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Unknotting tori in codimension one and spheres in codimension two

BY C. T. C. WALL
Mathematical Institute, Oxford

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We shall present this paper in the framework and terminology of differential topology though all our arguments are valid in the piecewise linear case also, under local unknottedness hypotheses. In particular we use $\mathbf{R}^p$ for Euclidean space of dimension p, S^{p-1} for the standard unit sphere in it, and D^p for the disc which it bounds.

Kosinski proved in (5), with certain restrictions on p and q, the following theorems.

I *Let T be a submanifold of S^{p+q+1}, diffeomorphic to $S^p \times S^q$. Then the closure of one of its complementary components is diffeomorphic to $D^{p+1} \times S^q$.*

II *Let T_1, T_2 be as in I. Then there is a diffeomorphism h of S^{p+q+1} with $h(T_1) = T_2$.*

III *Let S_1, S_2 be submanifolds of S^{p+q+1}, diffeomorphic to S^p. Then $S^{p+q+1} - S_1$ and $S^{p+q+1} - S_2$ are diffeomorphic.*

The present paper is motivated by the observation that all three theorems can be improved. To fix notation, we mention:

LEMMA 1. *Let T^{p+q} be a submanifold of S^{p+q+1}, with the homology of $S^p \times S^q$, and such that each component has Abelian fundamental group. Then $S^{p+q+1} - T$ splits as the disjoint union of two open sets, with closures C_p and C_q, each with boundary T. C_p is a homology S^p, and if $p \neq 1$ also a homotopy S^p (similarly for q).*

We shall call a manifold imbedded in a sphere, and diffeomorphic to a product P of a sphere with a disc or sphere, *unknotted* if there is a diffeomorphism of the larger sphere throwing the manifold onto the standard imbedded copy of P. A manifold diffeomorphic to the product of two spheres we call a *torus*. Now, with the notation of Lemma 1, our first main result is

THEOREM 2. *Assume $p+q+1 \neq 4$.*

(A) *Suppose C_p is a homotopy S^p. If $p+q+1 = 5$, or if $q = 1$ let T be a torus. If $p = 1$ and $q = 3$, assume the conjecture below. Then C_p is diffeomorphic to $S^p \times D^{q+1}$.*

(B) *Suppose $p, q \neq 1$ and, if $p+q+1 = 5$, that T is a torus. Then T is an unknotted torus.*

(C) *Suppose $q = 1$, and that T is a torus. If $p = 3$, assume the conjecture below. Then T is unknotted if and only if C_q is a homotopy S^1.*

Conjecture. Any h-cobordism of $S^3 \times S^1$ to itself is diffeomorphic to $S^3 \times S^1 \times I$.

It should perhaps be mentioned that although we formulate the conjecture in geometric terms, we have succeeded in doing the geometry, and reducing the problem to a purely algebraic (commutative) one. In particular, there is no connexion whatever between our conjecture and the unsolved cases of the Poincaré conjecture.

We now consider a submanifold $S \subset S^{p+2}$, diffeomorphic to S^p. It is not hard to see that the normal bundle is necessarily trivial, so the boundary T of a tubular neighbourhood is diffeomorphic to $S^p \times S^1$. We then have

THEOREM 3. *Let $p \neq 2$. Then S is unknotted if and only if T is.*

COLLORARY 3·1. *If $p \neq 2, 3, S$ is unknotted if and only if $S^{p+2} - S$ is a homotopy circle. If $p = 3$, this holds if the conjecture above is true.*

When $p \geqslant 4$, this corollary is due to Levine ((6)). (Levine excludes the case $p = 5$, but Browder has pointed out an easy way to fill the gap in the argument.)

Proof of Lemma 1. Let T be a submanifold of S^{p+q+1}, with the homology of $S^p \times S^q$; $p, q > 0$. By Alexander's duality theorem, $C = S^{p+q+1} - T$ has two components, and $H_i(C)$ vanishes for $i \neq 0,\ p, q$; if $p \neq q$, $H_p(C) \simeq \mathbf{Z}$, $H_q(C) \simeq \mathbf{Z}$, whereas if $p = q$, $H_p(C) \simeq \mathbf{Z} + \mathbf{Z}$. Let the components be C' and C''. Neither can be acyclic, for if C' was, its closure would be an acyclic manifold, and its boundary a homology sphere. So one component is a homology p-sphere, the other a homology q-sphere; we label their closures C_p and C_q.

