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EMNER, M.D.
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PROCEEDINGS

OF THE

ROYAL SOCIETY OF EDINBURGH.

VOL. IX. 1875–76. No. 94.

NINETY-THIRD SESSION.

Monday, 20th December 1875.

SIR WILLIAM THOMSON, President, in the Chair.

The following Communications were read:—

1. Vortex Statics. By Sir William Thomson.

(*Abstract.*)

The subject of this paper is *steady motion* of vortices.

1. Extended definition of "steady motion." The motion of any system of solid or fluid or solid and fluid matter is said to be steady when its configuration remains equal and similar, and the velocities of homologous particles equal, however the configuration may move in space, and however distant individual material particles may at one time be from the points homologous to their positions at another time.

2. Examples of steady and not steady motion:—

(1.) A rigid body symmetrical round an axis, set to rotate round any axis through its centre of gravity, and left free, performs steady motion. Not so a body having three unequal principal moments of inertia.

(2.) A rigid body of any shape, in an infinite homogeneous liquid, rotating uniformly round any, always the same, fixed line, and moving uniformly parallel to this line, is a case of steady motion.

(3.) A perforated rigid body in an infinite liquid moving in the manner of example (2.), and having cyclic irrotational motion of the liquid through its perforations, is a case of steady motion. To this case belongs the irrotational motion of liquid in the neighbourhood of any rotationally moving portion of fluid of the same shape as the solid, provided the distribution of the rotational motion is such that the shape of the portion endowed with it remains unchanged. The object of the present paper is to investigate general conditions for the fulfilment of this proviso; and to investigate, farther, the conditions of stability of distribution of vortex motion satisfying the condition of steadiness.

3. *General synthetical condition for steadiness of vortex motion.*—The change of the fluid's molecular rotation at any point fixed in space must be the same as if for the rotationally moving portion of the fluid were substituted a solid, with the amount and direction of axis of the fluid's actual molecular rotation inscribed or marked at every point of it, and the whole solid, carrying these inscriptions with it, were compelled to move in some manner answering to the description of example (2). If at any instant the distribution of molecular rotation* through the fluid, and corresponding distribution of fluid velocity, are such as to fulfil this condition, it will be fulfilled through all time.

4. *General analytical condition for steadiness of vortex motion.*—If, with (§ 24, below) vorticity and "impulse," given, the kinetic energy is a maximum or a minimum, it is obvious that the motion is not only steady, but stable. If, with same conditions, the energy is a maximum-minimum, the motion is clearly steady, but it may be either unstable or stable.

5. The simple circular Helmholtz ring is a case of stable steady motion, with energy maximum-minimum for given vorticity and given impulse. A circular vortex ring, with an inner irrotational annular core, surrounded by a rotationally moving annular shell (or endless tube), with irrotational circulation outside all, is a case of motion which is steady, if the outer and inner contours of the

* One of the Helmholtz's now well-known fundamental theorems shows that, *from the molecular rotation at every point of an infinite fluid* the velocity at every point is determinate, being expressed synthetically by the same formulæ as those for finding the "magnetic resultant force" of a pure electro-magnet. —*Thomson's Reprint of Papers on Electrostatics and Magnetism.*

section of the rotational shell are properly shaped, but certainly unstable if the shell be too thin. In this case also the energy is maximum-minimum for given vorticity and given impulse.

6. In these examples of steady motion, the "resultant impulse" (V. M.* § 8) is a simple impulsive force, without couple; the corresponding rigid body of example 3 is a circular toroid, and its motion is purely translational and parallel to the axis of the toroid.

5. We have also exceedingly interesting cases of steady motion in which the impulse is such that, if applied to a rigid body, it would be reducible, according to Poinsot's method, to an impulsive force in a determinate line, *and a couple with this line for axis.* To this category belong certain distributions of vorticity giving longitudinal vibrations, with thickenings and thinnings of the core travelling as waves in one direction or the other round a vortex ring, which will be investigated in a future communication to the Royal Society. In all such cases, the corresponding rigid body of § 2 example (2) has both rotational and translational motion.

