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HOMOLOGY OF GROUP SYSTEMS WITH APPLICATIONS TO KNOT THEORY

BY H. F. TROTTER

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Introduction

Let L be a tame link[1] in S^3 with components $L_1, \cdots, L_\mu$ and let V be the union of open tubular neighbourhoods of the L_i. Then $S^3 - V$ is a 3-manifold M with boundary B consisting of μ tori B_i which are also the boundaries of the closures $\bar{V}_i$ of the V_i. The *complete group system* of L consists of the fundamental groups $\pi_1(M)$, $\pi_1(B_i)$, $\pi_1(V_i)$ and the homomorphisms $\pi_1(V_i) \leftarrow \pi_1(B_i) \rightarrow \pi_1(M)$ induced by inclusion. Fox [8] has stressed the importance of the group system and conjectured that the known algebraic invariants of links depend only on the group system. It is now known [18] that in situations of interest in knot theory, M and the B_i are aspherical spaces so that the homomorphisms $\pi_1(B_i) \rightarrow \pi_1(M)$ (which we shall call the *external group system*) determine the homotopy type of the pair (M, B). Many of the known link invariants can be described in terms of the homology of various coverings of this pair. (Some, such as the quadratic form for links, depend on the homology of branched coverings; it seems likely that these are determined by the complete group system.) Thus Fox's conjecture appears to be substantially correct.

Our main concern in this paper is to show how these invariants can be explicitly calculated by purely algebraic methods. We consider only invariants that can be derived from the external group system. They are of particular interest since the external system depends only on the homeomorphism type of the complementary space $S^3 - L$.[2]

[1] A *link* of multiplicity μ consists of μ circles imbedded in the 3-sphere S^3. It is *tame* if there is a triangulation of S^3 in which it is a subcomplex. More generally, one may consider arbitrary tamely imbedded graphs in arbitrary 3-manifolds. For graphs the B_i would in general be surfaces of higher genus.

[2] The external group system may be defined intrinsically in terms of $S^3 - L$ by an elaboration of Fox's definition of peripheral subgroup [9]. Let $Y = S^3 - L$ and let X be its Freudenthal compactification [11] obtained by adding one "point at infinity" p_i for each component L_i. For each i, take an arc in X running from a basepoint p_0 in Y to p_i, and let a_i be the intersection of this arc with Y. For any compact $A \subset Y$, let X_i be the component of $X - A$ containing p_i, and let $Y_i = X_i \cap Y$. (The Y_i may not be distinct if A is not large enough.) Taking basepoints p_i^* in the final segments of the $Y_i \cap a_i$, we get a group system consisting of $\pi_1(Y, p_0)$ and the $\pi_1(Y_i, p_i^*)$, with maps specified by taking an element of $\pi_1(Y_i, p_i^*)$ represented by a loop based at p_i^* into the element of $\pi_1(Y, p_0)$ represented by a loop which starts at p_0, goes along a_i to p_i^*, around the given loop, and back along a_i to p_0. The compact subsets of Y are directed by inclusion, and we obtain a directed family of group systems which has the external group system of L as direct limit.

Section 1 is occupied with the basic definitions. From an abstract point of view a group system is a group G with a family of groups and homomorphisms φ_i: $G_i \to G$. For any such system one can construct a pair of spaces $(X, \bigcup Y_i)$ such that X and the Y_i are aspherical, $G = \pi_1(X)$, $G_i = \pi_1(Y_i)$ and the maps φ_i coincide with those induced by inclusion. The homology of this pair (with local coefficients in a G-module) is then the homology[3] of the given system. We give a purely algebraic construction which is in principle the same as Massey's [17], but permits the use of more "economical" complexes and is therefore better adapted to explicit calculation.[4] Since most of this section consists of slight modifications of standard propositions in homological algebra we have omitted most details of proofs.

Section 2 sets out a general algorithm for computing the low-dimensional homology of group systems given by generators and relations. Most of it is a reformulation of known relations between the free differential calculus and the homology of groups; the construction of the diagonal map, however, is new. Some convenient simplifications which are possible when the group systems and coefficients satisfy certain restrictions are discussed in § 3.

In §§ 4 and 5 we compute invariants of the group system of a knot which are related to the homology of cyclic coverings of the knot. Seifert [22, 23] showed by partly geometrical arguments that the latter could be calculated from a "linking matrix" obtained from an orientable surface having the knot as boundary, and we essentially reproduce his results in an algebraic setting. It is a consequence of our results that once an orientation of 3-space is specified, the Seifert invariants are determined by the external group system, and hence by the complementary space of the knot. The same is shown to hold for the quadratic form of a knot, which is interesting since methods of Kyle [14] can be used to show that a certain pair of 2-component links with homeomorphic complementary spaces (of the type described in [26]) have inequivalent quadratic forms.[5] In § 4 we deal with the algebraic analogue of the infinite cyclic covering, and obtain in Theorem 2 some invariant properties of the Seifert matrix which are slightly stronger than those previously known. An algorithm for computing the analogue of self-linking in an arbitrary fiinite covering of a knot is developed in § 5 and applied to obtain Seifert's results on finite cyclic coverings.

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[3] Here, and in similar contexts throughout, we write "homology" rather than "homology and cohomology".

[4] M. Auslander [29] uses a standard complex, and Takasu [30] an algebraic mapping cylinder construction.

[5] The computation is carried out in detail in [13].

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1. The homology of group systems

1.1 *Notations and definitions.* For a group homomorphism $f\colon G \to G'$, the same letter will denote the induced homomorphism $f\colon Z(G) \to Z(G')$ of the integer group rings. A left (right) G-module is a left (right) module over $Z(G)$; we write $\otimes_G$, Hom_G rather than $\otimes_{Z(G)}$, $\mathrm{Hom}_{Z(G)}$. Let A and A' be right modules over G and G' respectively. An additive homomorphism $p\colon A \to A'$ is a map under the homomorphism $f\colon G \to G'$ if $p(a \cdot g) = p(a) \cdot f(g)$ for all $a \in A$, $g \in G$. A map $q\colon A' \to A$ is under f if $q(a \cdot f(g)) = q(a) \cdot g$ for all $a \in A'$, $g \in G$. Similar definitions apply to left modules.

Let B, C and B', C' be left modules over G and G' respectively. Then maps $u\colon A \to A'$, $v\colon B \to B'$, $w\colon C' \to C$, all under f, induce maps $u \otimes_f v\colon A \otimes_G B \to A' \otimes_{G'} B'$ and $\mathrm{Hom}_f(v, w)\colon \mathrm{Hom}_{G'}(B', C') \to \mathrm{Hom}_G(B, C)$.

By a *resolution over a group G* we shall mean what is called in [1] a projective resolution of Z over $Z(G)$, i.e., it is a complex X with an augmentation ε such that

$$\longrightarrow X_n \xrightarrow{d_n} X_{n-1} \longrightarrow \cdots \longrightarrow X^0 \xrightarrow{\varepsilon} Z \longrightarrow 0$$

is exact, the X_n are projective left G-modules, and the d_n and ε are G-homomorphisms. (Z is considered a G-module under the trivial action of G.) A *resolution over a system* $\{G, G_i, \varphi_i\}$ is a pair of G-complexes (X, Y) such that:

(i) X is a resolution over G;

(ii) Y is the direct sum of complexes $Y_i = Z(G) \otimes_{G_i} \bar{Y}_i$ where $\bar{Y}_i$ is a resolution over G_i and $Z(G)$ is considered a right G_i-module *via* the map φ_i;

(iii) Y is a G-direct summand of X.

We shall call the $\bar{Y}_i$ referred to in (ii) the *auxiliary resolutions*.

A map of a system $\{G, G_i, \varphi_i\}$ into another $\{G', G'_j, \varphi'_j\}$ consists of homomorphisms $f\colon G \to G'$ and $f_i\colon G_i \to G'_{j(i)}$ (where for each i, $G'_{j(i)}$ may be any of the G'_j) such that the obvious commutativity relations $f\varphi_i = \varphi'_{j(i)} f_i$ hold. If $(X, Y), (X', Y')$ are resolutions over the two systems then a function $f^{\#}\colon X \to X'$ is a *chain-map under the map* $\{f, f_i\}$ if

(i) it is a chain-map and a module homomorphism under f,

(ii) for each i, $f^{\#}|Y_i = f\otimes_{f_i} f_i^{\#}: Y_i \to Y'_{j(i)}$ where $f_i^{\#}$ is a chain-map $\bar{Y}_i \to \bar{Y}'_{j(i)}$ under f_i and $f: Z(G) \to Z(G')$ is the given homomorphism.

Resolutions exist over any system, and for any map of systems there exist induced chain-maps of resolutions which are unique to within a chain-homotopy. The proofs are easy generalizations of standard arguments (for example, see [1, pp. 75–77]). One simply makes the constructions for the auxiliary resolutions first, takes the indicated tensor products, and then extends the construction to the entire complex. (Since Y is a direct summand of X there is no difficulty in extending the maps and homotopies.) It follows that any two resolutions over a system are naturally homotopy-equivalent.

Let (X, Y) be a resolution over $\{G, G_i, \varphi_i\}$. Since the sequence

$$0 \longrightarrow Y \longrightarrow X \longrightarrow X/Y \longrightarrow 0$$

splits, the sequences

$$0 \longrightarrow A\otimes_G Y \longrightarrow A\otimes_G X \longrightarrow A\otimes_G X/Y \longrightarrow 0 \tag{1.1}$$

and

$$0 \longleftarrow \mathrm{Hom}_G(Y, B) \longleftarrow \mathrm{Hom}_G(X, B) \longleftarrow \mathrm{Hom}_G(X/Y, B) \longleftarrow 0 \tag{1.2}$$

are also exact, for any right G-module A or left G-module B. The sequence (1.1) gives rise to a homology sequence which (up to natural isomorphism) depends only on A and the system. We write $H_n(G; A)$ for $H_n(X; A)$, $H_n(\{G_i\}; A)$ for $H_n(Y; A)$, and $H_n(\{\varphi_i\}; A)$ for $H_n(X/Y; A)$, and call the sequence

$$\longrightarrow H_n(\{G_i\}; A) \longrightarrow H_n(G; A) \longrightarrow H_n(\{\varphi_i\}; A) \longrightarrow H_{n-1}(\{G_i\}; A) \longrightarrow$$

the *homology sequence of* $\{G, G_i, \varphi_i\}$ *with coefficients in* A. We remark that $H_n(G; A)$ is the usual n^{th} homology group of G with coefficients in A, while $H_n(\{G_i\}; A)$ is naturally isomorphic to the direct sum of the homology groups $H_n(G_i; A)$, where A is treated as a G_i-module *via* the map φ_i.