The case $p = q = 0$ is trivial (T consists of four points lying on a circle); if $p > 0$, $q = 0$ a similar argument shows that we have three components; two acyclic (with union C_q, say), and a homology S^p, C^p. This reduces us to the case above, as again $\partial C_p = T = \partial C_q$. Also if $p = 1$, the Schönflies theorem (in the plane) shows

$$C_p \simeq S^1 \times D^1, \quad C_q \simeq S^0 \times D^2.$$

Now return to the general case, and suppose $\pi_1(T)$ Abelian. If $p, q \geqslant 2$ this shows that T is simply connected; van Kampen's theorem now shows that C_p and C_q are simply connected (for $1 = \pi_1(S^{p+q+1}) = \pi_1(C_p) * \pi_1(C_q)$), so they are homotopy spheres. A similar argument goes if $p \geqslant 2$, $q = 0$, as each component of T is simply-connected. Finally, let $p \geqslant 2$, $q = 1$, so $\pi_1(T) \simeq H_1(T) \simeq \mathbf{Z}$. The commutative diagram

$$\begin{array}{ccc} \pi_1(T) & \longrightarrow & \pi_1(C_q) \\ \big\downarrow \wr & & \big\downarrow \\ H_1(T) & \xrightarrow{\simeq} & H_1(C_q) \end{array}$$

shows that $\pi_1(T)$ is a 'retract' of $\pi_1(C_q)$; hence $\pi_1(C_p)$ is a retract of

$$\pi_1(C_p) *_{\pi_1(T)} \pi_1(C_q) = \pi_1(S^{p+q+1}) = \{1\},$$

so C_p is simply connected, hence a homotopy S^p.

Proof of Theorem 2. We have already considered the cases $p + q + 1 \leqslant 2$; the cases $p + q + 1 = 3$ are due essentially to Alexander ((1)).

Now suppose $p \geqslant q$ and $p + q \geqslant 5$. Then $p + q + 1 \geqslant 2q + 1$, so (if C_q is a homotopy S^q) we can imbed S^q in C_q by a homotopy equivalence. Moreover, S^q unknots in S^{p+q+1} (this is due to Whitney ((10)) if $p > q$ and to Wu ((11)) if $p = q$), and in particular has a trivial normal bundle. Since the codimension $p + 1 \geqslant 3$, and the dimension $p + q + 1 \geqslant 6$, a result of Smale ((7), Theorem 4·1) shows that C_q is a tubular neighbourhood of S^q, and hence an unknotted $D^{p+1} \times S^q$. Hence also $T = S^p \times S^q$ is unknotted, and C_p diffeomorphic to $S^p \times D^{q+1}$.

In the case $p+q = 4$, the only gap in this argument is the appeal to Smale's theorem, and the only gap in Smale's argument is the assertion that an h-cobordism between $S^p \times S^q$ and itself is diffeomorphic to a product. But for $p = q = 2$, this has been proved by Barden ((2)); for $p = 3, q = 1$ it is our conjecture above; for $p = 4, q = 0$ it was shown by Smale ((7)) that 'a contractible 5-manifold with boundary diffeomorphic to S^4 is diffeomorphic to D^5', from which the result follows.

We have now proved parts (B) and (C) of the theorem, and all cases of part (A) except $q = 1, p \geqslant 3$, when C_p is necessarily a homotopy S^p, but unknotting need not occur. In this case we assume T diffeomorphic to $S^p \times S^1$. Introduce corners on ∂C_p so that T is the product of S^p and a square. Then C_p is an h-cobordism of manifolds with boundary (diffeomorphic to $S^p \times I$ – the 'ends' of the square) which, on the boundary (the 'sides' of the square) is a product. Hence the h-cobordism is also a product in a neighbourhood of the sides. Remove the sides and apply Smale ((7), Theorem 3·1): it follows if $p+2 > 5$ that C_p is a product, $S^p \times I \times I$. Removing the corners again, we have $S^p \times D^2$.

We now give an alternative proof of this last case, valid for $p \geqslant 3$, modelled on Alexander's proof when $p = 1$, and not depending on any conjecture. Let $x \in S^p$. Then $x \times S^1$ is null-homotopic in C_p; as dim $C_p \geqslant 5$, $x \times S^1$ bounds an imbedded disc D^2 in C_p. This disc has a tubular neighbourhood $D^p \times D^2$ in C_p, meeting T in $D^p \times S^1$. Also, since the group $SO_p \subset SO_{p+1}$ and hence acts on S^p, we can change the diffeomorphism of T on $S^p \times S^1$ so that the framing induced by the tubular neighbourhood above coincides with the product framing of $x \times S^1$ in $S^p \times S^1$. Now delete $D^p \times S^1$ from T, and replace by $S^{p-1} \times D^2$, thus giving a manifold U, and round the corners. U is diffeomorphic to a sphere S^{p+1}, so (by Smale again) bounds a disc D^{p+2} in C_p. Now C_p is obtained by attaching $D^p \times D^2$ to D^{p+2}: we assert that the attaching sphere $S^{p-1} \times 0$ bounds a disc in U, so is unknotted. Hence C_p is a D^2-bundle over S^p; since it is parallelizable, $C_p \simeq S^p \times D^2$. In fact, let I be an arc in D^2 joining 0 to a point $y \in S^1 = \partial D^2$. Then $S^{p-1} \times 0$ bounds the disc $(S^{p-1} \times I) \cup (D^p \times y)$ in U.