7. To find illustrations, suppose, first, the vorticity (defined below, § 24) and the force resultant of the impulse to be (according to the conditions explained below, § 29) such that the cross section is small in comparison with the aperture. Take a ring of flexible wire (a piece of very stout lead wire with its ends soldered together answers well), bend it into an oval form, and then give it a right-handed twist round the long axis of the oval, so that the curve comes to be not in one plane (fig. 1). A properly-shaped twisted ellipse of this kind [a shape perfectly determinate when the vorticity, the force resultant of the impulse, and the rotational moment of the impulse (V. M. § 6), are all given] is the figure of the core in what we may call the first† steady mode of single and simple toroidal

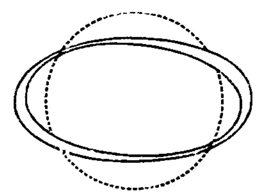

Fig. 1.

* My first series of papers on vortex motion in the "Transactions of the Royal Society of Edinburgh," will be thus referred to henceforth.

† First or gravest, and second, and third, and higher modes of steady motion to be regarded as analogous to the first, second, third, and higher fundamental modes of an elastic vibrator, or of a stretched cord, or of steady undulatory motion in an endless uniform canal, or in an endless chain of mutually repulsive links.

vortex motion with rotational moment. To illustrate the second steady mode, commence with a circular ring of flexible wire, and pull it out at three points, 120° from one another, so as to make it into as it were an equilateral triangle with rounded corners. Give now a right-handed twist, round the radius to each corner, to the plane of the curve at and near the corner; and, keeping the character of the twist thus given to the wire, bend it into a certain determinate shape proper for the data of the vortex motion. This is the shape of the vortex core in the second steady mode of single and simple toroidal vortex motion with rotational moment. The third is to be similarly arrived at, by twisting the corners of a square having rounded corners; the fourth, by twisting the corners of a regular pentagon having rounded corners; the fifth, by twisting the corners of a hexagon, and so on.

In each of the annexed diagrams of toroidal helixes a circle is introduced to guide the judgment as to the relief above and depression below the plane of the diagram which the curve represented in each case must be imagined to have. The circle may be imagined in each case to be the circular axis of a toroidal core on which the helix may be supposed to be wound.

To avoid circumlocution, I have said, "give a right-handed twist" in each case. The result in each case, as in fig. 1, illustrates a vortex motion for which the corresponding rigid body describes left-handed helixes, by all its particles, round the central axis of the motion. If now, instead of right-handed twists to the plane of the oval, or the corners of the triangle, square, pentagon, &c., we give left-handed twists, as in figs. 2, 3, 4, the result in each case will be a vortex motion for which the corresponding rigid body describes right-handed helixes. It depends, of course, on the relation between the directions of the force resultant and couple resultant of the impulse, with no ambiguity in any case, whether the twists in the forms, and in the lines of motion of the corresponding rigid body, will be right-handed or left-handed.

8. In each of these modes of motion the energy is a maximum-minimum for given force resultant and given couple resultant of impulse. The modes successively described above are successive solutions of the maximum-minimum problem of § 4; a determinate problem with the multiple solutions indicated above, but no other

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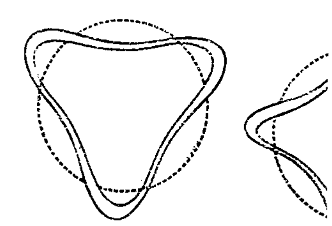

Fig. 2.

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solution, when the vorticity is given in a single simple ring of the liquid.

9. The problem of steady motion, for the case of a vortex line with infinitely thin core, bears a close analogy to the following purely geometrical problem :—

Find the curve whose length shall be a minimum with given

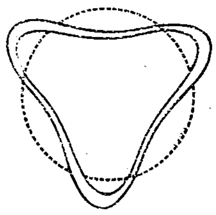

Fig. 2.

