Similarly, we see that \( f''(0) = m \). Thus \( M = m \). Therefore, \( f''(a) \) is a constant function in \([0, 2\pi]\).

Let \( f''(a) = \alpha \). Then \( f \) is given by the quadratic polynomial

\[
f(a) = \alpha a^2 / 2 + \beta a + \gamma.
\]

Our final step is to determine constants \( \alpha, \beta, \) and \( \gamma \) explicitly. Substitute (11) in (6) and then equate the coefficients of the term \( a \) and the constant term to obtain

\[
-\pi \alpha / 2 = \beta / 2, \quad \pi^2 \alpha / 2 + \beta \pi + 2 \gamma = \gamma / 2.
\]

Notice that the coefficient of the term \( a^2 \) vanishes. By (1) and by a basic theorem on differentiation under the integral sign we obtain \( f'(\pi / 2) = \pi / 2 \), which, with (11), implies

\[
\pi \alpha / 2 + \beta = \pi / 2.
\]

It follows from (12) and (13) that \( \alpha = -1, \beta = \pi, \) and \( \gamma = -\pi^2 / 3 \). Thus, (11) yields (5). This completes the proof of Theorem.

The values of the Euler integrals (2), (3) and (4) now follow from (1) and (5) by setting \( a \) equal to 0, \( \pi, \) and \( \pi / 2 \) respectively.

This work was supported in part by the Natural Sciences and Engineering Research Council of Canada Grant Nr. A-4012.

Reference


A HOMOLOGY VERSION OF THE BORSUK-ULAM THEOREM

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An involution on a topological space \( X \) is a continuous map from \( X \) to \( X \) which is its own inverse. For example the antipodal map, which maps a point to the opposite end of the diameter on which it lies, is an involution on the \( n \)-sphere \( S^n \).

Suppose \( X \) and \( Y \) are spaces equipped with involutions \( a \) and \( b \), respectively. A map \( f \) from \( X \) to \( Y \) is equivariant if it respects the involutions, i.e., \( b \circ f = f \circ a \).

One formulation of the Borsuk-Ulam theorem is that if \( m \) is greater than \( n \), then there is no map from \( S^m \) to \( S^n \) which is equivariant with respect to the antipodal map. Many sources, for example [1, § 7.2], include proofs of the Borsuk-Ulam theorem, as well as applications such as the “ham sandwich theorem.” We will use singular homology theory to prove a somewhat stronger theorem.

Our stronger theorem shows that the existence of any equivariant maps to \( S^n \) from any space \( X \) with an involution forces the existence of very special homology classes for \( X \), so special that \( X \) could not be a sphere of dimension greater than \( n \).

A few words about terminology: An elementary 0-chain is a singular 0-simplex with coefficient 1; loosely speaking, it’s just a single point. We will use reduced homology, which essentially means that we consider the empty set to be a singular simplex of dimension \(-1\), which is the boundary of every 0-simplex. It follows that \( \tilde{H}_{-1}(X) \) vanishes unless \( X \) is empty, and \( \tilde{H}_0(X) \) vanishes if \( X \) is path connected. Recall that each continuous map \( f \) induces a chain map \( f_* \), defined by composing \( f \) with singular simplices. In turn, such a chain map \( f_* \) induces a homology homomorphism \( f_* \).

THEOREM. Suppose \( X \) is a space with involution \( \nu \), and \( g: X \to S^n \) is an equivariant map. Then there exists an integer \( j \leq n \), and a homology class \( \beta \) of \( \tilde{H}_j(X; \mathbb{Z}/2) \) such that \( \beta \) is nonzero and \( \nu_* (\beta) = \beta \). Furthermore, if no such \( \beta \) exists for \( j \) less than \( n \), then \( \beta \) can be chosen such that \( g_* (\beta) \) is the nonzero element of \( \tilde{H}_n(S^n; \mathbb{Z}/2) \).
In this framework our theorem applied to an equivariant map \( g : S^m \to S^n \) with \( m > n \) would require the existence of a special nonzero element of \( \tilde{H}_j(S^m) \) for some \( j \leq n < m \). But \( \tilde{H}_j(S^m) = 0 \) for all \( j < m \) so our result generalizes the Borsuk-Ulam theorem.