Remark. If $p \geqslant q = 1$, *then* $S^p \times S^q$ *can knot in* S^{p+q+1}.

Examples are known ((3)) of imbeddings of S^p in S^{p+2} ($p \geqslant 1$) whose complement has non-Abelian fundamental group. If T is the boundary of a tubular neighbourhood of S^p, the corresponding $\pi_1(C_q)$ is also non-Abelian, so T is knotted.

Proof of Theorem 3. In one direction this is trivial; if S is unknotted, then T certainly is. Conversely, suppose T unknotted. The unknotting gives a diffeomorphism of T on an unknotted $S^p \times S^1$; for $x \in S^1$, $S^p \times x$ is then unknotted. Our idea is to prove S diffeotopic to $S^p \times x$.

As T is the boundary of a tubular neighbourhood of S, we have a diffeomorphism of T on $S \times S^1$, and S is diffeotopic to $S \times x$. We now need

LEMMA 4. *Let M and N be connected closed manifolds, h a homeomorphism of $M \times S^1$ on $N \times S^1$ with $h_* \pi_1(M) = \pi_1(N)$, $x \in S^1$. Then there is an h-cobordism W imbedded in $N \times S^1 \times I$, with ends $h(M \times x) \times 0$ and $(N \times x) \times 1$.*

Assuming this, we proceed as follows. We can regard S^{p+2} as formed of the tubular neighbourhood $S \times D^2$, a collar neighbourhood $T \times I$, and the complement $D^{p+1} \times S^1$.

Taking S^p and S for M and N in Lemma 4, we obtain an h-cobordism W in $T \times I$. If I is an arc in D^2 joining 0 to x, consider the union Δ^{p+1} of $S \times I$, W, and $D^{p+1} \times x$: this is a contractible manifold bounded by S in S^{p+2} (and can easily be made smooth by rounding corners). But if $p \geqslant 4$, the fact that Δ^{p+1} is contractible with boundary diffeomorphic to S^p implies that Δ^{p+1} is diffeomorphic to D^{p+1}; since D^{p+1} unknots in S^{p+2}, it follows that S is unknotted.

The case $p = 3$, as usual, offers more difficulty. Let us write Σ^n for a homotopy n-sphere, Δ^n for a compact contractible n-manifold. Then we will prove

LEMMA 5. (i) *Let* $\Sigma^4 = \partial\Delta_1^5 = \partial\Delta_2^5$. *Then there is a diffeomorphism of* Δ_1^5 *on* Δ_2^5 *inducing the identity on* Σ^4.

(ii) *Any two imbeddings of* Σ^4 *in* S^5 *are equivalent by a diffeomorphism of* S^5.

(iii) *Any two imbeddings of* Δ^4 *in* S^5 *are equivalent by a diffeomorphism of* S^5.

(iv) *Let* $\partial\Delta^4 = S^3$. *We can imbed* Δ^4 *in* S^5 *with its boundary unknotted.*

It follows immediately from (iii) and (iv) that any 3-sphere in S^5, which bounds a contractible 4-manifold in S^5, is unknotted; this completes the proof of Theorem 3.

Proof of Lemma 4. Our model S^1 is the unit circle in the complex plane; we may take $x = 1$. Define $e: \mathbf{R}^1 \to S^1$ by $e(t) = e^{2\pi i t}$; this is the projection of the universal covering. Also, for $t \in \mathbf{R}$, write $T(t) = t+1$. Then T generates the group of deck transformations, isomorphic to $\pi_1(S^1) = \mathbf{Z}$. We also write $e: N \times \mathbf{R} \to N \times S^1$ for the product with the identity, and correspondingly for T; $p_2: N \times \mathbf{R} \to \mathbf{R}$ for the projection.

Since $h_* \pi_1(M) = \pi_1(N)$, he factors through e, $he = eh'$. Since M is compact, for some i, $p_2(T^i h'(M)) = T^i(p_2 h'(M))$ consists only of positive numbers. Write $h'' = T^i h'$. Now $N = N \times 0$ certainly separates $N \times \mathbf{R}$; for the same reason so does $h''(M)$, which is disjoint from N. Hence there is a submanifold V of $N \times \mathbf{R}$ with boundary $h''(M) \cup N$. It is now easy to check that V is an h-cobordism (e.g. $H_*(V, N) = H_*(V \cup N \times \mathbf{R}-,$ $N \times \mathbf{R}-)$. But $N \times \mathbf{R}-$ is a deformation retract of $N \times \mathbf{R}$; so is

$$V \cup (N \times \mathbf{R}-) = h''(M \times \mathbf{R}-).$$

Also, $p_2(V)$ is non-negative.