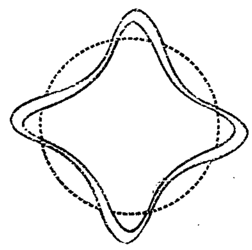

Fig. 3.

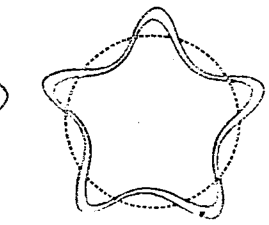

Fig. 4.

resultant projectional area, and given resultant areal moment (§ 27 below). This would be identical with the vortex problem if the energy of an infinitely thin vortex ring of given volume and given cyclic constant were a function simply of its apertural circumference. The geometrical problem clearly has multiple solutions answering precisely to the solutions of the vortex problem.

10. The very high modes of solution are clearly very nearly identical for the two problems (infinitely high modes identical), and are found thus :—

Take the solution derived in the manner explained above, from a regular polygon of N sides, when N is a very great number. It is obvious that either problem must lead to a form of curve like that of a long regular spiral spring of the ordinary kind bent round till its two ends meet, and then having its ends properly cut and joined so as to give a continuous endless helix with axis a circle (instead of the ordinary straight line-axis), and N turns of the spiral round its circular axis. This curve I call a toroidal helix, because it lies on a toroid * just as the common regular helix lies

* I call a circular toroid a simple ring generated by the revolution of any singly-circumferential closed plane curve round any axis in its plane not cutting it. A "tore," following French usage, is a ring generated by the revolution of a circle round any line in its plane not cutting it. Any simple

on a circular cylinder. Let a be the radius of the circle thus formed by the axis of the closed helix; let r denote the radius of the cross section of the ideal toroid on the surface of which the helix lies, supposed small in comparison with a; and let θ denote the inclination of the helix to the normal section of the toroid. We have

$$\tan\,\theta = \frac{2\pi a}{\mathrm{N}.2\pi r} = \frac{a}{\mathrm{N}r};$$

because $\frac{2\pi a}{\mathrm{N}}$ is as it were the step of the screw, and $2\pi r$ is the circumference of the cylindrical core on which any short part of it may be approximately supposed to be wound.

Let κ be the cyclic constant, I the given force resultant of the impulse, and μ the given rotational moment. We have (§ 28) approximately

$$\mathrm{I} = \kappa\pi a^2,\ \mu = \kappa \mathrm{N}\pi r^2 a\,.$$

Hence

$$a = \sqrt{\frac{\mathrm{I}}{\kappa\pi}},\ r = \sqrt{\frac{\mu}{\mathrm{N}\kappa^{\frac{1}{2}}\pi^{\frac{1}{2}}\mathrm{I}^{\frac{1}{2}}}},$$

$$\tan\,\theta = \sqrt{\frac{\mathrm{I}^{\frac{3}{2}}}{\mathrm{N}\mu\kappa^{\frac{1}{2}}\pi^{\frac{1}{2}}}}\,.$$

11. Suppose, now, instead of a single thread wound spirally round a toroidal core, we have two separate threads forming as it were a "two-threaded screw," and let each thread make a whole

ring, or any solid with a single hole through it, may be called a toroid; but to deserve this appellation it had better be not very unlike a tore.

The endless closed axis of a toroid is a line through its substance passing somewhat approximately through the centres of gravity of all its cross sections. An apertural circumference of a toroid is any closed line in its surface once round its aperture. An apertural section of a toroid is any section by a plane or curved surface which would cut the toroid into two separate toroids. It must cut the surface of the toroid in just two simple closed curves, one of them completely surrounding the other on the sectional surface: of course, it is the space between these curves which is the actual section of the toroidal substance, and the area of the inner one of the two is a section of the aperture.