**Proof of theorem:** The case \( n = 0 \) is straightforward, so assume that \( n \) is greater than \( 0 \). The proof will proceed by inductively constructing singular chains. (Similar methods were used on a higher level by P. A. Smith; see [2, chapter 13].) Bear in mind that signs can be ignored, since we are using coefficients in \( \mathbb{Z}/2 \).

It is convenient to define a "symmetrizer" chain map \( \theta = id_\ast + v_\ast \) on the singular chain complex of \( X \), where "id" denotes the identity map. We use the same notation for the chain map \( id_\ast + d_\ast \) on \( S^n \), where \( a \) is the antipodal map. These operators satisfy \( \theta \theta = 0 \) and \( \theta g_\ast = g_\ast \theta \), as one can easily verify.

Assume that for all \( j \) less than \( n \) and for all \( \beta \) in \( \tilde{H}_n(X; \mathbb{Z}/2) \), \( v_\ast(\beta) = \beta \) implies \( \beta = 0 \). (Otherwise, we have a \( \beta \) which satisfies the first part of the theorem.) Hence if \( x_j \) is a \( j \)-cycle such that \( \theta x_j = 0 \), then \( x_j \) must be a boundary. Our goal is to produce a nontrivial element \( \beta \) of \( \tilde{H}_n(X; \mathbb{Z}/2) \) such that \( v_\ast(\beta) = \beta \) and \( g_\ast(\beta) \neq 0 \). Our strategy will be to make some observations about \( j \)-dimensional hemispheres \( h_j \) in \( S^n \), construct chains \( c_j \) in \( X \) which behave much like the hemispheres, compare \( g_\ast c_j \) to \( h_j \), and finally show that \( \theta c_n \) is a cycle which determines the desired homology class.

First we choose singular \( j \)-chains \( h_j \) in \( S^n \), corresponding to hemispheres, such that

\[
\begin{align*}
h_0 & \text{ is an elementary 0-chain,} \\
\partial h_j & = \theta h_{j-1} \text{ for } 1 \leq j \leq n, \text{ and} \\
\theta h_n & \text{ generates } \tilde{H}_n(S^n; \mathbb{Z}/2).
\end{align*}
\]

Next, we will construct singular \( j \)-chains \( c_j \) in \( X \), for \( j \) ranging from \( 0 \) to \( n \), such that

\[
\begin{align*}
c_0 & \text{ is an elementary 0-chain, and} \\
\partial c_j & = \theta c_{j-1} \text{ for } 1 \leq j \leq n.
\end{align*}
\]

We have assumed that there is no nonzero \( \beta \) in \( \tilde{H}_{n-1}(X; \mathbb{Z}/2) \) such that \( v_\ast(\beta) = \beta \), so \( X \) is nonempty. Pick a point in \( X \), and let \( c_0 \) be the corresponding elementary 0-chain. Note that \( \theta c_0 \) is a cycle. Since \( \theta \theta c_0 = 0 \), there is a 1-chain \( c_1 \) such that \( \partial c_1 = \theta c_0 \).

Suppose that \( \partial c_j = \theta c_{j-1} \) for some \( j \) less than \( n \). We compute that \( \partial \theta c_j = \theta \partial c_j = \theta \theta c_{j-1} = 0 \), so \( \theta c_j \) is a cycle. Since \( \theta \theta c_j = 0 \), there exists a \((j + 1)\)-chain \( c_{j+1} \) such that \( \partial c_{j+1} = \theta c_j \). This completes the inductive definition of \( c_0, c_1, \ldots, c_n \).

Now we will inductively construct \( j \)-chains \( e_j \) in \( S^n \), for \( j \) ranging from \( 0 \) to \( n \), such that

\[
h_j - g_\ast c_j - \theta e_j \text{ is a cycle.}
\]

Note that \( h_0 - g_\ast c_0 \) is a cycle, since \( h_0 \) and \( c_0 \) were chosen to be elementary 0-chains. Therefore we can let \( e_0 = 0 \).

