ociety

'e construct $_nN$, whose

unifilar knot of $n-1$
vertical with a 2-gon,
-gon, we construct an
s.
orem A, art. 3.

which cd and ba are
their mid-points a in

s,

the right in H, till
low to a above past
the 2-gon pq from
We now read from
(a) on our right in
, we proceed along
gon on that pq; at
st (β) on our right
e

n pq, nor any of
ilar knot of $k+2$

knot $_nT$ we con-
r both inside or

ther on the right

the $(2+r)$-gonal
struct the uni-
2 of that face
or if otherwise

uct (Theo. D) by the 2-gon 12, not $_nP$ unifilar, but
us we have proof of
—If in the projection of any $(2+r)$-gonal mesh
f $n-2$ crossings we connect by a 2-gon the mid-
s of which one, and only one, is dotted (art. 2)
inside F, we construct an unifilar of n crossings by one of its odd 2-gons.

This is the constructing converse of Theorem B, art. 5, and is true when $r \gtreqless 0$.

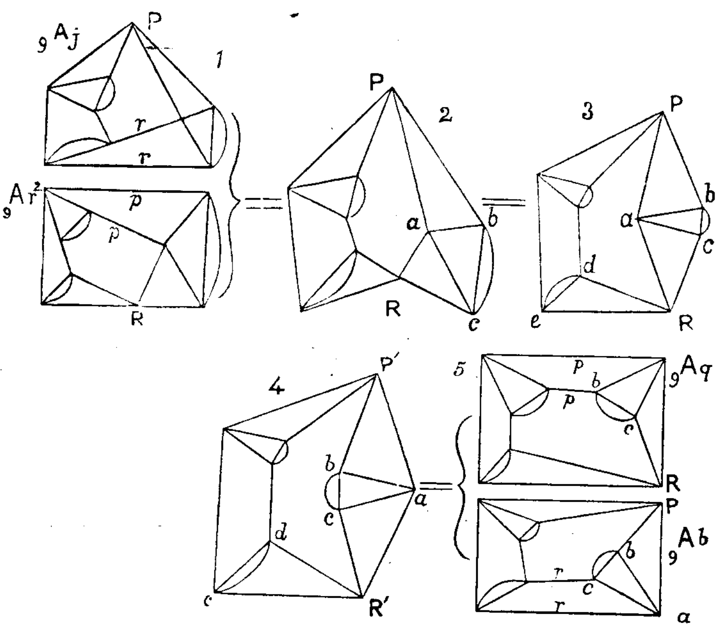
10. The constructing converse of Theorem C is

Theorem CC.—If in any unifilar knot of $n-2$ crossings we make, at any projected crossing r, either pair of opposed angles covertical with a double flap, adding two edges to each of the other pair of opposed meshes about r, we construct an unifilar of n crossings by one of its plural flaps.

No base which has a plural flap, not fixed, can be operated upon by Theorem AA or BB, unless the operation abolishes the plural flap. And every flap, single or plural, is fixed, if its deletion lays bare a section through two edges only. Such a fixed flap cannot compete for the lead, nor hinder an operation by AA or BB, which does not abolish the fixture. Every construction is by a leading flap (*vide* my paper, xvii.), and the leader has the most 2-gons.

2. On the Twists of Listing and Tait. By the Rev. Thomas P. Kirkman, M.A., F.R.S.

In the figure 1 following, the knot $_9Aj$ has a triangular section Prr cutting away on each side of it a $(3+r)$-gonal mesh, and $_9Ar^2$ has such a section Rpp, through one crossing only. Make in these knots creases at rr that approach to meet at R in 2, and creases pp that meet at P in 2. In 3, 2 is prepared for a rotation through two right angles about the fixed axis PR, through the crossing P and the crease-kiss R, or, as a more learned man would say, through the decussation P and the plicatorial osculation R—taking 2 for $_9Aj$: exchange here P and R if 2 is $_9Ar^2$. In 4 the rotation is effected, undoing the crossing P of 3, which has become a kiss P′ in 4, while the kiss R of 3 has become the crossing R′ in 4, if 3 is $_9Aj$: exchange here P and R, if 3 is $_9Ar^2$. In 5 the crease kiss is undone, and $_9Aj$ has been twisted into $_9Aq$, as $_9Ar^2$ has into $_9Ab$, both by the twist of Tait, which is above analysed. In Tait's twist the part rotated is not the same figure between P and R after rotation as before.



There may, of course, be any number of crossings in the rotating part about PR.

If here we consider P and R as both crossings, 3 and 4 are two unsolids of ten crossings, complementaries of each other about PR.