Define $F: N \times \mathbf{R}_+ \to N \times S^1 \times I$ by

$$F(n,t) = \left[n, e(t), \frac{t}{1+t}\right].$$

Since $t/(1+t)$ is strictly monotone, this is $(1-1)$; hence $F|V$ is an imbedding. Also, $F, h''(M) \subset h(M) \times I$. For $m \in M$, write $Fh''(m) = (h(m), u(m))$. Now define

$$G: M \times I \to h(M) \times I \subset N \times S^1 \times I \quad \text{by} \quad G(m,t) = (h(m), t + (1-t)u(m)).$$

This again is clearly an imbedding, and $G(m,0) = (h(m), u(m)) = Fh''(M)$. So $F(V)$ and $G(M \times I)$ fit together along $F(M)$: their union is an h-cobordism, of $F(N) = N \times 1 \times 0$ to $G(M \times 1) = h(M) \times 1$.

We now check that the images $F(V)$ and $G(M \times I)$ overlap only along $F(M) = G(M \times 0)$. But for $t > 0$, the point $G(m,t)$ has the same coordinates in $N \times S^1$ as $G(m,0) = Fh''(M)$, and a larger coordinate in I. Thus if it lies in the image of F, it must be the transform of $Fh''(M)$ by some T^i: $i > 0$. But if $i > 0$, $T^i h''(M)$ is clearly not contained in V; it is connected and disjoint from $\partial V (= N \times 0 \cup h''(M))$ so it is disjoint from V.

It follows that we can take W as $F(V) \cup G(M \times I)$, where the corner along $F(M)$ must be rounded (e.g. by the Cairns–Hirsch theorem).

Proof of Lemma 5. (i) Attach Δ_1^5, $\Sigma^4 \times I$, and Δ_2^5. The result is a homotopy 5-sphere so according to Kervaire and Milnor ((4)) bounds a contractible manifold. This gives an h-cobordism of Δ_1 and Δ_2, which is a product on the boundary hence (Smale ((7))) a product.

(ii) Given two imbeddings, let the closures of the complementary domains be Δ_1, Δ_1' or Δ_2, Δ_2'. By (i), we can extend the 'identity' of Σ^4 to diffeomorphisms

$$\Delta_1 \to \Delta_2, \quad \Delta_1' \to \Delta_2',$$

so the imbeddings are equivalent under a diffeomorphism of S^5.

(iii) Let Σ^4 be the double of Δ^4. Any imbedding of Δ^4 in S^5 induces one of Σ^4, for the imbedding must have trivial normal bundle, and so extend to an imbedding of $\Delta^4 \times I$, which has boundary Σ^4. (ii) now implies (iii).

(iv) Form Σ^4 from Δ^4 by attaching D^4 along the boundary. By Kervaire and Milnor ((4)) or Wall ((8)), Σ^4 bounds a Δ^5. The double of Δ^5 is a homotopy 5-sphere, hence ((7)) diffeomorphic to S^5. Thus Σ^4 imbeds in S^5, so there is an imbedding of Δ^4 in S^5 with $\partial\Delta^4$ bounding a D^4 in S^5, and hence unknotted.

It now seems appropriate to make a few comments on our conjecture.

Conjecture. Any h-cobordism H of $S^3 \times S^1$ to itself is diffeomorphic to $S^3 \times S^1 \times I$.

We have already drawn several consequences from this, the most interesting of which seems to be the unknotting criterion (3·1) for imbeddings of S^3 in S^5. We have also made several steps towards a proof of the conjecture, which may be summarized as follows.

Step 1. If we can find an h-cobordism of H to $S^3 \times S^1 \times I$, which is a product along the edges, the result will follow—essentially by Mazur's s-cobordism theorem (see (9), 6·3). So attach two 'edges' $S^3 \times S^1 \times I$ to the boundary components of H, and a further $S^3 \times S^1 \times I$ to join them up (giving W) and try to prove W bounds a homotopy $S^3 \times S^1$.

Step 2. Does W bound a framed manifold which retracts on $S^3 \times S^1$? The *a priori* obstruction to this lies in $\mathbf{Z}_2 + \mathbf{Z}_2$, but this turns out to be irrelevant. We then do surgery on F to make the retraction ϕ: $F \to S^3 \times S^1$ 3-connected.

Step 3. If we can make ϕ 4-connected, it is a homotopy equivalence, and we are finished. But $\pi_4(\phi) = H_4(\tilde{\phi})$ turns out to be a subgroup of $H_3(\tilde{F})$ and a free module over the group ring of $\pi_1(F)$; Poincaré duality defines a non-singular skew-Hermitian form on this module, and a study of this form will be necessary before we can complete the proof.

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