Suppose that \( e_j \) is a \( j \)-chain, where \( j \) is less than \( n \), such that \( h_j - g_\ast c_j - \theta e_j \) is a cycle. Since \( \tilde{H}_j(S^n; \mathbb{Z}/2) = 0 \), there is a \((j + 1)\)-chain \( e_{j+1} \) such that

\[
\partial e_{j+1} = h_j - g_\ast c_j - \theta e_j.
\]

Apply \( \theta \) and obtain

\[
\partial \theta e_{j+1} = \theta h_j - g_\ast \theta c_j.
\]

Since \( \theta h_j = \partial h_{j+1} \) and \( \theta c_j = \partial c_{j+1} \), this becomes

\[
\partial \theta e_{j+1} = \partial h_{j+1} - \partial g_\ast c_{j+1}.
\]

Therefore \( h_{j+1} - g_\ast c_{j+1} - \theta e_{j+1} \) is a cycle, as desired.

To complete the proof, we note that \( h_n - g_\ast c_n - \theta e_n \) is a cycle in \( S^n \), which is therefore
homologous to either zero or $\theta c_n$. In either case, when we apply $\theta$, we find that $\theta h_n - g_\theta c_n$ is homologous to zero. That is, $\theta h_n$ and $g_\theta c_n$ belong to the same homology class. Note that $\theta c_n$ is a cycle, because $\partial \theta c_n = \theta \partial c_n = \theta \theta c_{n-1} = 0$. Therefore, if $\beta$ is the homology class of $\theta c_n$, then $g_\theta(\beta)$ is the nonzero element of $\tilde{H}_n(S^n; \mathbb{Z}/2)$. It follows that $\beta$ is nonzero. Finally, the fact that $\theta \theta c_n = 0$ means that $v_\theta \theta c_n = \theta c_n$, so $v_\theta(\beta) = \beta$.

References


ON THE NUMBER OF MULTIPLICATIVE PARTITIONS

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I. A Number-Theoretic Function. In this note we show that if $f(n)$ is the number of essentially different factorizations of $n$, then

$$f(n) \leq 2n^{\sqrt{2}}.$$

In considering numbers that have exactly $k$ divisors, one is led to examine this function $f(n)$, the number of ways to write $n$ as the product of integers $\geq 2$, where we consider factorizations that differ only in the order of the factors to be the same. We call these representations of $n$ multiplicative partitions. For example, $f(12) = 4$, since

$$12 = 6 \cdot 2 = 4 \cdot 3 = 3 \cdot 2 \cdot 2$$

are the four multiplicative partitions of 12. From these four representations, we can conclude that a number has exactly 12 divisors if and only if its prime factorization is one of the following:

$$p^{11}, p^5q, p^3q^2, p^2qr.$$

This follows from the expression for $\tau(n)$, the number of divisors of $n = p_1^{a_1}p_2^{a_2} \cdots p_k^{a_k}$.

$$\tau(n) = \prod_{j=1}^{k} (1 + a_j).$$

For example, see [1].

The behavior of $f(n)$ is quite erratic, and apparently has not been previously studied in this form. We observe that if $q$ is prime, then $f(q^k) = p(k)$, the number of additive partitions of $k$. Also, if $q_1, q_2, \ldots, q_k$ are distinct primes, then $f(q_1q_2 \cdots q_k) = B(k)$, the $k$th Bell number. See [2].

More generally, $f(q_1^{a_1} \cdots q_k^{a_k})$ is the number of additive partitions of the “multi-partite number” $(a_1, a_2, \ldots, a_k)$, where addition is defined component-wise. See [3] for further details.

We will show that

$$f(n) \leq 2n^{\sqrt{2}}. \quad (1)$$

For a table of $f(n)$ for $1 \leq n \leq 100$, see the Appendix.

II. Proof of the Main Result. To prove (1) we first define an auxiliary function:

$$g(m, n) = \text{the number of multiplicative partitions of } n \text{ with all elements } \leq m.$$ 

Clearly $f(n) = g(n, n)$. We have the following