Surgery on Closed Manifolds

by

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Theorem (0.1): \((n > 4)\) Let \((f: N^n \to M^n, \hat{f})\) be a surgery problem where \(M^n\) is a closed, oriented manifold with \(\pi_1 M\) finite. Then,

\[
(f \times \text{Id}_{S^1}, \hat{f} \times \text{Id}_{S^1}) \quad \begin{cases} \text{index } N = \text{index } M, \text{ when } n = 0 \quad (4) \\ \text{for all nontrivial homomorphisms } \\ \mu : \pi \to \mathbb{Z}/2 \end{cases}
\]

is normal cobordant to a homotopy equivalence \(\begin{cases} \text{always } & \text{, when } n = 1 \quad (4) \\ \text{Arf}(f, \hat{f}) = 0 & \text{, when } n = 2 \quad (4) \\ \text{Arf}_\mu(f, \hat{f}) = 0 & \text{, when } n = 3 \quad (4) \end{cases}\)

\[
\text{Arf}_\mu(f, \hat{f}) = \text{Arf}(f_\mu : N^{n-1}_\mu \to M^{n-1}_\mu, \hat{f}_\mu), \text{ where } (f_\mu, \hat{f}_\mu) \text{ is the sub-surgery problem of } (f, \hat{f}) \text{ which is induced via transversality by the map } \begin{array}{c} M^n \to \text{B} \pi_1 M \xrightarrow{B\mu} \text{B} \mathbb{Z}_2 = \text{RP}^\infty. \end{array}
\]

Also, we can show

Theorem (0.2): For any closed manifold \(P^n\) with finite \(\pi_1\) and index = 0,

\((f: M^8 \to S^8, \hat{f}) \times (\text{Id}_P, \text{Id}_P)\) is normally cobordant to a homotopy equivalence - (where \((f, \hat{f}) = \text{Milnor surgery problem with index 8}).

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Section 1:

For any closed, compact, oriented manifold \( M \) with \( \pi_1 M \cong \pi \), we have the Sullivan-Wall structure sequence

\[
\Sigma M, G/TOP \xrightarrow{\partial} L_{n+1}^S (Z\pi) \rightarrow S(M) \rightarrow [M, G/TOP] \xrightarrow{\theta} L_n^S (Z\pi)
\]

There are also defined "intermediate" Wall groups \( L_{\pi}^X (Z\pi) \)
where \( X \subset \mathcal{K}_1 (Z\pi) \) or \( \{ \pi \} \subset X \subset \mathcal{K}_1 (Z\pi) \) is an involution invariant subgroup (see \([R]\)). There is a homomorphism \( L_{\pi}^S (Z\pi) \rightarrow L_{\pi}^X (Z\pi) \)
so we get maps

\[
\sigma^X : [M, G/TOP] \rightarrow L_n^X (Z\pi) \quad \text{and}
\]

\[
\theta^X : [\Sigma M, G/TOP] \rightarrow L_{n+1}^X (Z\pi)
\]

It follows from work of Quinn-Ranicki that there is a homomorphism

\[
A^X : \oplus H_{n-4} (B\pi; Z/2) \oplus H_{n-4} (B\pi; Z/2) \rightarrow L_n^X (Z\pi)
\]

where \((\quad) (2)\) denotes localization at 2, such that the 2 localizations of \( \sigma^X \) and \( \theta^X \) are given by composing \( A^X \) with a certain characteristic class formula that we worked out in \([T-W]\). We wrote out the one for \( \sigma^X \) (formula 1.7): to get the one for \( \theta^X \) use the same formula but replace \([M]\) by the homology suspension of the fundamental class. Indeed, given any compact, oriented manifold with bounding, \( W^n \), we get a formula for the map \([W/\partial W, G/TOP] \rightarrow L_n (Z\pi)\): replace \([M]\) with \([W, \partial W]\) in (1.7)*

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*Care is needed in \([T-W]\). The Wu class referred to there is the Morgan-Sullivan Wu class, \([M-S]\) p. 480-81. It is the inverse of the Wu class defined in Milnor-Stasheff \([Mi-S]\) 11,14. In particular, some of the polynomials on \([M-S]\) p. 481 are incorrect.
Recall that Wall ([W2]) has shown that $L^X_n(\mathbb{Z}/\pi) \to L^X_n(\mathbb{Z}/\pi)(2)$ is 1-1 for $\pi$ finite.

Periodicity implies that $A^X$ factors as

$$
\oplus \left( H_{n-41}^i(\pi, \mathbb{Z}/2) \oplus H_{n-41-2}^i(\pi, \mathbb{Z}/2) \right) \oplus J_{n-41}^{i} \oplus K_{n-41-2}^{i} \\
\oplus L_{n-41}^X(\mathbb{Z}/\pi)(2) \oplus L_{n}^X(\mathbb{Z}/\pi)(2)
$$

$J^*$ and $K^*$ are determined by the surgery obstructions of certain very special surgery problems. To be more specific, let $M^8 \to S^8$ denote the 8-dimensional Milnor surgery problem, and let $K^3 \to L^3$ denote the twisted Kervaire problem, i.e. the generator of $L^3_3(\mathbb{Z}e; \mathbb{Z}/2)$. Define homomorphisms

$$
\alpha^X_n : \Omega_n(B\pi) \to L^X_{n+8}(\mathbb{Z}/\pi)
$$

$$
\beta^X_n : \Omega_n(B\pi; \mathbb{Z}/2) \to L^X_{n+2}(\mathbb{Z}/\pi)
$$

by $\alpha^X_n(P)$ is the surgery obstruction for $M^8 \times P \to S^8 \times P$ and $\beta^X_n(P)$ is the surgery obstruction for the surgery problem induced along the bockstein of $K^3 \otimes P \to L^3 \otimes P$. (See [M-S]).