Similarly (1.2) gives rise to a sequence

$$\longleftarrow H^{n+1}(\{\varphi_i\}; B) \longleftarrow H^n(\{G_i\}; B) \longleftarrow H^n(G;B) \longleftarrow H^n(\{\varphi_i\}; B) \longleftarrow ,$$

the *cohomology sequence of* $\{G, G_i, \varphi_i\}$ *with coefficients in* B. $H^n(G; B)$ is usual n^{th} cohomology group of G with coefficients in B, while $H^n(\{G_i\}; B)$ is naturally isomorphic to the direct product of the groups $H^n(G_i; B)$.

1.2 *Equivalence of systems*; *fused systems*. In describing the group system of a topological space and subspaces, we spoke loosely of homomorphisms of fundamental groups "induced by inclusion". Such homomorphisms depend on the choice of base-points, and on the choice of paths joining the base-points of the subspaces to that of the space; they are well-defined only to within inner automorphisms.

Let $\{G, G_i, \varphi_i\}$ and $\{G, G_i, \varphi'_i\}$ be systems involving the same groups but possibly different maps. We say that they are *equivalent* if for each i there is an element g_i of G such that $\varphi'_i(x) = g_i\varphi_i(x)g_i^{-1}$, for all $x \in G_i$. The group system of a space and subspaces is thus well-defined up to equivalence. To show that the homology of the group system of a space and subspaces is a topological invariant, it suffices to show that equivalent systems have isomorphic homology. This is an immediate corollary of the following proposition.

PROPOSITION. *Equivalent group systems possess isomorphic resolutions.*

PROOF. We shall write $Z(G)_i$, $Z(G)'_i$ to denote $Z(G)$ considered as a right G_i-module *via* the maps φ_i, φ'_i respectively. For each i, let $\bar{Y}_i$ be a resolution over G_i and put

$$Y_i = Z(G)_i \otimes_{G_i} \bar{Y}_i = Z(G)_i \otimes_{G_i} Z(G_i) \otimes_{G_i} \bar{Y}_i \ ,$$
$$Y'_i = Z(G)'_i \otimes_{G_i} \bar{Y}_i = Z(G)'_i \otimes_{G_i} Z(G_i) \otimes_{G_i} \bar{Y}_i \ .$$

The elements $g \otimes e$, for $g \in G$, form an additive basis for both $Z(G)_i \otimes_{G_i} Z(G_i)$ and $Z(G)'_i \otimes_{G_i} Z(G_i)$; the additive isomorphism under which $g \otimes e$ corresponds to $gg_i^{-1} \otimes e$ is easily verified to be an isomorphism of left G-modules and right G_i-modules, and it follows that Y_i and Y'_i are isomorphic G-complexes. Consequently Y, the direct sum of the Y_i, and Y', the direct sum of the Y'_i, are isomorphic, and they can be extended in "parallel" fashion to give isomorphic resolutions (X, Y) and (X', Y').

Any system $\{G, G_i, \varphi_i\}$ has a naturally associated *fused* system $\{G, G_0, \varphi_0\}$ in which G_0 is the free product of the G_i and $\varphi_0\colon G_0 \to G$ is induced by the φ_i. The obvious natural map of the original system into the fused one induces maps of the homology and cohomology sequences, with any coefficients. For homology, the induced maps are isomorphisms except for $H_0(\{G_i\}; A) \to H_0(G_0; A)$ and $H_1(\{\varphi_i\}; A) \to H_1(\varphi_0; A)$, which are epimorphisms. For cohomology, the maps are isomorphisms except for monomorphisms $H^0(G_0; B) \to H^0(\{G_i\}; B)$ and $H^1(\varphi_0; B) \to H^1(\{\varphi_i\}; B)$. We indicate the proof for homology; a dual argument gives the proof for cohomology. It is easily seen that the induced map $H_*(G; A) \to H_*(G; A)$ is the identity. We show below that the map $H_*(\{G_i\}; A) \to H_*(G_0; A)$ is an isomorphism except in dimension zero, where it is an epimorphism. The "five lemma" then implies the result stated for the maps of the relative groups.

Examination of the constructions shows that the induced map $H_*(\{G_i\};A) \to H_*(G_0; A)$ is the same (when $H_*(\{G_i\}; A)$ is identified with the direct sum $\sum H_*(G_i; A)$) as the sum of the individual maps $H_*(G_i; A) \to H_*(G_0; A)$ induced by the inclusion of the G_i in G_0. What we need thus amounts to the following proposition. (The result below is well known to follow from a theorem of Whitehead [28], and the geometric definition of the homology of groups. I have not found a purely algebraic proof in the literature.)

PROPOSITION. *Let G be the free product of groups G_α and let A be a right G-module (and hence a G_α-module for all α). Let i_α: $H_*(G_\alpha; A) \to H_*(G; A)$ be induced by the inclusion $G_\alpha \to G$. Then except in dimension 0, the i_α give an isomorphism between the direct sum $\sum H_*(G_\alpha; A)$ and $H_*(G; A)$.*

PROOF. Let I_α, I be the augmentation ideals (fundamental ideals) in $Z(G_\alpha)$, $Z(G)$. By formula (4) of [1, p. 144], $H_n(G; A) \approx \mathrm{Tor}_{n-1}^{Z(G)}(A, I)$ for $n > 1$, with analogous results for $H_n(G_\alpha; A)$. $\bar{I}_\alpha = Z(G) \otimes_{G_\alpha} I_\alpha$ can be naturally identified with the left ideal of $Z(G)$ generated by I_α. Note that since G_α is a subgroup of G, $Z(G)$ is free over $Z(G_\alpha)$ and $\mathrm{Tor}_n^{Z(G_\alpha)}(Z(G), I_\alpha) = 0$ for $n > 0$. Theorem 3.1 of [1, p. 150] applies, and we conclude that $H_n(G_\alpha; A) \approx \mathrm{Tor}_{n-1}^{Z(G)}(A, \bar{I}_\alpha)$ for $n > 1$. To complete the proof we show that I is the direct sum of the $\bar{I}_\alpha$. Retractions p_α: $I \to \bar{I}_\alpha$ can be defined as follows. For a fixed α, let H be the subgroup of G generated by the G_β with $\beta \neq \alpha$. Each $g \in G$ then has a representation in the form $g = h_1 g_1 \cdots h_n g_n$ with the g_i in G_α and the h_i in H, which is unique if superfluous identity factors are absent. Elements of the form $g - e$ give an additive basis for I, and it is easy to verify that setting

$$p_\alpha(g - e) = h_1(g_1 - e) + h_1 g_1 h_2(g_2 - e) + \cdots + h_1 g_1 \cdots h_n(g_n - e)$$

yields a homomorphism $I \to \bar{I}_\alpha$ which is the identity on $\bar{I}_\alpha$ and 0 on all the other $\bar{I}_\beta$. The sum of all the p_α is the identity, and together with the inclusion maps i_α: $\bar{I}_\alpha \to I$ they yield a direct sum decomposition of I.

1.3 *Products*. The product of two pairs of topological spaces (U_1, V_1) and (U_2, V_2) is defined as the pair $(U_1 \times U_2, U_1 \times V_2 \cup V_1 \times U_2)$. If the spaces and subspaces are path-connected then (in the absence of pathology) the van Kampen theorem [2] implies that $\pi_1(U_1 \times V_2 \cup V_1 \times U_2)$ is isomorphic to the free product $\pi_1(U_1 \times V_2) * \pi_1(V_1 \times U_2)$, reduced by all relations $i_1(v) = i_2(v)$ where $v \in \pi_1(V_1 \times V_2)$ and i_1, i_2 are the maps induced by the inclusion of $V_1 \times V_2$ in $U_1 \times V_2$ and $V_1 \times U_2$. For group systems with a single auxiliary group (corresponding to a space and single connected subspace, in the topological picture) we define a product in analogy with the above, as follows.

The product of two systems φ: $G_0 \to G$ and φ': $G'_0 \to G'$ is a system φ'': $G''_0 \to G''$ in which G'' is the direct product $G \times G'$. G''_0 is the free product

$(G_0 \times G')*(G \times G_0')$, reduced by all relations $g \times \varphi'(g') = \varphi(g) \times g'$ where $g \times g'$ is an element of $G_0 \times G_0'$. The map φ'' is defined by

$$g_0 \times g' \longrightarrow \varphi(g_0) \times g' \qquad \text{for } g_0 \times g' \in G_0 \times G'$$
$$g \times g_0' \longrightarrow g \times \varphi'(g_0') \qquad \text{for } g \times g_0' \in G \times G_0' .$$

Since $Z(G \times G')$ is naturally isomorphic to $Z(G) \otimes Z(G')$ we may conclude from [1, Proposition XI, 1.1] that if X, X' are resolutions over G, G' respectively, then $X \otimes X'$ is a resolution over $G \times G'$. It can then be shown that if (X, Y), (X', Y') are resolutions over the systems $\varphi: G_0 \rightarrow G$, $\varphi': G_0' \rightarrow G'$ then $(X \otimes X', X \otimes Y' + Y \otimes X')$ is a resolution over the product system, as is to be expected from the topological analogy.