A section by any surface cutting every apertural circumference, each once and only once, is called a cross section of the toroid. It consists essentially of a simple closed curve.

number of turns round the toroidal core. The two threads, each endless, will be two helically tortuous rings linked together, and will constitute the core of what will now be a double vortex ring. The formulæ just now obtained for a single thread would be applicable to each thread, if κ denoted the cyclic constant for the circuit round the two threads, or twice the cyclic constant for either, and N the number of turns of either alone round the toroidal core. But it is more convenient to take N for the number of turns of both threads (so that the number of turns of one thread alone is $\frac{1}{2}$N), and κ the cyclic constant for either thread alone, and thus for very high steady modes of the double vortex ring

$$I = 2\kappa\pi a^2, \ \mu = \kappa N\pi r^2 a,$$

$$\tan\theta = \sqrt{\frac{(\frac{1}{2}I)^{\frac{3}{2}}}{N\mu\kappa^{\frac{1}{2}}\pi^{\frac{1}{2}}}}.$$

Lower and lower steady modes will correspond to smaller and smaller values of N, but in this case, as in the case of the single vortex core, the form will be a curve of some ultratranscendent character, except for very great values of N, or for values of θ infinitely nearly equal to a right angle (this latter limitation leading to the case of infinitely small transverse vibrations).

12. The gravest steady mode of the double vortex ring corresponds to N = 2. This with the single vortex core gives the case of the twisted ellipse (§ 7). With the double core it gives a system which is most easily understood by taking two plane circular rings of stiff metal linked together. First, place them as nearly coincident as their being linked together permits (fig. 5). Then separate them a little, and incline their planes a little, as shown in the diagram. Then bend each into an unknown shape determined by the strict solution of the transcendental problem of analysis to which the hydro-kinetic investigation leads for this case.

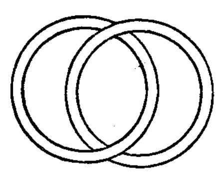

Fig. 5.

13. Go back now to the supposition of § 11, and alter it to this:—

Let each thread make one turn and a half, or any odd number of half turns, round the toroidal core: thus each thread will have an end coincident with an end of the other. Let these coincident ends be united. Thus there will be but one endless thread making an odd number N of turns round the toroidal core. The cases of N = 3 and N = 9 are represented in the annexed diagrams (fig. 9).*

Imagine now a three-threaded toroidal helix, and let N denote the whole number of turns round the toroidal core, we have

$$I = 3\kappa\pi a^2, \ \mu = \kappa N \pi r^2 a,$$

$$\tan\theta = \sqrt{\frac{(\frac{1}{3}I)^{\frac{3}{2}}}{N\mu\kappa^{\frac{1}{2}}\pi^{\frac{1}{2}}}}.$$

Suppose now N to be divisible by 3: then the three threads form three separate endless rings linked together. The case of N = 3 is illustrated by the annexed diagram (fig. 6), which is repeated from the diagram of V. M. § 58. If N be not divisible by 3, the three threads run together into one, as illustrated for the case of N = 14 in the annexed diagram (fig. 7).

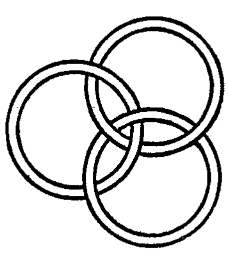

Fig. 6.

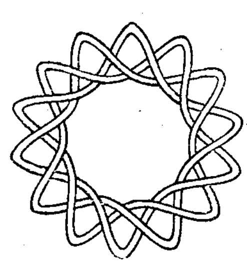

Fig. 7.

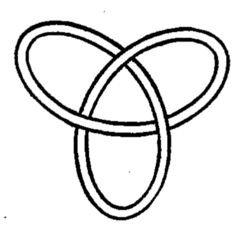

Fig. 8. "Trefoil Knot."

14. The irrotational motion of the liquid round the rotational cores in all these cases is such that the fluid velocity at any point is equal to, and in the same direction as, the resultant magnetic force at the corresponding point in the neighbourhood of a closed gal-

* The first of these was given in § 58 of my paper on vortex motion. It has since become known far and wide by being seen on the back of the "Unseen Universe."