The map $J^X_n$ is determined by $\alpha^X_n$ and the $J^X_r$ for $r < n$. The map $K^X_n$ is determined by $\beta^X_n$ and the $K^X_r$ and $J^X_r$ for $r < n$.

The precise relation between the $\alpha$'s and the $J$'s is supplied by (1.7) in [T-W]: to wit, if $g : P \to B\pi$,

$$
\alpha^X_n(P) = \sum J_{n-41}^* g_*(\xi_P \cap [P])
$$
where $\mathcal{I}$ is the Morgan-Sullivan $\mathcal{I}$ class [M-S]. With a bit more work, one can show

$$\beta_n^X(P) = \sum \chi_{n-41} g_*(y_P^2 \cap [P])$$

$$+ \sum \chi_{n-41-2} g_*(5(y_P^1 S_q^1 y_P \cap [P]))$$

where $5$ denotes the bockstein $0 \to \mathbb{Z}_2(2) \to \mathbb{Z}_2 \to \mathbb{Z}/2 \to 0$ and $y_P$ denotes the total Wu class of the oriented tangent bundle to $P$.

Theorem (0.1) follows from Ranicki's product formula

$$l_{n+1}^h(\mathbb{Z}(\pi \times \mathbb{Z})) = l_n^p(\mathbb{Z}\pi) \oplus l_n^h(\mathbb{Z}\pi)$$ (see [R]), plus the following result.

**Theorem 1.1.** Assume $\pi$ is finite.

(a) $\theta_0^P$ is 1-1

(b) For $j > 0$, $\theta_j$ is trivial, where

$$\mathcal{C}_1(\pi) = \ker(\mathcal{K}_1(\mathbb{Z}\pi) \to \mathcal{K}_1(\mathbb{Z}\pi) \oplus \mathcal{K}_1(\mathbb{Q}\pi))$$ and $\mathcal{C}_1(\pi) = \{\mathcal{C}_1(\pi), \pm \pi\}$.

(c) $\chi_0^p$ and $\chi_1^p$ are 1-1.

(d) For $j > 1$, $\chi_j^p$ is trivial.

Theorem (0.2) follows from (1.1)(b).

We can improve on 1.1 for some groups.
Theorem 1.2. Let \( \pi \) be a finite group whose 2 Sylow group is abelian. Then

(a) \( J_j^s \) is trivial for \( j > 0 \)

(b) \( \kappa_j^h \) is trivial for \( j > 2 \)

(c) \( \kappa_j^c \) is trivial for \( j > 3 \)

Remarks: (i) The result for \( \kappa_j^h \) is due to Morgan-Pardon, but the \( s \) result seems new.

(ii) See Theorem 4.1 for results on generalized quaternionic and semi-dihedral groups.

(iii) Using results of Quillen [Q] and the naturality of the \( s \), and \( \kappa \), one can prove the same result for the dihedral groups; the symmetric and alternating groups; and many others.

(iv) When we sketch the proof of 1.2 we will also determine \( \kappa_2^h \).
Section 2:

Following Wall ([WI], Theorem 12) it is easy to reduce Theorem 1.1 to the result for finite 2-groups.

Relative Detection Theorem 2.1: If $\pi$ is a finite 2-group, then

(a) $K_i(Z\pi \to \hat{Z}_2\pi) \to \bigoplus_{\text{special subquotients}} K_i(ZG \to \hat{Z}_2G)$ is 1-1 for all $i$.

(b) $L_i^{\epsilon}(Z\pi \to \hat{Z}_2\pi) \to \bigoplus_{\text{special subquotients}} L_i^{\epsilon}(ZG \to \hat{Z}_2G)$ is 1-1 for all $i$ and $\epsilon = 0$ or 1.

$$(Cl^{\epsilon}(\pi) = \ker \tilde{K}^{\epsilon}_i(Z\pi) \to \tilde{K}^{\epsilon}_i(\hat{Z}\pi) \oplus \tilde{K}^{\epsilon}_i(\hat{Q}\pi))$$

Remarks:

1. A subquotient of $\pi$ is a quotient group $G = H/N$ where $H$ is a subgroup of $\pi$.

2. A 2-group $G$ is special if all normal abelian subgroups of $G$ are cyclic. A special group is either cyclic, generalized quaternionic, dihedral, or semi-dihedral.