Mapping g into $g \times g$ gives a natural diagonal map of a system into its product with itself. This permits the definition of cup products

$$\smile_\lambda : H^p(G; B) \otimes H^q(G; B') \longrightarrow H^{p+q}(G; B'')$$
$$\smile_\lambda : H^p(\varphi; B) \otimes H^q(\varphi; B') \longrightarrow H^{p+q}(\varphi; B'')$$

for any G-homomorphism $\lambda: B \otimes B' \rightarrow B''$. ($B \otimes B'$ is considered a G-module under the action $g(b \otimes b') = gb \otimes gb'$.) If $B = B'$ and λ is symmetric so that $\lambda(b_1 \otimes b_2) = \lambda(b_2 \otimes b_1)$ then the usual commutativity relation

(1.3.1) $$u \smile_\lambda v = (-1)^{pq} v \smile_\lambda u$$

holds for cohomology classes of dimensions p, q.

2. Presentations of group systems, and their related resolutions

In this section we show how to construct the first few terms of a resolution over a system given by generators and relations. We shall consider only group systems $\{G, G_0, \varphi\}$ with a single auxiliary group; we lose little generality, since for most purposes a group system can be replaced by its fused version, and we gain substantially in simplicity. (The more general construction is not difficult, but is made complicated by the need to use more than one generator in the 0-dimensional term of the resolution.)

2.1 *Presentations and identities.* In dealing with presentations of groups we follow the terminology and notation of [6, 7]. Let F be a free group with generators $\mathbf{x} = \{x_i\}$, and $\mathbf{r} = \{r_j\}$ a set of words (the *relators*) in F. The *consequence* of $\mathbf{r}$ in F is the smallest normal subgroup of F containing $\mathbf{r}$. Then $\{\mathbf{x}; \mathbf{r}\}$ is a *presentation* for the group $G = F/R$ where R is the consequence of $\mathbf{r}$ in F. We need to consider identities between the relators of a presentation [16, 19]. Let P be a free group with generators ρ_j corresponding to the relators r_j, and define $\psi: P*F \rightarrow F$ by $\psi(\rho_j) = r_j$, $\psi(x_i) = x_i$. A word $s \in P*F$ is an *identity* (of the presentation $\{\mathbf{x}; \mathbf{r}\}$) if it is in the kernel of ψ, *and* in the consequence of P in $P*F$.

(This last condition amounts to saying that s is of the form $\prod_{m=1}^{n} w_m \rho_{j_m}^{\varepsilon_m} w_m^{-1}$ where the w_m are in F and $\varepsilon_m = \pm 1$.)

By a presentation of a system $\{G, G_0, \varphi\}$ we mean what Fox [7] calls a presentation of the homomorphism φ. Let $F = F' * F_0$ where F' is free with generators $\mathbf{x}'$ and F_0 is free with generators $\mathbf{x}_0$. (We allow $\mathbf{x}'$ or $\mathbf{x}_0$ to be empty, with the convention that the trivial group is freely generated by the empty set.) Let $\mathbf{r}_0$ be a set of words in F_0 and $\mathbf{r}'$ a set of words in F (not necessarily in F'). Let R_0 be the consequence of $\mathbf{r}_0$ in F_0 and R the consequence of $\mathbf{r}' \cup \mathbf{r}_0$ in F. Then $\{\mathbf{x}', \mathbf{x}_0; \mathbf{r}', \mathbf{r}_0\}$ is a presentation for the system $\{G, G_0, \varphi\}$ where $G = F/R$, $G_0 = F_0/R_0$, and φ is induced by the inclusion $F_0 \rightarrow F$. Writing $\mathbf{x}$ for $\mathbf{x}' \cup \mathbf{x}_0$ and $\mathbf{r}$ for $\mathbf{r}' \cup \mathbf{r}_0$ we observe that $\{\mathbf{x}; \mathbf{r}\}$ is a presentation for G. To define identities we introduce the free group P in the form $P' * P_0$ where P' and P_0 are free with generators corresponding to the elements of $\mathbf{r}'$, $\mathbf{r}_0$ respectively, and define the map $\psi: P * F \rightarrow F$ as before. An identity is then an element of $P * F$ which is an identity of the presentation $\{\mathbf{x}; \mathbf{r}\}$; if it happens to lie in $P_0 * F_0$ it is also an identity of the auxiliary presentation $\{\mathbf{x}_0, \mathbf{r}_0\}$.

2.2 *The topological picture.* Before going on to describe the chain-complexes associated with presentations of groups and group systems we shall discuss briefly the topological picture involved. Strictly speaking, this picture is irrelevant to the algebraic development, but the reader may find it helpful in motivating the constructions of the next section. Since this discussion is purely heuristic we give no proofs.

The construction of a 2-dimensional cell-complex having a fundamental group G with presentation $\{\mathbf{x}; \mathbf{r}\}$ is well known [20]. There is a single vertex p, each generator x_i is represented by a loop based at p, and each relator r_j corresponds to a 2-cell adjoined so that its boundary represents the relator. What we shall call the associated complex of the presentation has a 2-skeleton isomorphic to the chain-complex of the universal covering space of this cell-complex.[6] The group of 0-chains, considered as a G-module, has a single generator e corresponding to a base-point chosen in the universal covering space. The 1-chains have a basis $\{a_i\}$ where each a_i corresponds to the 1-cell in the covering space which begins at e and covers the loop representing x_i. More generally, any word w in the free group F is represented by a path in the 1-skeleton of the original complex,

[6] We shall actually define a 3-dimensional chain-complex associated with a presentation and set of identities. The identities may be interpreted topologically as elements of the second homotopy group of the 2-complex associated with the presentation [19], and our algebraic construction corresponds to adjoining 3-cells which "kill" the second homotopy group.

and this is covered by a unique path starting at e in the universal covering space. Considering this path as a 1-chain, we obtain a function (from F to the group of 1-chains in the covering complex) which corresponds to the crossed homomorphism α defined in the next section.

2.3 *Resolutions derived from presentations.* In this section and the next we shall have to describe a number of functions $f: F \to M$ where F is a free group on a set of generators $\{x_i\}$ and M is a left F-module. These maps will satisfy some relation of the form

$$f(uv) = f(u) + uf(v) + g(u, v) \qquad u, v \in F \tag{2.3.1}$$

and

$$f(1) = 0 ,$$

where $g: F \times F \to M$ is a given function of two variables. Since $f((uv)w)$ must be the same as $f(u(vw))$, g has to satisfy

$$g(u, vw) - g(uv, w) - g(u, v) + ug(v, w) = 0 \tag{2.3.2}$$

and

$$g(u, 1) = g(1, v) = 0 \qquad u, v, w \in F .$$

We shall need the following converse.

LEMMA *Given any function $g: F \times F \to M$ satisfying* (2.3.2), *there exists a unique function $f: F \to M$ which satisfies* (2.3.1) *and has arbitrary prescribed values on the generators x_i.*

PROOF. The uniqueness of f is obvious, for setting $u = v = 1$ in (2.3.1) shows that $f(1) = 0$, and setting $v = u^{-1}$ in (2.3.1) gives

$$f(u^{-1}) = -u^{-1}f(u) - u^{-1}g(u, u^{-1}) . \tag{2.3.3}$$

Thus if f is given on the generators x_i, it is determined on the inverses x_i^{-1}, and an obvious induction on the length of the word w shows that (2.3.1) determines $f(w)$ for all $w \in F$. Conversely, to construct f given the values $f(x_i)$ we first define $f(x_i^{-1})$ by (2.3.3). A straightforward induction on the lengths of the words involved shows that f can be consistently defined on the semi-group of (unreduced) words in the x_i and their inverses in such a way that (2.3.1) is satisfied. By a trivial calculation, $f(x_i x_i^{-1}) = f(x_i^{-1}x_i) = 0$ for any generator x_i. For any u, v, w in the semi-group

$$f(uvw) = f(u) + uf(v) + uvf(w) + g(u, vw) + ug(v, w) . \tag{2.3.4}$$

If $v = x_i x_i^{-1}$ or $x_i^{-1}x_i$ then $f(v) = g(v, w) = 0$, and $f(uvw) = f(u) + uf(w) + g(u, w) = f(uw)$. From this it follows that f has the same value on any two words in the semi-group which represent the same element of F, so that f is well-defined on F.

If f satisfies (2.3.1) with g identically zero, it is called a *crossed homomorphism*. Thus as a special case of this lemma we have the well-known result that crossed homomorphisms of a free group can be arbitrarily prescribed on the generators.

Let $\{\mathbf{x}; \mathbf{r}\}$ be a presentation of a group G and $\mathbf{s} = \{s_k\}$ a set of identities of the presentation. The *associated complex of* $\{\mathbf{x}; \mathbf{r}; \mathbf{s}\}$ is a free augmented G-complex X,

$$X_3 \xrightarrow{d} X_2 \xrightarrow{d} X_1 \xrightarrow{d} X_0 \xrightarrow{\varepsilon} Z \longrightarrow 0 \tag{2.3.5}$$

defined as follows. X_0 has a single basis element e; X_1 has a basis $\{a_i\}$ whose elements correspond to the generators x_i; X_2 has a basis $\{b_j\}$ whose elements correspond to the relators r_j; and X_3 has a basis $\{c_k\}$ whose elements correspond to the identities s_k. We consider X as an F-module *via* the quotient map $F \to G$, and as a $P*F$-module *via* the map $P*F \xrightarrow{\psi} F \to G$. Let α: $F \to X_1$ be the crossed homomorphism such that $\alpha(x_i) = a_i$ for all i, and $\beta: P*F \to X_2$ the crossed homomorphism such that $\beta(\rho_j) = b_j$ for all j and $\beta(x_i) = 0$ for all i. Then the augmentation and boundary operator are determined by giving their values on the generators as follows:[7]

$$\varepsilon(e) = 1\ , \qquad d(a_i) = (x_i - 1)e\ ,$$
$$d(b_j) = \alpha(r_j)\ , \qquad d(c_k) = \beta(s_k)\ .$$

We remark that in the notation of the free differential calculus [5], $\alpha(w) = \sum_i (\partial w/\partial x_i) a_i$ and $\beta(w) = \sum_j (\partial w/\partial \rho_j) b_j$. It follows from Lemma 5.1 and § 1 of [16] that $d^2 = \varepsilon d = 0$ so that X actually is a complex. (Note that α corresponds exactly to the "differential" of [16], while β is the same as the correspondence described in Lemma 5.2 of [16].) Indeed Lemma 5.1 of [16] states that (2.3.5) is exact at X_1, X_0 and Z. Let us say that s is a *complete* set of identities if (2.3.5) is also exact at X_2. Corollary 5.3 of [16] then implies that any presentation has a complete set of identities. We can summarize these remarks in the following proposition.