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15. To understand the motion of the liquid in the rotational core itself, take a piece of Indian-rubber gas-pipe stiffened internally with wire in the usual manner, and with it construct any of the forms with which we have been occupied, for instance the symmetrical trefoil knot (fig. 8, § 13), uniting the two ends of the tube carefully by tying them firmly by an inch or two of straight cylindrical plug, then turn the tube round and round, round its sinuous axis. The rotational motion of the fluid vortex core is thus represented. But it must be remembered, that the outer form of the core has a motion perpendicular to the plane of the diagram, and a rotation round an axis through the centre of the diagram, and perpendicular to the plane in each of the cases represented by the preceding diagrams. The whole motion of the fluid, rotational and irrotational, is so related in its different parts to one another, and to the translational and rotational motion of the shape of the core, as to be everywhere slipless.

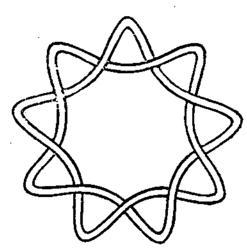

Fig. 9. "Nine-leaved Knot."

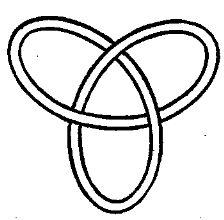

Fig. 8. "Trefoil Knot."

16. Look to the preceding diagrams, and, thinking of what they represent, it is easy to see that there must be a determinate particular shape for each of them which will give steady motion, and I think we may confidently judge that the motion is stable in each, provided only the core is sufficiently thin. It is more easy to judge of the cases in which there are multiple sinuosities by a synthetic view of them (§ 3) than by consideration of the maximum-minimum problem of § 8.

17. It seems probable that the two- or three- or multiple-threaded toroidal helix motions cannot be stable, or even steady, unless I, μ, and N are such as to make the shortest distances between different positions of the core or cores considerable in

comparison with the core's diameter. Consider, for example, the simplest case (§ 12, fig. 5) of two simple rings linked together.

18 Go back now to the simple circular Helmholtz ring. It is clear that there must be a shape of absolute maximum energy for given vorticity and given impulse, if we introduce the restriction that the figure is to be a figure of revolution, that is to say, symmetrical round a straight axis. If the given vorticity be given in this determinate shape the motion will be steady, and there is no other figure of revolution for which it would be steady (it being understood that the impulse has a single force resultant without couple). If the given impulse, divided by the cyclic constant, be very great in comparison with the two-thirds power of the volume of liquid in which the vorticity is given, the figure of steadiness is an exceedingly thin circular ring of large aperture and of approximately circular cross section. This is the case to which chiefly attention is directed by Helmholtz. If, on the other hand, the impulse divided by the cyclic constant be very small compared with the two-thirds power of the volume, the figure becomes like a long oval, bored through along its axis of revolution and with the ends of the bore rounded off (or trumpeted) symmetrically, so as to give a figure something like the handle of a child's skipping-rope, but symmetrical on the two sides of the plane through its middle perpendicular to its length. It is certain that, however small the impulse, with given vorticity the figure of steadiness thus indicated is possible, however long in the direction of the axis and small in diameter perpendicular to the axis and in aperture it may be. I cannot, however, say at present that it is certain that this possible steady motion is stable, for there are figures not of revolution, deviating infinitely little from it, in which, with the same vorticity, there is the same impulse and the same energy, and consideration of the general character of the motion is not reassuring on the point of stability when rigorous demonstration is wanting.

19. Hitherto I have not indeed succeeded in rigorously demonstrating the stability of the Helmholtz ring in any case. With given vorticity, imagine the ring to be thicker in one place than in another. Imagine the given vorticity, instead of being distributed in a symmetrical circular ring, to be distributed in a ring still,

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with a circular axis, but thinner in one part than in the rest. It is clear that with the same vorticity, and the same impulse, the energy with such a distribution is greater that when the ring is symmetrical. But, now let the figure of the cross section of the ring, instead of being approximately circular, be made considerably oval. This will diminish the energy with the same vorticity and the same impulse. Thus, from the figure of steadiness we may pass continuously to others with same vorticity, same impulse, and same energy. Thus, we see that the figure of steadiness is, as stated above, a figure of maximum-minimum, and not of absolute maximum, nor of absolute minimum energy. Hence, from the maximum-minimum problem we cannot derive proof of stability.

