will not be unique up to diffeomorphism.)

([W2], problem F16, p. 388) Let Γ be a Poincaré duality group of dimension ≥ 3. Is the 'fundamental group at infinity' of Γ necessarily trivial? This is known in many cases, e.g. if Γ has a finitely presented normal subgroup Γ' of infinite index and either Γ' or Γ/Γ' has one end. In dimensions ≥ 5 it is equivalent to having the universal cover of a compact $K(\Gamma, 1)$ manifold homeomorphic to euclidean space.

For more on the subsequent history of these problems, see [FRR] and [D].

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