Any presentation $\{\mathbf{x}; \mathbf{r}\}$ *of a group G has a complete set of identities* $\mathbf{s}$. *The associated complex of* $\{\mathbf{x}; \mathbf{r}; \mathbf{s}\}$ *is the 3-skeleton of a resolution over G.*

For a presentation with identities $\{\mathbf{x}', \mathbf{x}_0; \mathbf{r}', \mathbf{r}_0; \mathbf{s}', \mathbf{s}_0\}$ of a system $\{G, G_0, \varphi\}$ we define an associated pair of complexes (X, Y) as follows. X is simply the associated complex of $\{\mathbf{x}; \mathbf{r}; \mathbf{s}\}$ where $\mathbf{x} = \mathbf{x}' \cup \mathbf{x}_0$, etc., and Y is the

[7] The topological interpretation suggests the formula $d\alpha(w) = (w-1)e$. An algebraic proof is easy. Both $d\alpha$ and the map $w \to (w-1)e$ are crossed homomorphisms $F \to X_0$; they coincide on the generators of F and are therefore identical.

submodule generated by e and those a_i, b_j and c_k which correspond to elements of $\mathbf{x}_0$, $\mathbf{r}_0$ and $\mathbf{s}_0$. It is easy to see that Y is a subcomplex and a direct summand, and that in fact $Y \approx Z(G) \otimes_{G_0} \bar{Y}$ where $\bar{Y}$ is the associated complex of the presentation $\{\mathbf{x}_0; \mathbf{r}_0; \mathbf{s}_0\}$ of G_0. We say that the set of identities is complete for the presentation of the system if $\mathbf{s}_0$ is complete for $\{\mathbf{x}_0; \mathbf{r}_0\}$ and $\mathbf{s}$ is complete for $\{\mathbf{x}; \mathbf{r}\}$. It is obvious that the pair of complexes associated with a system presentation and complete set of identities is the 3-skeleton of a resolution over the presented system.

2.4 *The diagonal chain-map.* Suppose that (X, Y) is a resolution over a system $\{G, G_0, \varphi\}$ whose 3-skeleton is associated with a presentation and complete set of identities for the system. In this section we present explicit formulas[8] which define on the 3-skeleton of (X, Y) a chain map $D^\#: (X, Y) \to (X \otimes X, Y \otimes Y)$ under the diagonal map $D: \{G, G_0, \varphi\} \to \{G, G_0, \varphi\} \times \{G, G_0, \varphi\}$.

Routine calculation shows that the function $g: F \times F \to X_1 \otimes X_1$ defined by $g(u, v) = \alpha(u) \otimes u\alpha(v)$ satisfies condition (2.3.2). Hence there is a unique function $\gamma: F \to X_1 \otimes X_1$ such that

$$\gamma(uv) = \gamma(u) + u\gamma(v) + \alpha(u) \otimes u\alpha(v) \tag{2.4.1}$$

and

$$\gamma(x_i) = 0 \qquad \text{for all } i .$$

The map $D^\#$ is defined on the basis elements of X as follows:

$$\begin{aligned} D^\#(e) &= e \otimes e \\ D^\#(a_i) &= e \otimes a_i + a_i \otimes x_i e \\ D^\#(b_j) &= e \otimes b_j + b_j \otimes e + \gamma(r_j) \\ D^\#(c_k) &= e \otimes c_k + c_k \otimes e + \textstyle\sum_1^n \varepsilon_m(\alpha(w_m) \otimes w_m b_{j_m} + w_m b_{j_m} \otimes \alpha(w_m)) \\ &\quad + \textstyle\sum_1^n \delta_m w_m(b_{j_m} \otimes \alpha(r_{j_m})) - \sum_{1 \leq l < m \leq n} \varepsilon_l w_l b_{j_l} \otimes \varepsilon_m w_m \alpha(r_{j_m}) , \end{aligned}$$

where c_k corresponds to the identity

$$s_k = \textstyle\prod_{m=1}^n w_m \rho_{j_m}^{\varepsilon_m} w_m^{-1}, \qquad \varepsilon_m = \pm 1 \qquad \text{and} \qquad \delta_m = \frac{1}{2}(\varepsilon_m - 1) .$$

(Fortunately in using this map for computing cup products we shall be interested chiefly in the component of $D^\#(c_k)$ lying in $X_1 \otimes X_2$, which is

[8] The method of contracting homotopies for constructing diagonal maps described in XI, 5 of [1] is of little use in the present context. Even in dimension 0, explicit description of a contracting homotopy for the complex associated with a presentation apparently requires solution of the word problem for the given presentation. Thus it seems likely that there is no effectively computable general algorithm for constructing such contracting homotopies. Our formulas were suggested by those of Kyle [15].

simply $\sum_1^n \varepsilon_m(\alpha(w_m) \otimes w_m b_{j_m})$.)

These formulas, plus the requirement that $D^\#$ be a homomorphism under the diagonal map $G \to G \times G$ determine the map. We must now verify that it is a chain map. It is sufficient to check that $\varepsilon' D^\# = \varepsilon$ and $d' D^\# = D^\# d$ on all the basis elements. (We write d', ε' for the boundary operator and augmentation in $X \otimes X$.)

In dimension 0,

$$\varepsilon' D^\#(e) = \varepsilon'(e \otimes e) = 1 = \varepsilon(e) .$$

In dimension 1,

$$\begin{aligned} d' D^\#(a_i) &= e \otimes (x_i - 1)e + (x_i - 1)e \otimes x_i e \\ &= x_i e \otimes x_i e - e \otimes e = D^\# d(a_i) . \end{aligned}$$

Define $\zeta\colon F \to X_1 \otimes X_1$ by $\zeta(u) = e \otimes \alpha(u) + \alpha(u) \otimes ue$. Calculation using (2.4.1) shows that $d'\gamma + \zeta$ is a crossed homomorphism $F \to X_1 \otimes X_1$. $D^\#\alpha$ is also a crossed homomorphism, and for each i, $D^\#\alpha(x_i) = (d'\gamma + \zeta)(x_i)$. It follows that $D^\#\alpha = d'\gamma + \zeta$. Thus in dimension 2,

$$\begin{aligned} d' D^\#(b_j) &= e \otimes \alpha(r_j) + \alpha(r_j) \otimes e + d'\gamma(r_j) \\ &= e \otimes \alpha(r_j) + \alpha(r_j) \otimes r_j e + d'\gamma(r_j) \\ &= (\zeta + d'\gamma)(r_j) = D^\#\alpha(r_j) = D^\# d(b_j) . \end{aligned}$$

In the preceding calculation we used the fact that for any $r \in R$ (the kernel of the map $F \to G$), $rx = x$ for any $x \in X$. Using the same fact, the following consequences of (2.4.1) are easily seen to be valid for any r, $r_m \in R$.

$$\text{(2.4.2)} \quad \begin{aligned} &\text{(i)} \ \gamma(r^\varepsilon) = \varepsilon\gamma(r) - \delta\alpha(r) \otimes \alpha(r) \ \text{ where } \varepsilon = \pm 1,\ \delta = \tfrac{1}{2}(\varepsilon - 1) \\ &\text{(ii)} \ \gamma(wrw^{-1}) = w\gamma(r) + \alpha(w) \otimes w\alpha(r) - w\alpha(r) \otimes \alpha(w) \\ &\text{(iii)} \ \gamma\left(\prod_1^n r_m\right) = \sum_1^n \gamma(r_m) + \sum_{1 \le l < m \le n} \alpha(r_l) \otimes \alpha(r_m). \end{aligned}$$

(Formula (i) with $\varepsilon = -1$ is obtained by expanding $0 = \gamma(r^{-1}r)$.) Hence if $s_k = \prod_1^n w_m \rho_{j_m}^{\varepsilon_m} w_m^{-1}$ is an identity for the presentation,

$$\begin{aligned} &\gamma\left(\prod_1^n w_m r_{j_m}^{\varepsilon_m} w_m^{-1}\right) \\ \text{(2.4.3)} \quad &= \gamma(1) = 0 \\ &= \sum_1^n \gamma(w_m r_{j_m}^{\varepsilon_m} w_m^{-1}) + \sum_{1 \le l < m \le n} \varepsilon_l w_l \alpha(r_{j_l}) \otimes \varepsilon_m w_m \alpha(r_{j_m}) \\ &= \sum_1^n \varepsilon_m w_m \gamma(r_{j_m}) + \sum_1^n \varepsilon_m\left(\alpha(w_m) \otimes w_m \alpha(r_{j_m}) - w_m \alpha(r_{j_m}) \otimes \alpha(w_m)\right) \\ &\quad - \sum_1^n \delta_m w_m\left(\alpha(r_{j_m}) \otimes \alpha(r_{j_m})\right) \\ &\quad + \sum_{1 \le l < m \le n} \varepsilon_l w_l \alpha(r_{j_l}) \otimes \varepsilon_m w_m \alpha(r_{j_m}) . \end{aligned}$$

Now,

$$\begin{aligned} d'D^{\sharp}c_k = {} & e \otimes \sum_1^n \varepsilon_m w_m b_{j_m} + \sum_1^n \varepsilon_m w_m b_{j_m} \otimes e \\ & + \sum_1^n \varepsilon_m w_m \alpha(r_{j_m}) \otimes \alpha(w_m) + \sum_1^n \varepsilon_m w_m b_{j_m} \otimes (w_m - 1)e \\ & - \sum_1^n \varepsilon_m \alpha(w_m) \otimes w_m \alpha(r_{j_m}) + \sum_1^n (w_m - 1)e \otimes \varepsilon_m w_m b_{j_m} \\ & + \sum_1^n \delta_m w_m(\alpha(r_{j_m}) \otimes \alpha(r_{j_m})) \\ & - \sum_{1 \leqq l < m \leqq n} \varepsilon_l w_l \alpha(r_{j_l}) \otimes \varepsilon_m w_m \alpha(r_{j_m}) , \end{aligned}$$

and taking (2.4.3) into account, this simplifies to

$$\begin{aligned} d'D^{\sharp}c_k & = \sum_1^n \{\varepsilon_m w_m(e \otimes b_{j_m}) + \varepsilon_m w_m(b_{j_m} \otimes e) + \varepsilon_m w_m \gamma(r_{j_m})\} \\ & = D^{\sharp}(\sum_1^n \varepsilon_m w_m b_{k_m}) = D^{\sharp} d c_k . \end{aligned}$$

To complete the discussion we need only remark that $D^{\sharp}$ carries Y into $Y \otimes Y$ (and *a fortiori* into $X \otimes Y + Y \otimes X$) and coincides with the result of "lifting" a diagonal map $\bar{Y} \rightarrow \bar{Y} \otimes \bar{Y}$, where $\bar{Y}$ is the auxiliary resolution over G_0.