When the knot to be twisted is such that the rotated part is the same after rotation as before, 3 has no complementary, and the result is simply one pair of convertibles, viz., K on the left instead of $_9Aj$, turned into K′ on the right instead of $_9Aqj$, and not two pairs as here, $_9Aj$ turned into $_9Aq$ and $_9Ar^2$ into $_9Ab$.

If, in the above process, we have only two summits a and b between P and R in 3, making the 2-gon ab collateral with the triangles abP and abR, we have the simplest possible case of twisting; and in this the rotation makes no change, 4 and 3 being as projections identical.

This is the twist of Listin
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gracefully flowing curves.
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complete disguise the manœu
The figures 3 and 4, consi
the complementary pair C71
we had obtained $_9Aj$, $_9Aq$, $_9A$
to be regularly constructed by
the minute comparisons of th
solids—the first pair by unk
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crossings, having each a linear
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have to do is to take first e
unsolids, and by unkissing, as
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and by unkissing first at P and
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at either in each, every 9-fold n
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include every unsolid before ha
P and R unlike, to be handled
it has one, two, &c., different
alike. And every 9-fold so g
having a triangular section at
reflected image. It will som
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be found. This means that the
Prr and Stt, at which it can be
more at which it can be twisted
B′C. For the truth is, that the
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This is the twist of Listing, who found it in his study of knots of seven crossings, under what seems to me the needless disguise of gracefully flowing curves. It was sharp wit in Listing so to observe it; it was still more acute in Tait to detect under a more complete disguise the manœuvre of his twist of greater complexity.

The figures 3 and 4, considered as unsolids of ten crossings, are the complementary pair C71 and C74. If we had these two before we had obtained ${}_9Aj$, ${}_9Aq$, ${}_9Ar^2$, and ${}_9Ab$, all of which are subsolids to be regularly constructed by their leading flaps, we could, without the minute comparisons of that process, at once draw these subsolids—the first pair by unkissing at R and P′, the second by unkissing at P and R′. And if we had all the unsolids of ten crossings, having each a linear section PR, which cuts away on both hands a $(3+r)$-gonal mesh, we could with the same ease draw every knot of nine crossings that has a triangular section P*rr*. All we have to do is to take first every complementary pair of 10-fold unsolids, and by unkissing, as just shown, write down from it two couples of 9-fold convertibles; next to take every unsolid 10-fold without complementary, which has a linear section PR, above described, of which the two crossings P and R are not identical, and by unkissing first at P and then at R, to obtain from each such unsolid one couple of convertibles. The remaining unsolids in which those crossings P and R are identical will give, by unkissing at either in each, every 9-fold not already found which has a linear section P*rr*; where the words just used, "remaining unsolids," include every unsolid before handled as having a section PR with P and R unlike, to be handled again once, twice, &c., according as it has one, two, &c., different linear sections PR with P and R alike. And every 9-fold so got from P and R alike is a 9-fold having a triangular section at which it can be twisted into its reflected image. It will sometimes happen that amongst our couples obtained of convertibles, AB, BC, &c., the couple CC will be found. This means that the knot C has two different sections, P*rr* and S*tt*, at which it can be twisted into itself, besides one or more at which it can be twisted into B or B′ of the couples BC and B′C. For the truth is, that the number of different crossings P in the linear sections PR on all the knots of n crossings is exactly the number of different triangular sections P*rr*, above described, on all the knots of $n - 1$ crossings. We shall obtain repetitions of some nine-folds, but no vain repetition; for when the unkissing is finished we shall have grouped all the convertibles in twos, threes, sixes, &c., without ever attempting to perform a twist, as well as have given an accurate account of the number of different triangular sections on the grouped ones and on the uniques, without ever trying to count these sections, or to distinguish them by their symmetry.

In all that precedes, no distinction is made or supposed between unifilar and plurifil knots; nor do I know any reason why unifilars only should be considered.

The crossing P in fig. 1 may stand for any tessarace in the projection of any knot, or of any n-acron whatever, through which lies the triangular section Prr there described. The twisting can take place in them all, no matter what the faces and the other $n - 1$ summits may be, and the groups can be formed of which every figure can be so twisted into one or more of the others.

The question here presents itself—Will it be profitable, supposing that the census of all the knots of n crossings is wanted, to employ the method of unkissing above opened? I am of opinion that it will.

Let the subsolids (which have no linear section PR) be divided into S_n, all that have no triangular section Prr cutting away on both hands a $(3 + r)$-gonal mesh, and T_n, all those which admit one or more such sections Prr, and let U_n comprise all the unsolids, which have each one or more linear sections PR, cutting away on each hand a $(3 + r)$-gonal mesh.