3. The maps in (2.1) are compositions of restriction maps associated to subgroups $H \subseteq \pi$ and projection maps associated to quotients $H \to H/N = G$. 
4. \( \mathcal{L}_1(\mathbb{Z} \pi) \rightarrow 0 \quad (\mathbb{Z} \pi \rightarrow \mathbb{Z}_2 \pi) = L_1(\mathbb{Z} \pi \rightarrow \mathbb{Z}_2 \pi) \)

\[ \mathcal{L}_1(\mathbb{Z} \pi \rightarrow \mathbb{Z}_2 \pi) = L_1(\mathbb{Z} \pi \rightarrow \mathbb{Z}_2 \pi) \]

where \( \mathcal{L}_1 \) = the L-groups defined by Wall in [W2]. \( L_1 \neq \mathcal{L}_1 \) in general (see [W2] Section 5.4).

5. \( \mathcal{L}_0(\mathbb{Z} \pi) \rightarrow 0 \quad (\mathbb{Z} \pi \rightarrow \mathbb{Z}_2 \pi) = L_1(\mathbb{Z} \pi \rightarrow \mathbb{Z}_2 \pi) \).

6. Theorem 2.1 was motivated by the calculations of Wall [W2], Section (5.2), Carlsson-Milgram [C-M], Pardon [P], Bak-Kolster [Kl], [B-K], [K2], and especially Milgram-Hambleton [M-H].

Theorem 1.1 (b) is reduced to the result for special 2-groups as follows:

A is induced by a map of spectra \( A \) which fits into a commutative diagram

\[
\begin{align*}
\mathbb{L}(\mathbb{Z}) \wedge B\pi^+ & \xrightarrow{A} \mathbb{L}(\mathbb{Z} \pi) \\
\mathbb{L}(\mathbb{Z}_2) \wedge B\pi^+ & \xrightarrow{A_2} \mathbb{L}(\mathbb{Z}_2 \pi)
\end{align*}
\]

If we localize at (2), then \( \mathbb{L}(\mathbb{Z}) \rightarrow \mathbb{L}(\mathbb{Z}_2) \) is equivalent to

\[ \pi K(\mathbb{Z}(2); 4i+1) \times \pi K(\mathbb{Z}/2; 4i+2) \xrightarrow{\text{project}} \pi K(\mathbb{Z}/2; 4i+2) \xrightarrow{\text{include}} \pi K(\mathbb{Z}/2; 4i+2) \]

\[ \pi K(\mathbb{Z}/2; 4i+2) \]
This implies that \( \tilde{\mathcal{S}}_j^X \) lifts to a map

\[
\tilde{\mathcal{S}}_j^X : H_j(\pi, \mathbb{Z}_2) \rightarrow L_{j+1}^X(\mathbb{Z} \pi \rightarrow \mathbb{Z}_2 \pi)_j(2) \quad \text{for all } x.
\]

Apply Theorem 2.1 with \( \epsilon = 1 \) and \( x = \text{Cl} \rightarrow 0 \).

Theorem 1.1(d) is reduced to the result for 2-groups by the following theorem.

**Absolute Detection Theorem 2.2**: If \( \pi \) is a finite 2-group, then

\[
L^P_1(\mathbb{Z} \pi) \rightarrow \bigoplus_{\text{special subquotients}} L^P_1(\mathbb{Z} G) \text{ is 1-1.}
\]

The proof of (2.2) relies on Wall's reduction theorem which implies that \( L^P_1(\mathbb{Z}_2 \pi) = L^P_1(\mathbb{ZZ}/2) \).
Section 3: Proof of the Relative Detection Theorem

\[ \pi = \text{finite 2-group} \]
\[ \mathcal{Q}\pi = \bigtimes_{\rho} A_{\rho}, \text{where } \rho \text{ varies over the } \mathbb{Q}\text{-irreducible} \]
\[ \text{representations of } \pi \text{ and } A_{\rho} = \text{simple} \]
\[ \mathbb{Q}\text{-algebra.} \]

Let \( n_{\rho} = \text{image } (\mathbb{Z}\pi \to \mathcal{Q}\pi \to A_{\rho}) \), \( n = \bigtimes_{\rho} n_{\rho} \). \( n_{\rho} \) is a \( \mathbb{Z} \)-order of \( A_{\rho} \).

**Proposition 3.1**

\[ K_1(\mathbb{Z}\pi \to \hat{\mathbb{Z}}_2 \pi) \cong K_1(n \to \hat{n}_2) \cong \bigoplus_{\rho} K_1(n_{\rho} \to \hat{n}_{\rho}(2)) \]

**Proof:** Consider the following commutative diagram with exact rows:

\[ \cdots \to K_1(\mathbb{Z}\pi \to \hat{\mathbb{Z}} \pi) \to K_1(\mathbb{Z}\pi \to \hat{\mathbb{Z}}_2 \pi) \to K_1(\hat{\mathbb{Z}} \pi \to \hat{\mathbb{Z}}_2 \pi) \to \cdots \]

\[ \downarrow f_1 \quad \downarrow g_1 \quad \downarrow h_1 \]

\[ \cdots \to K_1(n \to \hat{n}) \to K_1(n \to \hat{n}_2) \to K_1(\hat{n} \to \hat{n}_2) \to \cdots \]

\[ \downarrow k_1 \downarrow l \]

\[ K_1(\mathcal{Q}\pi \to \hat{\mathcal{Q}} \pi) \quad K_1(n_{\text{odd}} \to \hat{n}_{\text{odd}} \pi) \]