3. Quasi-resolutions

This section is concerned with simplifications which are sometimes possible if the coefficient modules used satisfy certain restrictions. It is difficult to find a complete set of identities for a presentation (indeed there may not exist any finite complete set of identities [27]) so that in general the methods of the preceding section are effective for computing homology groups only in dimensions 0 and 1. We shall show how this difficulty can be largely evaded for the group systems of interest in knot theory.

Let a map $Z(G) \rightarrow \Lambda$ be given, where Λ is an arbitrary ring. Then any right (left) Λ-module can be considered a G-module *via* this map, and can be used as a coefficient module for homology of G.[9] Because of the natural isomorphisms $A \otimes_G X \approx (A \otimes_\Lambda \Lambda) \otimes_G X \approx A \otimes_\Lambda (\Lambda \otimes_G X)$ and $\mathrm{Hom}_G(X, B) \approx \mathrm{Hom}_G(X, \Lambda \otimes_\Lambda B) \approx \mathrm{Hom}_\Lambda(\Lambda \otimes_G X, B)$, valid for any right Λ-module A and left Λ-modules B, X [1; II, 5], the homology or cohomology of G with coefficients in any Λ-module can be obtained from $\Lambda \otimes_G X$, where X is any resolution over G. More generally, if there is a homotopy equivalence between $\Lambda \otimes_G X$ and some Λ-complex $\bar{X}$, then $\bar{X}$ will do just as well, even though there may be no resolution X' over G such that $\bar{X} \approx \Lambda \otimes_G X'$. (The simplification this may make possible is illustrated in §4.) We shall call any such $\bar{X}$ a Λ-*resolution* over G. To calculate cup-products using a Λ-resolution, one starts with a diagonal map for a resolution X over G. This induces a diagonal map for $\Lambda \otimes_G X$, which can then be transferred to a diagonal map for $\bar{X}$ *via* the postulated homotopy equivalence. The

[9] For simplicity we speak only of groups, but the results of this section apply equally well to group systems.

Λ-complex associated with a given presentation and set of identities is defined to be $\Lambda \otimes_G X$, where X is the usual associated complex. It is a free Λ-module on the "same" generators as X, and the boundary operator is given by the same formulas, except that the coefficients must be interpreted as their images in Λ rather than as their images in $Z(G)$.

We call a Λ-module K *residual* if for some finite n, $\Lambda^n \oplus K \approx \Lambda^n$, and Λ itself *non-residual* if there are no non-trivial residual Λ-modules. If K is a residual G-module, it follows from the distributivity of tensor product over direct sums that $\Lambda \otimes_G K$ is a residual Λ-module.

A *quasi-resolution* over a group G is a projective augmented G-complex (X, d) such that the augmentation ε maps X_0 onto Z, and the kernel of d_n (kernel of ε for $n = 0$) is the direct sum of the image of d_{n+1} and a residual G-module K_n. Let K'_n be isomorphic to K_n, and construct a new complex (X', d') with $X'_0 = X_0$, $X'_n = X_n \oplus K'_{n-1}$ for $n \geqq 1$, $d'|X_n = d$, and $d'|K'_n$ an isomorphism onto K_n. The result is acyclic, and is actually a resolution over G since residual modules are direct summands of free modules and are therefore projective. If Λ is a non-residual ring, $\Lambda \otimes_G X'$ and $\Lambda \otimes_G X$ are isomorphic, and consequently *if X is a quasi-resolution over G and Λ is non-residual then $\Lambda \otimes_G X$ is a Λ-resolution over G.* The point is that, as will be shown below, a quasi-resolution is sometimes easier to obtain than a resolution.

Any Λ which is finite-dimensional and torsion-free over an integral domain R is non-residual, for it is easily seen that a residual Λ-module would be torsion-free and zero-dimensional over R and hence trivial. In particular, group rings of finite groups and rings of matrices with coefficients in an integral domain are non-residual. If H is a free abelian group, $Z(H)$ is an integral domain. If H is the direct product of a free abelian group H_1 and a finite group H_2, then $Z(H)$ has dimension equal to the order of H_2 over the integral domain $Z(H_1)$ and is therefore non-residual. In particular, the group ring of any finitely generated abelian group is non-residual. Conceivably, all group rings are non-residual, but this appears to be a very difficult question. An affirmative answer would of course make the distinction between resolutions and quasi-resolutions vacuous.

We call a group presentation aspherical if, when it is taken with the empty set of identities, the associated complex (2.3.5) is acyclic. (The terminology is suggested by the fact that a presentation is aspherical if and only if the geometric complex described in 2.2 is an aspherical space.)

In general let K be the kernel of $d_2: X_2 \to X_1$ in the complex X associated with a presentation and empty set of identities, so that $K = 0$ if and only if the presentation is aspherical. We shall be concerned with the change produced in K when the presentation is altered by adding a new generator and defining relator (Tietze operation [6] of type I) or adding a new relator which is a consequence of the others (Tietze operation of type II). (We shall not use the inverse operations of deleting generators or relators.) Whichever the type of operation, X may be considered a subcomplex of X'. If the operation is of type I, then $K' = K$. If it is of type II, then $K' = K \oplus Z(G)$ and the direct sum splitting may be chosen in such a way that the summand $Z(G)$ is generated by $\beta(s)$, where s is any given identity exhibiting the new relator as a consequence of the others. We omit the proofs, which are straightforward.

From now on we shall consider only finite presentations and finitely presentable groups. The *deficiency of a presentation* is the number of generators minus the number of relators. The deficiency of a group is the maximum of the deficiencies of its finite presentations [25]. Note that a Tietze operation of type I does not change the deficiency of a presentation, while a type II operation decreases the deficiency by one.

For any two finite presentations of the same group, there exists a third presentation obtainable from either of the first two by a finite sequence of operations of types I and II [6, p. 198]. Let K, K', K'' be the respective associated kernels, and n, n', n'' the respective deficiencies. There must be $n - n''$ type II operations used in going from the first presentation to the third, and $n' - n''$ in going from the second to the third. Consequently $K'' \approx K \oplus [Z(G)]^{n-n''} \approx K' \oplus [Z(G)]^{n'-n''}$. If the first presentation is aspherical, $K = 0$ and $[Z(G)]^{n-n''} \approx K' \oplus [Z(G)]^{n'-n''}$. Tensoring with Z over G gives $Z^{n-n''} \approx Z \otimes_G K' \oplus Z^{n'-n''}$, which implies $n \geqq n'$ and shows that *any aspherical presentation has maximum deficiency.* If the first presentation is aspherical and the second has the same deficiency, K' is a residual G-module. This gives the partial converse: *if G has an aspherical presentation then the complex associated with any presentation of maximum deficiency is a quasi-resolution over G.*

Papakyriakopoulos [18] defines a non-empty proper closed subset B of S^3 to be (geometrically) *splittable* if there is a semi-linear 2-sphere $S^2 \subset S^3 - B$, such that both components of $S^3 - S^2$ contain points of B. Let B be a tame unsplittable graph in S^3 with μ components and 1-dimensional Betti number r. Then $\pi_1(S^3 - B)$ is an aspherical group with deficiency $r - \mu + 1$. This follows readily from the asphericity of $S^3 - B$ [18], since $S^3 - B$ has the homotopy type of a finite 2-complex (which must also be aspherical) and, by Alexander duality, has Euler characteristic $\mu - r$. A complex of the same homotopy type with a single vertex must have $r - \mu + 1$ more 1-cells than 2-cells to have the proper characteristic. The derived presentation (which will be aspherical) has therefore $r - \mu + 1$ more generators then relators; i.e., it has deficiency $r - \mu + 1$. Since the presentation is aspherical it has maximum deficiency and actually gives the deficiency of the group.

4. Group systems of knots; abelian coefficients

By an *abelian* G-module A we mean a module such that $xya = yxa$ for all $x, y \in G$ and $a \in A$. An abelian G-module may be considered a module over $Z(H)$, where H is the commutator quotient group of G, and for calculating homology or cohomology with abelian coefficients, we may apply the results of the preceding section by taking Λ to be $Z(H)$.

We shall obtain a presentation for the group system of a knot such that the associated complex X is a quasi-resolution and $Z(H) \otimes_G X$ has a particularly simple form. Since H is a finitely generated abelian group, $Z(H)$ is a non-residual ring, so that we obtain a $Z(H)$-resolution over the system, in the sense of §3.