Suppose that S_n and U_n are found, and that the subsolids T_n, in general more numerous than S_n, are wanting. If we can obtain U_{n+1}, we shall readily get by the simple process of unkissing, not only the missing T_n, but every pair of convertibles possible out of T_n and the unsolids U_n that admit a triangular section Prr, and every fact required for our table of uniques and grouped convertibles.

I believe that even when U_{n+1} is more numerous than T_n, it can be more easily found than T_n, which comprises only subsolids to be obtained by the minute and often many comparisons that determine the leading flap; whereas U_{n+1} is rapidly put together without minute comparisons by laying 2 upon $n - 1$, 3 upon $n - 2$, &c., by the simplest marginal sections *fc* only.

The reader, having before him in the plates of vol. xxxii. *Trans. R.S.E.* nearly all the knots unifilar or not of fewer than ten crossings, can amuse himself by considering the question above proposed. He can begin with $S_5 = {}_5B + {}_5C$. Compounds like ${}_6E$ are rejected all through. But the student will not be able to confine his attention to unifilars only.

Observe that in the linear sections PR neither P nor R can be a non-terminal crossing of a plural flap. Also, if in PR R is a crossing of the 2-gon RR′, PR and PR′ are for our purpose the same linear section, because we get the same n-fold knot whether we unkiss at R or at R′ upon the $(n+1)$-fold. Hence it follows that Prr and P$r'r'$ are the same triangular section when rRr and r'Rr' are covertical angles at the crossing R.

It is not difficult to give simple rules whereby S_n is found without error or repetition from S_{n-1} and S_{n-2} by the leading flap of each knot of S_n. But a definition of a fixed flap must be made and stuck to.

I conclude with two useful little theorems.

a. On every unsolid unifilar V in U_{n+1} each of its linear sections PR (P and R unlike) lies through four angles at P and R, which are all odd or all even.

b. The couples of n-folds obtained by unkissing on V, or on V and its complementary, are all unifilars or not, according as these angles (in a) are odd or not.

My objection to the twistings, that they put a twist upon the tape, has been answered by Tait. In the case of Listing's twist, I have satisfied myself that his answer is sufficient. It appears to me that it ought to be formally demonstrated as sufficient in all cases.

After all, as it is certainly not on record who invented kissing, it may come to be forgotten who invented unkissing.

P.S. Nov. 7.—I have learned how to form readily all the unifilars only of U_{n+1}, required for unifilar couples, by operating on unifilars only of n and of fewer crossings. I shall soon present to Professor Tait the requisite unifilars in U_{12}, and thus I hope to save him much time and trouble in grouping the unifilar convertibles of eleven crossings.

$$= (a+3b)\begin{vmatrix} 1 & -b & -b \\ 1 & 4a & -2b \\ 1 & 3a & a \end{vmatrix} - 2b\begin{vmatrix} 1 & 4a & -2b \\ 1 & 3a & a \\ 1 & 2a & a \end{vmatrix}$$

$$= (a+3b,\begin{vmatrix} 1 & -b & -b \\ 1 & 3a & -2b \\ 1 & 2a & a \end{vmatrix} + (a+3b)\begin{vmatrix} 1 & 0 & -b \\ 1 & a & -2b \\ 1 & a & a \end{vmatrix} - 2b\begin{vmatrix} 1 & 4a & -2b \\ 1 & 3a & a \\ 1 & 2a & a \end{vmatrix}$$

$$= (a+3b,\begin{vmatrix} 1 & -b & -b \\ 1 & 3a & - \\ 1 & 2a & a \end{vmatrix} + (a+b)a(a+2b)\,,$$

for the last two determinants in the preceding line are each equal to $a(a+2b)$. Thus we have

$$S_4 = \frac{\begin{vmatrix} 1 & -b & -b \\ 1 & 3a & -2b \\ 1 & 2a & a \end{vmatrix}}{a(a+b)(a+2b)} + \frac{1}{a+3b}$$

$$= \qquad S_3 \qquad + \frac{1}{a+3b}$$

as it should be. And this method of showing that the validity of the fourth case is dependent on that of the third is applicable in other case.

8. Sevenfold Knottiness. By Prof. Tait.

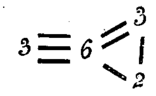
(Abstract.)