The Meyer-Vietoris sequences associated to the arithmetic squares

\[
\begin{array}{ccc}
\mathbb{Z}\pi & \to & \mathcal{Q}\pi \\
\downarrow & & \downarrow \\
\hat{\mathbb{Z}} \pi & \to & \hat{\mathcal{Q}} \pi \\
\end{array}
\begin{array}{ccc}
n & \to & \mathcal{Q}\pi \\
\downarrow & & \downarrow \\
\hat{n} & \to & \hat{\mathcal{Q}} \pi \\
\end{array}
\]

imply that \( f_1, k_1 \circ f_1 \) and \( k_1 \) are isomorphisms. Since \( \pi \) is a
2-group, \( \hat{\mathbb{Z}}_{\text{odd}} \pi \) is a maximal \( \hat{\mathbb{Z}}_{\text{odd}} \) order (see [Re]),
\[ \hat{\mathbb{Z}}_{\text{odd}} \pi = \hat{\mathbb{Z}}_{\text{odd}}(\text{odd}) \], and \( h_1 \) is an isomorphism. Apply the 5-lemma.

Let
\[
K_1^f(\mathbb{Z}\pi \to \hat{\mathbb{Z}}_2 \pi) = \bigoplus \rho \ K_1(\eta_\rho \to \hat{n}_\rho(2)) \\
\text{faithful}
\]
\[
K_1^u(\mathbb{Z}\pi \to \hat{\mathbb{Z}}_2 \pi) = \bigoplus \rho \ K_1(\eta_\rho \to \hat{n}_\rho(2)) \\
\text{unfaithful}
\]

Proposition 3.2 :

(a) \( K_1^f(\mathbb{Z}\pi \to \hat{\mathbb{Z}}_2 \pi) \hookrightarrow K_1(\mathbb{Z}\pi \to \hat{\mathbb{Z}}_2 \pi) \hookrightarrow K_1(\pi/N \to \hat{\mathbb{Z}}_2 \pi/N) \) is a trivial map, for any proper normal subgroup \( N \).

(b) \( K_1^u(\mathbb{Z}\pi \to \mathbb{Z}_2 \pi) \hookrightarrow K_1(\mathbb{Z}\pi \to \hat{\mathbb{Z}}_2 \pi) \to \bigoplus \frac{K_1(\mathbb{Z}\pi/N \to \hat{\mathbb{Z}}_2 \pi/N)}{N \not\triangleleft \pi} \) is 1-1.

Proposition 3.3 : Assume \( \pi \) is a 2-group which is not special.

Then \( \pi \) contains an index 2 subgroup \( \pi_o \) such that

(a) For any \( \Phi \)-irreducible faithful representative \( \rho \) of \( \pi \),
\[ \rho|_{\pi_o} = \rho_1 + \rho_2 \] where \( \rho_1 \) and \( \rho_2 \) are nonisomorphic \( \Phi \)-irreducible representations.

(b) \( \rho_1^\pi = \rho_2^\pi = \rho \), and

(c) \( K_1^f(\mathbb{Z}\pi \to \hat{\mathbb{Z}}_2 \pi) \hookrightarrow K_1(\mathbb{Z}\pi \to \hat{\mathbb{Z}}_2 \pi) \to K_1(\mathbb{Z}\pi_o \to \hat{\mathbb{Z}}_2 \pi_o) \)
is 1-1.

Theorem 2.1 (a) then follows from (3.2) and (3.3) by induction on the order of \( \pi \). The proof of 2.1 (b) is similar.
In particular,
\[
\text{Cl}_0(\pi) \to 0^{\text{L}_1} \bigl( \mathbb{Z}_\pi \to \mathbb{Z}_2 \bigr) = \bigoplus_{\rho} \text{L}_1 \tilde{K}_0(\pi) \to \tilde{I}_1(\pi) \to \tilde{K}_0(\hat{\pi}_\rho(2)).
\]
where \( I_\rho = \text{image} (\tilde{K}_0(\pi) \to \tilde{K}_0(\hat{\pi}_\rho(2))). \)

and
\[
\text{Cl}_1(\pi) \to 0^{\text{L}_1} \bigl( \mathbb{Z}_\pi \to \mathbb{Z}_2 \bigr) = \bigoplus_{\rho} \text{L}_1 \tilde{S}_1 \to \tilde{S}_1(\pi) \to \tilde{S}_1(\hat{\pi}_\rho(2)).
\]
Section 4: Special 2-Groups

Theorem 4.1

(a) If $\pi$ is a special 2-group and $j > 0$, then
\[ \overline{J^I}_j \text{ and } \overline{J}^s_j \text{ are trivial.} \]

(b) If $\pi$ is cyclic or dihedral, then $\overline{X}^s_j = 0$ for $j > 1$.

If $\pi$ is quaternionic, then $\overline{X}^s_j = 0$ for $j \neq 0, 1, 3$ and $\overline{X}^p_3 = 0$.

If $\pi$ is semi-dihedral, then $\overline{X}^p_j = 0$ for $j > 1$.

Remark: Cappell and Shaneson [C-S] have shown that $\overline{X}^h_3 \neq 0$ when $\pi$ = quaternion group.