Let F be a tame oriented surface with boundary the oriented knot K, and let h be its genus. Seifert [22] showed how such a surface could be constructed for any tame knot; for a more detailed description see [3]. F is topologically equivalent to a disc with h pairs of bands. Its fundamental group is free on $2h$ generators $u_1, \cdots, u_{2h}$ which may be chosen, as shown in Figure 1, in such a way that the product of commutators $y =$

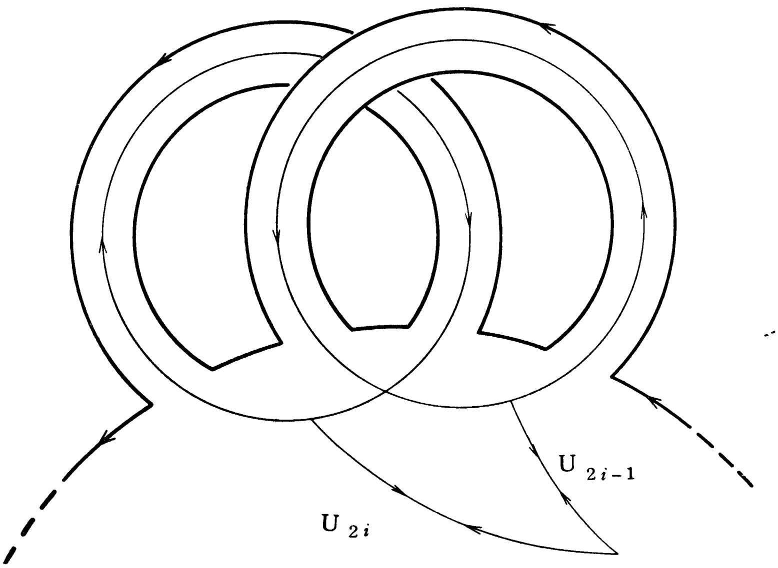

Figure 1

$\prod_{i=1}^{h}[u_{2i-1}, u_{2i}]$ is represented by a loop running parallel to the boundary of the surface. A presentation of $G = \pi_1(S^3 - K)$ can be obtained by expressing $S^3 - K$ as the union of two open subsets, and applying the van Kampen theorem [2]. We take one subset, V, to be $S^3 - F$, and the other, U, to be a neighborhood of F in $S^3 - K$ which has F as a strong deformation retract, together with a tubular neighbourhood of a small circle linking K once. Then the intersection of U and V is $U - F$ and consists of one layer "above" F and one "below", joined by a bridge in $S^3 - F$. (It is to be understood that "above" and "below" are defined in terms of the orientation of F and S^3.)

Take a base-point p in the upper layer of $U \cap V$ and let q be a point directly below it in the lower layer. The group $\pi_1(U)$ is free on $2h + 1$ generators. One generator, x, is represented by a loop which runs around the bridge in $U \cap V$ from p to q and then returns to p by cutting directly through F, and the other $2h$ generators may be identified with the generators $u_1, \cdots, u_{2h}$ of $\pi_1(F)$. The group $\pi_1(U \cap V)$ is free on $4h$ generators $u_1^\sharp, \cdots, u_{2h}^\sharp, u_1^\flat, \cdots, u_{2h}^\flat$ where $u_i^\sharp$ is represented by a loop running parallel to u_i in the upper layer, and $u_i^\flat$ is represented by a loop which goes *via* the bridge to q, runs parallel to u_i in the lower layer, and returns to p *via* the bridge. The space V has the homotopy type of the complement in S^3 of a connected graph with 1-dimensional Betti number $2h$. Hence, by the last paragraph of § 3, $\pi_1(V)$ has a presentation of deficiency $2h$. By Alexander duality we can choose $2h$ generators $v_1, \cdots, v_{2h}$ so that the linking number of the cycles represented by u_i and v_j is δ_{ij}, while the remaining generators $v_{2h+1}, \cdots, v_{2h+k}$ can be chosen to lie in the commutator subgroup of $\pi_1(V)$. There will then be k relators in each of which the v_i with $i \leqq 2h$ occur with net index zero.

According to the van Kampen theorem, one obtains a presentation for $G = \pi_1(U \cup V)$ by taking presentations for $\pi_1(U)$ and $\pi_1(V)$ and adjoining relators $i_*(u)i'_*(u^{-1})$ for each generator u of $\pi_1(U \cap V)$ where i_* and i'_* are the homomorphisms of the fundamental groups induced by the inclusions of $U \cap V$ in V and U respectively. Under i'_*, $u_i^\sharp$ is identified with u_i and $u_i^\flat$ with xu_ix^{-1}. We shall simply write $u_i^\sharp, u_i^\flat$ for $i_*(u_i^\sharp), i_*(u_i^\flat)$ and from now on consider them as abbreviations for certain words in the generators v_i. Then the u_i may be eliminated along with the relators identifying u_i with $u_i^\sharp$. The final result is a presentation for G with $2h + k + 1$ generators $x, v_1, \cdots, v_{2h+k}$ and $2h + k$ relators r_i. For $i = 1, \cdots, 2h$, the relators are $r_i = xu_i^\sharp x^{-1}(u_i^\flat)^{-1}$, while the last k relators are those of the presentation of $\pi_1(V)$ and do not contain the generator x.

A peripheral subgroup [9] is generated by x and $y = \prod_{i=1}^{h}[u_{2i-1}^\sharp, u_{2i}^\sharp]$, and is presented by these generators and the relator $r_s = [x, y]$. We thus obtain a presentation of the group system consisting of G, the peripheral subgroup, and the inclusion map, by adding the generator y, the defining relator $r_y = y^{-1}\prod_{i=1}^{h}[u_{2i-1}^{\sharp}, u_{2i}^{\sharp}]$, and the relator r_s to the presentation of G given above. If r_s is omitted, the result is a presentation of G of deficiency 1, which is maximum deficiency for a knot group. Thus it is only necessary to find an identity expressing r_s as a consequence of the other relators to obtain a presentation and set of identities whose associated complex will be a quasi-resolution over the system.

The identity arises from the obvious geometric fact that $\prod_{i=1}^{h}[u_{2i-1}^{\sharp}, u_{2i}^{\sharp}]$ and $\prod_{i=1}^{h}[u_{2i-1}^{\flat}, u_{2i}^{\flat}]$ represent the same element of $\pi_1(V)$. Hence (using F, P, ψ and ρ_j as in 2.3) there is an R in the consequence of P in $P*F$ with the property that $\psi(R) = \prod_{i=1}^{h}[u_{2i-1}^{\flat}, u_{2i}^{\flat}](\prod_{i=1}^{h}[u_{2i-1}^{\sharp}, u_{2i}^{\sharp}])^{-1}$. This R does not contain x or any of the ρ_j with $j \leqq 2h$.

Define

$$w_i = \prod_{j=1}^{i}[u_{2j-1}, u_{2j}] \qquad \text{(with } w_0 = 1\text{)}$$

and

$$W_i = w_{i-1}\rho_{2i-1}w_{i-1}^{-1}\cdot w_{i-1}u_{2i-1}\rho_{2i}(w_{i-1}u_{2i-1})^{-1}\cdot \cdot(w_iu_{2i})\rho_{2i-1}^{-1}(w_iu_{2i})^{-1}\cdot w_i\rho_{2i}^{-1}w_i^{-1} .$$

Direct calculation gives

$$\psi(W_i) = w_{i-1}x[u_{2i-1}^{\sharp}, u_{2i}^{\sharp}]x^{-1}w_i^{-1} ,$$

so that

$$\psi(\prod_{i=1}^{h}W_i) = x(\prod_{i=1}^{h}[u_{2i-1}^{\sharp}, u_{2i}^{\sharp}])x^{-1}w_h^{-1} .$$

From this it follows that

$$(4.1) \qquad (\prod_{i=1}^{h}W_i)\cdot R\cdot y\rho_y y^{-1}\cdot yx\rho_y^{-1}(yx)^{-1}\cdot \rho_s^{-1}$$

is an identity.

Let (X, Y) be the pair of $Z(H)$-complexes associated with this presentation and identity which, as was remarked above, is a $Z(H)$-resolution over the system. X is a free $Z(H)$-module with one generator e in dimension 0, generators $a_x, a_y, a_1, \cdots, a_{2h+k}$ corresponding to $x, y, v_1, \cdots, v_{2h+k}$ in dimension 1, generators $b_y, b_s, b_1, \cdots, b_{2h+k}$ corresponding to $r_y, r_s, r_1, \cdots, r_{2h+k}$ in dimension 2, and a generator c in dimension 3 corresponding to the identity (4.1). The subcomplex Y is generated by e, a_x, a_y and b_s. H is an infinite cyclic group; x maps into a generator of H which we shall denote by t, while all other generators of the presentation map into the identity in H. As a consequence, crossed homomorphisms into X behave as multiplicative-to-additive homomorphisms when applied to words not involving the generator x. Because of this, the boundary operator can be described by comparatively simple formulas as follows.

$$\begin{aligned}
&\varepsilon(e) = 1 \\
&d(a_x) = (t-1)e \qquad d(a_y) = d(a_i) = 0 \\
&d(b_y) = -a_y \qquad d(b_s) = (t-1)a_y \\
&d(b_i) = \textstyle\sum_{j=1}^{2h}(p_{ij}t - p'_{ij})a_j + \sum_{j=2h+1}^{2h+k}(q_{ij}t - q'_{ij})a_j \qquad (i \leqq 2h) \\
& = \textstyle\sum_{j=2h+1}^{2h+k} q_{ij}a_j \qquad (i > 2h) \\
&d(c) = \textstyle\sum_{i=2h+1}^{2h+k} q_i b_i - b_s - (t-1)b_y \,.
\end{aligned}$$

In these formulas, for $i \leqq 2h$, p_{ij} and q_{ij} give the net index of v_j in $u_i^\sharp$, while p'_{ij} and q'_{ij} give the net index of v_j in $u_i^\flat$. For $i > 2h$, q_{ij} is the net index of v_j in r_i; as remarked before, this is zero for $j \leqq 2h$. The q_i arise from the term R in (4.1).

Because of the particular choice of the generators v_j, p_{ij} is the linking number of $u_i^\sharp$ with u_j (or $u_j^\flat$) so that *the matrix with entries p_{ij} is identical with the linking matrix defined by Seifert* [22.] We shall call it V in agreement with his notation. The linking number of $u_i^\flat$ with u_j is p'_{ij}. This is obviously the same as the linking number of u_i with $u_j^\sharp$, so the transposed matrix V' has entries p'_{ij}. The difference $p_{ij} - p'_{ij}$ is the intersection number of u_i with u_j on F, so that $V - V'$ is the skew-symmetric unimodular matrix S (the negative of Seifert's Δ [23]) consisting of h copies of $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ along the main diagonal.