From the point of view of the Hypothesis of Vortex Atoms, it becomes a question of great importance to find how many distinct forms there are of knots with a given amount of knottiness. The enormous numbers of lines in the spectra of certain elementary substances show that the form of the corresponding Vortex Atoms cannot be regarded as very simple. But this is no objection against, it is rather an argument in favour of the truth of, the Hypothesis.

For not only are the great majority of possible knots not stable forms for vortices; but altogether independently of the question of kinetic stability, the number of distinct forms with each degree of knottiness is exceedingly small,—very much smaller than I was prepared to find it. I have already stated that for three, four, five, and sixfold knottiness, the numbers are only 1, 1, 2, 4. For a reason given in my first paper, knots whose number of crossings is a multiple of 6 form an exceptional class: so I thought it might be useful to discover and to figure all the distinct forms with seven-fold knottiness. Eight and higher numbers are not likely to be attacked by a rigorous process until the methods are immensely simplified. The method of partitions, supplemented by the graphic formulæ of my last paper, is to some extent tentative. I have verified the present results by means of it, and have extended it to 8-fold knottiness, but I am not certain that I have got *all* the possible forms of the latter.

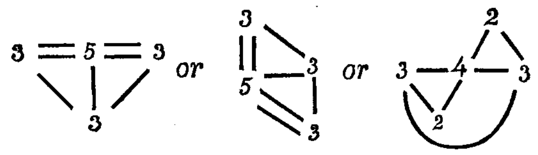
As I did not see how to abridge the process, I wrote out all the admissible permutations of the seven letters in the even places of the scheme. These I found to be 579, five of which were, of course, unique. The others (as 7 is a prime number) were divisible into 82 groups—those of each group being mutually equivalent. On examination, it was found that only 22 of the 87 selected arrangements satisfied the criterion for possible knots (see I § (*b*) of my paper, *ante* p. 238), and several even of these were repetitions. These repetitions were of two kinds—1st, the mere inversion of the order of the scheme; 2d, the relative positions of a 3-fold and a 4-fold knot which in certain cases were found combined as a 7-fold form. Clearing off these repetitions, and along with them a form really belonging to 6-fold knots (because consisting of two trefoil knots and one nugatory intersection), there remain only *eleven* distinct forms of the 7th order. These are as follows:—

1.



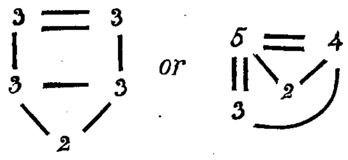
This has a great many forms, with correspondingly different symbols, being a mere compound of a 3-fold and a 4-fold knot, which may have *any* relative positions on the string.

2.

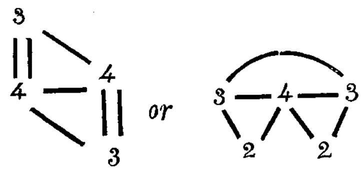


This is one of Listing's knots.

3.

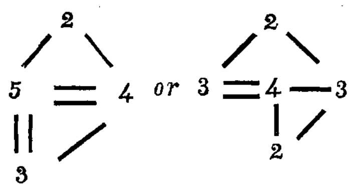


4.



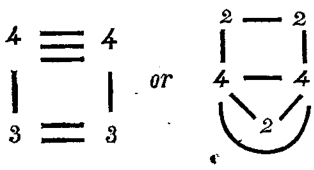
Listing has shown that this is deformable into 2 above.

5.

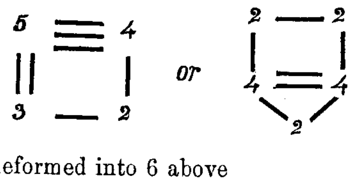


I find that this can be deformed into 3 above. It is figured in my paper on *Links*, *ante*, p. 325, first woodcut.

6.

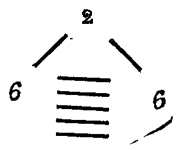


7.



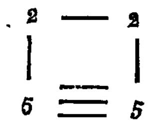
This can be deformed into 6 above

8.



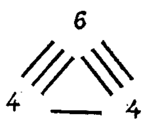
This species of knot occurs for *all* numbers of intersections greater than 2.

9.

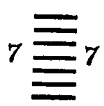


This is the 7 knot which Listing does not sketch. *Ante*, p. 311.

10.



11.



This is the simple twist, which occurs for every *odd* number of intersections.

As 2 and 4, 3 and 5, 6 and 7 are capable of being deformed into one another, three of them are not independent forms, and thus the number of distinct forms of seven-fold knots is only *eight*.

Drawings of various forms of each of these knots were given, as well as indications of the modes in which they can be formed from knottinesses of lower orders.