The following result of Oliver [O] is used to improve $\overline{C}^I_1$-results to $s$-results.

Theorem 4.2: If $\pi$ is a special 2-group, then $\overline{C}^I_1(\pi)$ is trivial.

For the proof of 4.1 (b) we need to analyze what happens to $X_j$ under products.

For any pair of groups $\pi_1$ and $\pi_2$, we get a pairing of spectra

\[ \mu: L_0(Z\pi_1) \wedge L_0(Z\pi_2) \rightarrow L_0(Z(\pi_1 \times \pi_2)) \text{ (see [R])} \]

such that the following diagram commutes.
If we introduce coefficients by doing surgery on \( \mathbb{Z}/2 \)-manifolds, then we get an analogous diagram.

By using the techniques of [T-W], one can analyse

\[
\mu : \mathbb{L}_0(\mathbb{Z}; \mathbb{Z}/2) \wedge \mathbb{L}^0(\mathbb{Z}; \mathbb{Z}/2) \to \mathbb{L}_0(\mathbb{Z}; \mathbb{Z}/2)
\]

localized at 2. This yields the following commutative diagram.

\[
\begin{array}{ccc}
\mathbb{L}_0(\mathbb{Z}/2; \mathbb{Z}/2) \times \mathbb{H}^j(\mathbb{Z}/2; \mathbb{Z}/2) & \to & \mathbb{L}^0(\mathbb{Z}/2; \mathbb{Z}/2) \\
\overline{X}_1 \times \mathcal{J}^j(\mathbb{Z}/2) & \to & \overline{X}_{i+j} \\
\mathbb{L}_{i+2}(\mathbb{Z}/2; \mathbb{Z}/2) \times \mathbb{L}^j(\mathbb{Z}/2; \mathbb{Z}/2) & \to & \mathbb{L}_{i+j+2}(\mathbb{Z}/2; \mathbb{Z}/2)
\end{array}
\]

where \( \mathcal{J}^j(\mathbb{Z}/2) \) is induced by

\[
\mathbb{K}(\mathbb{Z}/2; 0) \wedge \mathbb{B}\pi_2^+ \to \mathbb{L}^0(\mathbb{Z}; \mathbb{Z}/2) \wedge \mathbb{B}\pi_2^+ \xrightarrow{\mathcal{A}^*} \mathbb{L}^0(\mathbb{Z}; \mathbb{Z}/2)
\]

where \( \overline{X}_1 = X_1 \) reduced mod 2.

**Action by the Center**

Suppose \( C \) is the center of a group \( \pi \), then multiplication

\( \alpha : C \times \pi \to \pi \) is a homomorphism which induces a map \( B\alpha : B(C \times \pi) \to B\pi \)

plus a commutative diagram
If we combine 4.4 and 4.5, then we get the following commutative diagram

\[
\begin{array}{cccccc}
\mathbb{H}_1(C;\mathbb{Z}/2) \times \mathbb{H}_j(\pi;\mathbb{Z}/2) & \xrightarrow{\alpha_*} & \mathbb{H}_{i+j}(\pi;\mathbb{Z}/2) \\
\downarrow & & \downarrow \\
\mathbb{L}_{i+2}(\mathbb{Z}, C;\mathbb{Z}/2) \times \mathbb{L}_j(\mathbb{Z}, \pi;\mathbb{Z}/2) & \xrightarrow{\alpha_*} & \mathbb{L}_{i+j+2}(\mathbb{Z}, \pi;\mathbb{Z}/2)
\end{array}
\]

The proof of 4.1 also involves the following result.

**Theorem 4.7:** If \( \pi \) is a special 2-group, then there is an exact sequence

\[
\begin{array}{cccccc}
\mathcal{K}_0 \cap I_{\phi}(\hat{n}_{\phi}(2)) & \to & L_1^P(\mathbb{Z}, \pi) & \to & \oplus \ L_1^P(\mathbb{Z}, G) \\
\mathcal{L}_1^1(\hat{n}_{\phi}(2)) & \to & L_1^1(\mathbb{Z}, \pi) & \to & \oplus \ L_1^1(\mathbb{Z}, G)
\end{array}
\]

where \( \phi \) is the unique faithful, \( \mathbb{Q} \)-irreducible representation of \( \pi \), and \( I_{\phi} = \text{Image} : (\hat{\mathcal{K}}_0(\hat{n}_{\phi}) \to \hat{\mathcal{K}}_0(\hat{n}_{\phi}(2)). \) Also, there is an exact sequence,

\[
\begin{array}{cccccc}
\mathcal{C}^1(\hat{n}_{\phi}(2)) & \to & \mathcal{C}^1(\mathbb{Z}, \pi) & \to & \oplus \mathcal{C}^1(\mathbb{Z}, G) \\
\mathcal{L}_1(\hat{n}_{\phi}(2)) & \to & L_1(\mathbb{Z}, \pi) & \to & \oplus L_1(\mathbb{Z}, G)
\end{array}
\]

where \( \mathcal{C}^1 \) represents the group of homomorphisms from \( \mathbb{Z} \) to \( \mathbb{Z} \) with special subquotients.
Proof of (4.1) when \( \pi = \mathbb{Z}/2 \)

Facts

1. \( \text{tor} \mathcal{C}l_1^0 (\mathbb{Z} \mathbb{Z}/2 \to \mathbb{Z}_2 \mathbb{Z}/2) = 0 \) unless \( j = 1(4) \).