The homology of G with simple integer coefficients is that of the complex $Z \otimes_H X$ which is a free abelian group on the "same" generators as X, with boundary operator defined by the formulas above with t replaced by 1. Since $H_1(G; Z) \approx Z$, the $k \times k$ matrix Q with entries q_{ij}, $i, j = 2h+1, \cdots, 2h+k$, must be unimodular, for if any of its elementary divisors were different from ± 1 then $H_1(G; Z)$ would have at least one additional cyclic factor. This fact and the relation $d^2(c) = 0$ imply that all q_i are zero, so that actually

$$d(c) = -b_s - (t-1)b_y \,.$$

Let $\bar{X}$ be the submodule of X generated by the basis elements other than the a_i, b_i with $i > 2h$, and make it a complex by using the same formulas as above for the boundary operator, replacing these a_i and b_i by zero wherever they occur. We assert that *there is a homotopy equivalence between the pairs of $Z(H)$-complexes (X, Y) and $(\bar{X}, Y)$.* To exhibit the equivalence we define $f: X \to \bar{X}$ to be the identity on $\bar{X}$ while $f(a_i) = f(b_i) = 0$ for $i > 2h$. The inverse map $g: \bar{X} \to X$ is the injection except in dimension 2, where it is defined by

$$g(b_i) = b_i + \sum_{j,j'=2h+1}^{2h+k} (q_{ij}t - q'_{ij})(Q^{-1})_{jj'} b_{j'} .$$

It is easily verified that f and g are chain maps, and that fg is the identity on $\bar{X}$. A homotopy between gf and the identity on X is obtained by sending a_i into $\sum_{j=2h+1}^{2h+k}(Q^{-1})_{ij}b_j$ for $i > 2h$, and sending all the other basis elements of X into zero.

To calculate cup products in dimension 3 we need to know the component of $D^\sharp(c)$ lying in $X_1 \otimes X_2$. The formula of § 2.4 applied to (4.1) simplifies greatly since in the $Z(H)$-module X, $\alpha([u_{2j-1}, u_{2j}])$ reduces to 0 for all j. We obtain

$$\begin{aligned}(D^\sharp(c))_{1,2} = \sum_{j=1}^{h}\{-\alpha(u^\flat_{2j}) \otimes b_{2j-1} + \alpha(u^\flat_{2j-1}) \otimes b_{2j}\} \\ + a_y \otimes b_y - (a_y + a_x) \otimes tb_y \\ + \text{terms involving } b_j \quad \text{with } j > 2h .\end{aligned}$$

The last terms (arising from R in (4.1)) are mapped into zero under the homotopy equivalence of the preceding paragraph. The first terms can be written as $\sum_{i,j=1}^{2h}(VS)_{ij}(a_i \otimes b_j)$. As will appear, use of the identity $S = (V' - V)^{-1}$ results in a more invariant form for this expression. We summarize these calculations in

PROPOSITION 4.1. *Let K be a tame oriented knot in S^3, and $\{G, G', i\}$ its group system, with $G = \pi_1(S^3 - K)$, G' a peripheral subgroup, and i the inclusion map. Let t be a generator of $H = G/[G,G] = H_1(S^3 - K)$. Then if V is the linking matrix obtained from a Seifert surface for K of genus h, the pair of $Z(H)$-complexes (X, Y) given below is a $Z(H)$-resolution over the system $\{G, G', i\}$.*

X is freely generated over $Z(H)$ by a generator e in dimension 0, $2h + 2$ generators a_x, a_y, $a_1, \cdots, a_{2h}$ in dimension 1, $2h + 2$ generators b_y, b_s, $b_1, \cdots, b_{2h}$ in dimension 2, and one generator c in dimension 3. The boundary operator is given by

$$\begin{array}{lll} d(e) = 0 & d(a_x) = (t-1)e & d(a_y) = d(a_i) = 0 \\ d(b_y) = -a_y & d(b_s) = (t-1)a_y & \\ d(b_i) = \sum_{j=1}^{2h}(tV - V')_{ij}a_j & & d(c) = -b_s - (t-1)b_y \end{array}$$

where $i = 1, 2, \cdots, 2h$. The subcomplex Y is generated by e, a_x, a_y and b_s. The image of c under the diagonal map has

$$\sum_{i,j=1}^{2h}[V(V' - V)^{-1}]_{ij}(a_i \otimes b_j) + a_y \otimes b_y - (a_y + a_x) \otimes tb_y \tag{4.2}$$

for its component in $X_1 \otimes X_2$.

The first homology group of the infinite cyclic covering of $S^3 - K$ may be identified with $H_1(G; Z(H)) = H_1(X)$, and calculated immediately from the above. The 1-cycles are generated by a_y and the a_i, and the boundaries are generated by a_y and the $d(b_i)$. Hence $tV - V'$ is a relation matrix for $H_1(G; Z(H))$ (considered as a $Z(H)$-module).

It follows that the elementary ideals [5] of $tV - V'$ are invariants of the group G. In particular, the elementary ideal of deficiency 0, which is the principal ideal generated by $\det(tV - V')$, is an invariant. The generator of this ideal is determined up to a unit factor $\pm t^n$ and we define $\Delta(t)$, the Alexander polynomial of K, to be the unique generator which has no terms of negative degree, a non-zero constant term, and the value $+1$ for $t = 1$. (Seifert [22] chose the sign so as to make the leading coefficient positive; our choice works out more naturally in some formulas.) If V is non-singular, $\Delta(t) = \det(tV - V')$ since the constant term is $\det(-V') \neq 0$ and $\Delta(1) = \det(V - V') = +1$ since the matrix is unimodular and skew-symmetric.

Seifert made use of a matrix $\Gamma = -VS = -V(V' - V)^{-1}$. Since S is unimodular,

$$-(tV - V')S = (V' - V + V - tV)(V' - V)^{-1} = E - \Gamma + t\Gamma$$

is also a relation matrix for the first homology group of the infinite cyclic covering of $S^3 - K$; this is one of Seifert's results [22].

The formulas of Proposition 4.1 can be applied to any square integer matrix V such that $V - V'$ is unimodular to yield a pair of $Z(H)$-complexes (X, Y) with a selected element (4.2) in $X_1 \otimes X_2$. We shall say that two such matrices are *h-equivalent* if there is a homotopy equivalence between their related complexes which carries the selected element of one onto the selected element of the other. In examining sufficient conditions for h-equivalence, it is clearly permissible to restrict attention to the subcomplex generated by the a_i and b_i.

If a change of basis is made, so that $a_i = \sum_k P_{ik}\bar{a}_k$, $b_j = \sum_k Q_{jk}\bar{b}_k$ (P, Q unimodular), then the boundary operator has matrix $P'(tV - V')(Q^{-1})'$ with respect to the new basis $\{\bar{a}_i\}$, $\{\bar{b}_j\}$ and the selected element is $\sum_{ij}[P'V(V' - V)^{-1}Q]_{ij}\bar{a}_i \otimes \bar{b}_j$. Taking $Q = (P^{-1})'$ and setting $W = P'VP$, the matrix for the boundary operator is $tW - W'$, and for the selected element is $W(W' - W)^{-1}$. We have shown that *if P is unimodular, V and $P'VP$ are h-equivalent*. We next show that if V is singular, it is h-equivalent to a matrix U partitioned in the form

$$U = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & q & W \end{bmatrix} \tag{4.3}$$

where U is $2n \times 2n$, W is $2n-2 \times 2n-2$ and q is $2n-2 \times 1$.[10] For any V there exist unimodular Q, R such that QVR has diagonal form; and if V is singular, the first row can be made to be 0. Then QVQ' also has a zero first row. $QVQ' - (QVQ')' = Q(V - V')Q'$ is skew-symmetric and unimodular, and its first column is the same as the first column of QVQ' since the first row of the latter is zero. The first entry in this column is 0, and because of the unimodularity the g.c.d. of the other entries in the column must be 1. Hence there exists a sequence of elementary row operations on QVQ', not affecting the first row, which makes the second entry of the first column -1 and the rest 0. These operations can be effected by premultiplication by a unimodular P; postmultiplication by P' does not affect the first column, and so $PQVQ'P'$ agrees with U in its first row and column. Postmultiplication by a suitable R' will add appropriate multiples of the first column to the others, and reduce all but the first entry of the second row to 0. Premultiplication by R will have no effect since the first row is zero, and we finally obtain $U = (RPQ)V(RPQ)'$ in the required form.

Let A, B be the submodules of the complex constructed from U generated by $a_1, \cdots, a_{2n}$ and $b_1, \cdots, b_{2n}$ respectively. Consider the submodules A', B' generated by $a_3, \cdots, a_{2n}$ and $b_3, \cdots, b_{2n}$ as a complex with boundary operator given by the matrix $tW - W'$. Define $f\colon A, B \to A', B'$ by $f(b_1) = f(b_2) = f(a_2) = 0$, $f(a_1) = -t\sum_3^n q_i a_i$, and $f(a_i) = a_i$, $f(b_i) = b_i$ for $i \geqq 3$. Define $g\colon A', B' \to A, B$ by $g(a_i) = a_i$, $g(b_i) = b_i - t^{-1}q_i b_1$, $i \geqq 3$. Then f and g are chain maps, fg is the identity, and by sending a_1 into b_2, a_2 into $-t^{-1}b_1$, and all the other generators into zero, one obtains a homotopy between gf and the identity on A, B. Hence f is a homotopy equivalence. The matrix $U(U' - U)^{-1}$ partitions into

$$\begin{bmatrix} 0 & 0 & 0 \\ -q'(W' - W)^{-1}q & -1 & 0 \\ -q + W(W' - W)^{-1}q & 0 & W(W' - W)^{-1} \end{bmatrix},$$

from which it is easy to see that f takes the selected element determined by U into the selected element determined by W. Consequently if (4.3) holds, then U and W are h-equivalent. Together with the result of the preceding paragraph, this shows that *every linking matrix is h-equivalent to a non-singular matrix.*[11] Note that the proof provides an

[10] To avoid special treatment of an obvious trivial case we make the convention that there is a 0×0 matrix, which is symmetric, skew-symmetric, and has determinant $+1$.

[11] It is an easy consequence of this remark and those following Theorem 3 in §5 that Fox's results on the homology of cyclic coverings of knots of genus one [10] generalize to all knots with Alexander polynomial of degree 2.

algorithm for reducing such a matrix to a non-singular one.