20. The known phenomena of steam rings and smoke rings show us enough of, as it were, the natural history of the subject to convince us beforehand that the steady configuration, with ordinary proportions of diameters of core to diameter of aperture, is stable, and considerations connected with what is rigorously demonstrable in repect to stability of vortex columns (to be given in a later communication to the Royal Society) may lead to a rigorous demonstration of stability for a simple Helmholtz ring if of thin enough core in proportion to diameter of aperture. But at present neither natural history nor mathematics gives us perfect assurance of stability when the cross section is considerable in proportion to the area of aperture.

21. I conclude with a brief statement of general propositions, definitions, and principles used in the preceding abstract, of which some appeared in my series of papers on vortex motion communicated to the Royal Society of Edinburgh in 1867–68 and 69, and published in the Transactions for 1869. The rest will form part of the subject of a continuation of that paper, which I hope to communicate to the Royal Society before the end of the present session.

Any portion of a liquid having vortex motion is called *vortex core*, or, for brevity, simply "core." Any finite portion of liquid which is all vortex core, and has contiguous with it over its whole boundary irrotationally moving liquid, is called *a vortex*. A vortex thus defined is essentially a ring of matter. That it must be so was first discovered and published by Helmholtz. Sometimes the word *vortex* is extended to include irrotationally moving liquid circulating round or moving in the neighbourhood of vortex core; but as different portions of liquid may successively come into the neighbourhood of the core, and pass away again, while the core always remains essentially of the same substance, it is more proper to limit the substantive term *a vortex* as in the definition I have given.

22. *Definition I.*—The circulation of a vortex is the circulation [V.M. § 60 (*a*)] in any endless circuit once round its core. Whatever varied configurations a vortex may take, whether on account of its own unsteadiness (§ 1 above), or on account of disturbances by other vortices, or by solids immersed in the liquid, or by the solid boundary of the liquid (if the liquid is not infinite), its "circulation" remains unchanged [V. M. § 59, Prop. (1)]. The circulation of a vortex is sometimes called its *cyclic constant.*

Definition II.—An axial line through a fluid moving rotationally, is a line (straight or curved) whose direction at every point coincides with the axis of molecular rotation through that point [V. M. § 59 (2)].

Every axial line in a vortex is essentially a closed curve, being of course wholly without a vortex.

23. *Definition III.*—A closed section of a vortex is any section of its core cutting normally the axial line through every point of it. Divide any closed section of a vortex into smaller areas; the axial lines through the borders of these areas form what are called vortex tubes. I shall call (after Helmholtz) a vortex filament any portion of a vortex bounded by a vortex tube (not necessarily infinitesimal). Of course, a complete vortex may be called therefore a vortex filament; but it is generally convenient to apply this term only to a part of a vortex as just now defined. The boundary of a complete vortex satisfies the definition of a vortex tube.

A complete vortex tube is essentially endless. In a vortex filament infinitely small in all diameters of cross sections "rotation" varies [V. M. § 60 (*e*)] from point to point of the length of the filament, and from time to time inversely as the area of the cross section. The product of the area of the cross section into the rotation is equal to the circulation or cyclic constant of the filament.

24. Vorticity will be used to designate in a general way the distribution of molecular rotation in the matter of a vortex. Thus, if we imagine a vortex divided into a number of infinitely thin vortex filaments, the vorticity will be completely given when the volume of each filament and its circulation, or cyclic constant, are given; but the shapes and positions of the filaments must also be given in order that, not only the vorticity, but its distribution, can be regarded as given.

25. The vortex density at any point of a vortex is the circulation of an infinitesimal filament through this point divided by the volume of the complete filament. The vortex density remains always unchanged for the same portion of fluid. By definition it is the same all along any one vortex filament.