2. \( H_k(\mathbb{Z}/2, \mathbb{Z}_2(2)) = 0 \) for \( k \neq 0 \).

3. \( \mathcal{L}_j^S (\mathbb{Z} \mathbb{Z}/2) \xrightarrow{P \oplus i} \mathcal{L}_j^S (\mathbb{Z}e) \oplus \mathcal{L}_j^S (\mathbb{Z}e) \) is 1-1 for \( j \neq 3(4) \). (\( P : \mathbb{Z}/2 \to e, i : e \to \mathbb{Z}/2 \)).

4. The Pontryagin product \( \alpha_* : H_2i(\mathbb{Z}/2 ; \mathbb{Z}/2) \times H_1(\mathbb{Z}/2 ; \mathbb{Z}/2) \to H_{2i+2} (\mathbb{Z}/2 ; \mathbb{Z}/2) \) is onto. \( 2 \mathcal{L}_j^S (\mathbb{Z} \mathbb{Z}/2) = 0 \).

Facts 1 and 2 imply that \( \mathcal{L}_j^1 = 0 \) for \( j > 0 \). Fact 3 plus naturality of \( \xi_j^S \), imply \( \xi_j^S = 0 \) for \( j > 0 \) and \( j \neq 1(4) \). Fact 4 plus commutativity of (4.6) imply \( \xi_j^S = 0 \) for \( j = 1(4) \) and \( j > 1 \).

Proof of (4.1) when \( \pi = D_n \), the dihedral group:

Lemma 4.8: If \( A = \mathbb{Z}/2 \) or \( \mathbb{Z}(2) \), then

\[ \oplus \mathbb{H}_1(E;A) \to \mathbb{H}_1(D_n;A) \text{ is onto.} \]

Proof: (See Quillen \([Q], 4.6\))

In Section 5, we show that for \( \pi = E \), \( \xi_j^S = 0 \) for \( j > 1 \), and \( \mathcal{L}_j^1 = 0 \) for \( j > 0 \).
Proof of (4.1) when \( \pi = \mathbb{Z}/2^i \) (\( i \geq 1 \)) or \( \text{SD}_n \):

**Lemma 4.9**: If \( \pi = \mathbb{Z}/2^i \) (\( i \geq 1 \)) or \( \text{SD}_n \), then
\[
\text{tor} \; L_1(n_\phi - \hat{n}_{\phi}(2)) = (0).
\]

Apply (4.7). \( L_*^{\mathbb{Z}}(\mathbb{Z}/2^i) \rightarrow L_*^{\mathbb{Z}}(\mathbb{Z}/2^i) \) is 1-1. (See [B]).

Proof of (4.1) for \( \pi = \mathbb{Q}_n \), generalized quaternionic:

**Facts**

1. \( \text{tor} \; L_{j+1}^{\mathbb{C}l_{1} - 0}(\mathbb{Z}Q_n \rightarrow \mathbb{Z}_2 Q_n) \xrightarrow{\partial_*} L_{j+1}^{\mathbb{C}l_{1} - 0}(\mathbb{Z}D_n \rightarrow \mathbb{Z}D_n) \) is 1-1 for \( j \neq 1 \) or 2(4). \( \partial_* : Q_n \rightarrow Q_n/C = D_n \) (see (4.7)).

2. \( H_{4k+2}(Q_n, \mathbb{Z}(2)) = 0 \).

3. \( \bigoplus_{\text{cyclic subgroups}} H_{4k+1}(\mathbb{H}, \mathbb{Z}(2)) \rightarrow H_{4k+1}(Q_n, \mathbb{Z}(2)) \) is onto.

4. The Pontryagin product \( H_{4i}(\mathbb{C}; \mathbb{Z}/2) \times H_{2i}(\mathbb{Q}_n, \mathbb{Z}/2) \rightarrow H_{4i+2}(\mathbb{Q}_n, \mathbb{Z}/2) \) is onto for \( i \leq 3 \). 2 \( \text{tor} \; L_{1}^{\mathbb{Z}}(\mathbb{Z}Q_n) = 0 \) for \( i \neq 1(4) \).

5. \( \text{tor} \; L_{0}^{\mathbb{C}l_{1}}(\mathbb{Z}Q_n) = 0 \).

6. \( L_{1}^{\mathbb{C}l}(\mathbb{Z}Q_n) \xrightarrow{\partial_* \oplus 1} L_{1}^{\mathbb{C}l}(\mathbb{Z}D_n) \oplus L_{1}^{\mathbb{C}l}(\mathbb{Z}Q_n) \) is 1-1

    where \( \partial_* : Q_n \rightarrow Q_n/C = D_n \), and \( 1 : Q_n \rightarrow Q_n \) is the inclusion map.

7. \( H_{4k+3}(Q_n; \mathbb{Z}/2) \xrightarrow{t_*} H_{4k+3}(Q_{n+1}; \mathbb{Z}/2) \) is trivial for all \( k \).