26. Divide a vortex into infinitesimal filaments inversely as their densities so that their circulations are equal; and let the circulation of each be $\frac{1}{n}$ of unity. Take the projection of all the filaments on one plane. $\frac{1}{n}$ of the sum of the areas of these projections is (V. M. §§ 6, 62) equal to the component impulse of the vortex perpendicular to that plane. Take the projections of the filaments on three planes at right angles to one another, and find the centre of gravity of the areas of these three sets of projections. Find, according to Poinsot's method, the resultant axis, force, and couple of the three forces equal respectively to $\frac{1}{n}$ of the sums of the areas, and acting in lines through the three centres of gravity perpendicular to the three planes. This will be the resultant axis; the force resultant of the impulse, and the couple resultant of the vortex.

The last of these, that is to say, the couple is also called the rotational moment of the vortex (V. M. § 6).

27. *Definition IV.*—The moment of a plane area round any axis is the product of the area multiplied into the distance from that axis of the perpendicular to its plane through its centre of gravity.

Definition V.—The area of the projection of a closed curve on the plane for which the area of projection is a maximum will be called the area of the curve, or simply the area of the curve. The area of the projection on any plane perpendicular to the plane of the resultant area is of course zero.

Definition VI.—The resultant axis of a closed curve is a line through the centre of gravity, and perpendicular to the plane of its resultant area. The resultant areal moment of a closed curve is the moment round the resultant axis of the areas of its projections on two planes at right angles to one another, and parallel to this axis. It is understood, of course, that the areas of the projections on these two planes are not evanescent generally, except for the case of a plane curve, and that their zero values are generally the sums of equal positive and negative portions. Thus their moments are not in general zero.

Thus, according to these definitions, the resultant impulse of a vortex filament of infinitely small cross section and of unit circulation is equal to the resultant area of its curve. The resultant axis of a vortex is the same as the resultant axis of the curve, and the rotational moment is equal to the resultant areal moment of the curve.

28. Consider for a moment a vortex filament in an infinite liquid with no disturbing influence of other vortices, or of solids, immersed in the liquid. We now see from the constancy of the impulse (proved generally in V. M. § 19) that the resultant area, and the resultant areal moment of the curve formed by the filament, remain constant, however its curve may become contorted; and its resultant axis remains the same line in space. Hence, whatever motions and contortions the vortex filament may experience, if it has any motion of translation through space this motion must be on the average along the resultant axis.

29. Consider now the actual vortex made up of an infinite number of infinitely small vortex filaments. If these be of volumes inversely proportional to their vortex densities (§ 25), so that their circulations are equal, we now see from the constancy of the impulse that the sum of the resultant areas of all the vortex filaments remains constant; and so does the sum of their rotational moments: and the resultant areal axis of them all regarded as one system is a fixed line in space. Hence, as in the case of a vortex filament, the translation, if any, through space is on the average along its resultant axis. All this, of course, is on the supposition that there is no other vortex, and no solid immersed in the liquid, and no bounding surface of the liquid near enough to produce any sensible influence on the given vortex.

2. Experiments illustrating Rigidity produced by Centrifugal Force. By John Aitken, Esq.

If an endless chain is hung over a pulley and the pulley driven at a great velocity, it is well known that the motion so communicated to the chain has almost no tendency to change the form of the curve in which the chain hangs, and that the principal effect of the motion is to confer on the chain a quasi-rigidity which enables it to resist any force tending to alter its curvature.

This is only true in a general sense, and possibly may be true of some ideal form of chain; but in all chains we can experiment on there are forces in action in the moving chain which tend to cause the chain to depart from the form which it has while at rest.

I shall refer to these disturbing forces later on. As the disturbing forces in most chains are very small, we shall neglect them, and for the present suppose the centrifugal force just balances the tension at all points. The following experiments were made to illustrate the balance of these forces, to show that into whatever curves we may bend the chain when in motion, the centrifugal force has no tendency to alter these curves: that all forms are forms of stability, as far as the centrifugal force is concerned.

The first experiments were to show the effect of destroying the balance between the tension and the centrifugal force. In these experiments the links on the descending side of the loop were