8. \( H_{4k+3}(Q_n; \mathbb{Z}) \rightarrow H_{4k+3}(Q_n; \mathbb{Z}/2) \) is onto for all \( k \).
Facts 1, 2, and 3 plus naturality imply $\tilde{f}_j = 0$ for $j > 0$.

Fact 4 plus the commutativity of (4.6) imply $x^S_j = 0$ for $j > 3$ and $j \neq 3$ (4). Facts 4 and 8 plus the commutativity of

\[
\begin{align*}
H_{4k}(\mathbb{Z}/2; \mathbb{Z}/2) \times H_3(\mathbb{Q}_n) & \xrightarrow{\alpha_*} H_{4k+3}(\mathbb{Q}_n; \mathbb{Z}/2) \\
L_2(\mathbb{Z}; \mathbb{Z}/2) \times L^3(\mathbb{Z}; \mathbb{Q}_n) & \xrightarrow{\alpha_*} L_1(\mathbb{Z}; \mathbb{Q}_n)
\end{align*}
\]

imply $x^S_{4k+3} = 0$ for $k > 0$.

\[
\begin{align*}
(\tilde{\jmath}^* \text{ is induced by}) \\
K(\mathbb{Z}; 0) \wedge BQ_\mathbb{Q}_n^+ & \xleftarrow{\iota^*} L^* (\mathbb{Z}) \\
& \xrightarrow{\iota^*} L^* (\mathbb{Z}; \mathbb{Q}_n)
\end{align*}
\]

Fact 5 implies $x^S_2 = 0$.

Facts 6 and 7 imply $x^P_2$ is trivial.
Section 5: Proof of Theorem 1.2

As always it suffices to assume that \( \pi \) is a 2-group. We first do the case of an elementary abelian 2-group, 
\( E = \mathbb{Z}/2 \oplus \ldots \oplus \mathbb{Z}/2 \).

Lemma 5.1: \( J^S_j : H_j(\mathbb{Z}E ; \mathbb{Z}/2) \rightarrow L^S_j(\mathbb{Z}E)(2) \) is trivial for \( j > 0 \).

Proof: \( SK_1(\mathbb{Z}E) = 0 \), so 1.1 (b) proves the result.

The fact that \( SK_1(\mathbb{Z}E) = 0 \) shows that \( L^S_*(\mathbb{Z}E) \rightarrow L'_*(\mathbb{Z}E) \) is an isomorphism so by Wall's calculations [W2] the torsion in \( L^S_*(\mathbb{Z}E) \) has exponent 2.

Lemma 5.2: \( \chi^S_j : H_j(\mathbb{Z}E ; \mathbb{Z}/2) \rightarrow L^S_{j+2}(\mathbb{Z}E)(2) \) is trivial for \( j > 1 \).

Proof. \( H_1(\mathbb{Z}E/2 ; \mathbb{Z}/2) \otimes H_j(\mathbb{Z}E ; \mathbb{Z}/2) \rightarrow H_{1+j}(\mathbb{Z}(E \times \mathbb{Z}/2) ; \mathbb{Z}/2) \)

\[ \downarrow \chi_1 \times \chi^j_j(\mathbb{Z}/2) \]

\[ \downarrow \chi_{1+j} \]

\[ L^S_{1+j}(\mathbb{Z}E/2 ; \mathbb{Z}/2) \otimes L^j(\mathbb{Z}E ; \mathbb{Z}/2) \rightarrow L^S_{1+j+2}(\mathbb{Z}(E \times \mathbb{Z}/2) ; \mathbb{Z}/2) \]

commutes. Since the result is true for \( \mathbb{Z}/2 \) we can begin an induction.

Since \( L^S_0(\mathbb{Z}[\mathbb{Z}/2 \times \mathbb{Z}/2]) \) is torsion-free [W2], \( \chi^S_2 \) must be trivial for \( \mathbb{Z}/2 \times \mathbb{Z}/2 \). It is not hard to finish.

We need a generalization of a trick in Stein [S].
Lemma 5.3: Let \( \pi_1 \) and \( \pi_2 \) be finite groups and suppose the torsion in \( L^X_*(\mathbb{Z} (\pi_1 \times \pi_2)) \) is annihilated by \( \mathbb{Z}/2^r \). Assume further that \( H_*(B\pi_1; \mathbb{Z}/2^r) \) is a free \( \mathbb{Z}/2^r \) module.

Then, if \( J^X_j \) is trivial for \( j > 0 \) for \( \pi_1 \) and for \( \pi_2 \), then \( J^X_j \) is trivial for \( j > 0 \) and \( \pi_1 \times \pi_2 \).

Proof: By the universal coefficients theorem

\[
\bigoplus_{s \geq r} H_1(B\pi_1; \mathbb{Z}/2^s) \otimes H_{n+1-i}(B\pi_2; \mathbb{Z}/2^s) \rightarrow \bigoplus_{s} H_{n+1}(B\pi_1 \times \pi_2; \mathbb{Z}/2^s) \xrightarrow{\beta} H_n(B\pi_1 \times \pi_2; \mathbb{Z}/2)
\]

is onto the torsion in \( H_n \).

The lemma follows from the commutativity of

(see next page)
\[ H_1(\mathbb{B}_{\pi_1}; \mathbb{Z}/2^g) \otimes H_{n+1-1}(\mathbb{B}_{\pi_2}; \mathbb{Z}/2^g) \rightarrow H_n(\mathbb{B}(\pi_1 \times \pi_2); \mathbb{Z}_2) \]

\[ \downarrow \beta \otimes 1 + 1 \otimes \beta \]

\[ H_{1-1}(\mathbb{B}_{\pi_1}; \mathbb{Z}_2) \otimes H_{n+1-1}(\mathbb{B}_{\pi_2}; \mathbb{Z}/2^g) \oplus H_1(\mathbb{B}_{\pi_1}; \mathbb{Z}/2^g) \otimes H_{n-1}(\mathbb{B}_{\pi_2}; \mathbb{Z}_2) \]

\[ \downarrow j^x_{n-1} \otimes j^{n+1-1} \oplus j^1 \otimes j^x_{n-1} \]

\[ L^x_{1-1}(\mathbb{Z}_{\pi_1}; \mathbb{Z}_2) \otimes L^{n+1-1}(\mathbb{Z}_{\pi_2}; \mathbb{Z}/2^g) \oplus L^1(\mathbb{Z}_{\pi_1}; \mathbb{Z}/2) \otimes L^x_{n-1}(\mathbb{Z}_{\pi_2}/\mathbb{Z}_2) \rightarrow L^x_n(\mathbb{Z}[\pi_1 \times \pi_2]; \mathbb{Z}/2^g). \]
Lemma 5.4: If $A$ is an abelian 2-group, then

$$\theta_j^S : H_j(BA; \mathbb{Z}/2) \to L_j^S(ZA)$$

is trivial for $j > 0$.

Proof: The lemma follows from the Stein trick (lemma 5.3) and induction on the rank of $A$ once we observe:

(i) the result is true if $A$ is elementary abelian (5.1)

(ii) by Wall [W2], $L_*^S(ZA)$ has torsion of exponent at most 4.

We now take up the results for $X_j$. To fix notation let $A$ be our abelian group. Let $i : E \to A$ be the inclusion of the subgroup of elements of order $\leq 2$. Let $j : \mathbb{Z}^r \to A$ be a map of a free abelian group of rank $r = \text{rank of } A$ which is onto.

Then we have

$$(i) \quad \oplus H_i(BE; \mathbb{Z}/2) \otimes H_{n-i}(BZ^r; \mathbb{Z}/2) \to H_n(BA; \mathbb{Z}/2)$$

is onto, where the map is defined using the $H$-space structure of $BA$

$$(ii) \quad \begin{array}{c}
H_i(BE; \mathbb{Z}/2) \otimes H_{n-i}(BZ^r; \mathbb{Z}/2) \\
\downarrow X \otimes \nu
\end{array} \to H_n(BA; \mathbb{Z}/2) \quad \begin{array}{c}
\downarrow X \\
L_{i+2}(ZE)(2) \otimes L_{n-1}(Z[Z^r])(2) \\
\downarrow X
\end{array} \to L_{n+2}(ZA)(2)$$

commutes.
An easy induction plus 5.2 shows that any $c \in H_j(\text{BA}; \mathbb{Z}/2)$ such that $K^h_j(c) = 0$ must be equal to $j_*(\overline{c})$ for the unique element $\overline{c} \in H_j(\text{BA}; \mathbb{Z}/2)$ such that $j_*(\overline{c}) = c$.

**Lemma 5.4:** The maps

$$K^h_j : H^j(\text{BA}; \mathbb{Z}/2) \to L^h_{j+2}(\mathbb{Z}A)(2)$$

are trivial for $j > 2$.

**Proof:** Bak [B] shows $L^h_*(\mathbb{Z}A) \to L^h_*(\mathbb{Z}A)$ is monic so we prove the result for $K^h_j$.

The result just above the lemma implies that it is enough to show that the problem $(T^2 \to S^2) \times T^j$ is solvable for $j > 2$ over BA.

We can write our problem as $(T^2 \to S^2) \times T^{j-1} \times S^1$ where $j-1 > 1$.

Now 1.1(d) plus Ranicki's result [R1] that

$$L^h_{j-1+2}(\mathbb{Z}[G]) \to L^h_{j+2}(\mathbb{Z}[G \times \mathbb{Z}])$$

factors through $L^p_{j-1+2}(\mathbb{Z}[G])$ finishes the proof.

An entirely similar trick shows 1.2 (c). We now do the promised determination of $K_2$.

**Theorem 5.5:** The sequence

$$H_2(\text{BE}; \mathbb{Z}/2) \xrightarrow{i_*} H_2(\text{BA}; \mathbb{Z}/2) \xrightarrow{K^h_2} L^h_0(\mathbb{Z}[A])$$

is exact.
Proof: Naturality of $\kappa_2^h$ plus 5.2 shows that we have a zero sequence. Naturality again reduces exactness for $A$ to exactness for $\mathbb{Z}/2 \times \mathbb{Z}/4$.

For $\mathbb{Z}/2 \times \mathbb{Z}/4$ the cokernel of $i_*$ is $\mathbb{Z}/2$. Morgan-Pardon showed that $\kappa_2^h \neq 0$ by example.
References


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