In deriving Proposition 4.1, we used a canonical set of generators for $\pi_1(F)$, and the matrix V can be thought of as giving a bilinear form on $H_1(F)$, relative to a canonical basis. If another basis is chosen, the linking matrix obtained will have the form $P'VP$ for some unimodular P, and will therefore be h-equivalent to V. We have proved:

PROPOSITION 4.2. *The conclusions of Proposition* 4.1 *hold if V is any matrix h-equivalent to the linking matrix obtained from any basis for the first homology group of any Seifert surface for K. It is always possible to reduce such a V to a non-singular one.*

We shall call a matrix of the type referred to in Proposition 4.2 a *Seifert matrix for K*. The actual matrix is of course not an invariant of K, since if V is a Seifert matrix for K, then so is any $\bar{V}$ which is congruent to it. The rest of this section is devoted to the proof of a partial converse of this remark (Theorem 2).

From now on we suppose V to be a non-singular $2h \times 2h$ Seifert matrix for K. Then $\det(tV - V') = \Delta(t)$ is an invariant of K and so is its leading coefficient $\det(V)$. Let R be an integral domain in which $\det(V)$ is a unit. (For example, we can always take R to be the rational numbers, or the p-adic integers for any prime not dividing $\det(V)$, while if $\det(V) = \pm 1$ we can take $R = Z$.) V is then a unimodular matrix over R.

We shall consider the cohomology of the relative complex X/Y (which is isomorphic to the subcomplex of X generated by the a_i, the b_i, b_y and c, with $d(b_y)$ redefined as 0 and $d(c)$ redefined as $-(t-1)b_y$) with coefficients in the exact sequence

$$(4.4) \qquad 0 \longrightarrow R(H) \xrightarrow{\Delta} R(H) \xrightarrow{j} \Lambda \longrightarrow 0 ,$$

where $\Delta\colon R(H) \to R(H)$ is multiplication by $\Delta(t)$, and Λ is the quotient of $R(H)$ by the image of Δ. The 1-cochains and 2-cocycles with coefficients in $R(H)$ can be represented by $2h$-dimensional column vectors with entries in $R(H)$ by making the column vector with i^{th} entry δ_{ij} correspond to the cochain which is 1 on a_j (or b_j) and 0 on the other generators. (Since R is an integral domain, $(t-1)$ is not a divisor of zero in $R(H)$, so every 2-cocycle must vanish on b_y.) Denoting these column vectors by bold-face type, the coboundary is given by the formula $\delta(\mathbf{a}) = (tV - V')\mathbf{a}$. Define $T = V'V^{-1}$. Then for any 2-cocycle $\mathbf{b}$, $(t - T)\mathbf{b} = (tV - V')V^{-1}\mathbf{b} = \delta(V^{-1}\mathbf{b})$, so that $t\mathbf{b} \equiv T\mathbf{b}$ modulo coboundaries. It follows that every $\mathbf{b}$ is cohomologous to one with entries in R (considered as a subring of $R(H)$). On the other hand, since V and V' are non-singular there is no non-trivial coboundary with all entries in R. Hence $H^2(X/Y; R(H))$ can be identified with the $2h$-dimensional column vectors over R and is a free $2h$-dimensional R-module. It also has the structure of an $R(H)$-module, scalar multiplication being defined by $p(t)\cdot\mathbf{b} = p(T)\mathbf{b}$. (This shows that the similarity class of T as a matrix over R is an invariant.) Since V and V' are non-singular, $H^1(X/Y;\, R(H)) = 0$, and part of the exact sequence derived from 4.4 is

$$0 \longrightarrow H^1(X/Y;\, \Lambda) \xrightarrow{\delta^*} H^2(X/Y;\, R(H)) \xrightarrow{\Delta^*} H^2(X/Y;\, R(H))\ .$$

$\Delta(t)$ is a scalar multiple of the characteristic polynomial of T, so the Cayley-Hamilton theorem implies that Δ^* annihilates $H^2(X/Y;\, R(H))$, and it follows by exactness that δ^* is an isomorphism onto. We claim that ${\delta^*}^{-1}(\mathbf{b}) = \operatorname{adj}(tV - V')\mathbf{b}$ in the sense that the right side is a 1-cochain with coefficients in $R(H)$ which represents the required element of $H^1(X/Y;\, \Lambda)$. By the definition of δ^*, this follows immediately from the equation

$$\delta\big(\operatorname{adj}(tV - V')\mathbf{b}\big) = (tV - V')\operatorname{adj}(tV - V')\mathbf{b} = \Delta(t)\mathbf{b}\ .$$

Next we define a bilinear form on $H^2(X/Y;\, R(H))$ by a modification of Kyle's procedure.[12] The first step is to define a homomorphism $\lambda\colon \Lambda \otimes R(H) \to R$ under the diagonal map of G into $G \times G$, considering Λ and $R(H)$ as $Z(H)$-modules and hence G-modules in the natural way, and R as a G-module with trivial action. For $p(t) \in R(H)$, let $\bar{p}(t)$ be the "conjugate" polynomial $p(t^{-1})$ and define $\lambda_1\colon \Lambda \otimes R(H) \to \Lambda$ by $\lambda_1(q(t) \otimes p(t)) = q(t)\bar{p}(t)$, where $q(t)$ and $q(t)\bar{p}(t)$ are taken modulo $\Delta(t)$. We have

$$\lambda_1\big(t(q(t) \otimes p(t))\big) = \lambda_1\big(tq(t) \otimes tp(t)\big) = tq(t)t^{-1}\bar{p}(t) = \lambda_1\big(q(t) \otimes p(t)\big)\ ,$$

which shows that λ_1 is a map under the diagonal when its range is considered a G-module with trivial action. The residue classes of $1, t, \cdots, t^{2h-1}$ form an R-basis for Λ, which is canonical in the sense that it depends only on the choice of the generator t in H. This gives a canonical R-homomorphism $\lambda_2\colon \Lambda \to R$ with $\lambda_2(1) = 1$ and $\lambda_2(t^i) = 0$ for $i = 1, 2, \cdots, 2h-1$. We now take $\lambda = \lambda_2\lambda_1$, and consider the associated $\smile_\lambda\colon H^1(X/Y;\, \Lambda) \otimes H^2(X/Y;\, R(H)) \longrightarrow H^3(X/Y;\, R)$.

$H_3(X/Y;\, R)$ is isomorphic to R and is generated by the fundamental cycle c, which is uniquely determined (up to a choice of sign) as the image of the generator of $H_3(X/Y;\, Z)$ under the map induced by the inclusion

[12] See [15]. The possibility of extending these methods to local coefficients is mentioned in [13], but the idea is not developed there. In an unpublished manuscript, *Linking invariants of manifolds with operators*, Fox has discussed an analogous form defined in terms of classical intersection theory.

$Z \to R$. The choice of sign of course corresponds to a choice of orientation in S^3. We obtain an R-valued bilinear form on $H^2(X/Y; R(H))$ by defining

$$L(b_1, b_2) = (\delta^{*-1}(b_1) \smile_\lambda b_2)(c) .$$

The explicit calculation is quite simple. Referring to (4.2) we note first that if $\mathbf{a}, \mathbf{b}$ are column vectors over $R(H)$ representing elements of $H^1(X/Y; \Lambda)$, $H^2(X/Y; R(H))$, then $(\mathbf{a} \smile_\lambda \mathbf{b})(c)$ is obtained by evaluating the matrix product $\mathbf{a}' V(V' - V)^{-1}\bar{\mathbf{b}}$ in $R(H)$ ($\bar{\mathbf{b}}$ denoting the conjugate of $\mathbf{b}$), reducing the result modulo $\Delta(t)$, and then applying λ_2. Thus if $\mathbf{b}_1$, $\mathbf{b}_2$ are column vectors over R representing $b_1, b_2 \in H^2(X/Y; R(H))$, we must consider the product $\mathbf{b}_1'[\operatorname{adj}(tV - V')]' V(V' - V)^{-1}\bar{\mathbf{b}}_2$. (Since $\mathbf{b}_2$ has entries in R, $\bar{\mathbf{b}}_2 = \mathbf{b}_2$.) The matrix $\operatorname{adj}(tV - V')$ consists of $(2h - 1)$-dimensional minors of $tV - V'$ and does not contain powers of t higher than $2h - 1$ or less than 0. Hence the result of applying λ_2 to the product is simply to replace t by 0. Since V is even-dimensional $\det(V) = \det(-V') = \operatorname{adj}(-V')(-V')$. Hence $[\operatorname{adj}(-V')]' = -\det(V)V^{-1}$ and $(Lb_1, b_2) = -\det(V)\mathbf{b}_1'(V' - V)^{-1}\mathbf{b}_2$.

The formula shows that L is skew-symmetric, whereas one might expect symmetry by analogy with the self-linking of 1-cycles or their dual 2-cocycles in a 3-manifold. The explanation lies in the fact that λ is not a symmetric pairing. From its definition, L should be invariant under the action of G, that is $L(tb_1, tb_2)$ should equal $L(b_1, b_2)$. This property may be verified directly from the formula. It is equivalent to $T'(V' - V)^{-1}T = (V' - V)^{-1}$, or $T^{-1}(V' - V)(T')^{-1} = V' - V$, which follows at once from the definition of T as $V'V^{-1}$.

We summarize these results as

THEOREM 1. *Let K be a tame knot in S^3 with group system $\{G, G', i\}$. Let t be a generator of $H = G/[G, G] = H_1(S^3 - K)$. Suppose V is a non-singular $2h \times 2h$ Seifert matrix for K, and let R be an integral domain in which* $\det(V)$ *is a unit. Then the $R(H)$-module $H^2(\{i\}; R(H))$ carries an R-valued bilinear form L which is uniquely determined by choice of the generator t and an orientation for S^3. As an R-module, $H^2(\{i\}; R(H))$ is free on $2h$ generators and possesses a basis with respect to which multiplication by t has the matrix $T = V'V^{-1}$, and L has the matrix* $-\det(V)(V' - V)^{-1}$.

If a different R-basis is chosen for M, the action of t will be given by $P^{-1}TP$ and the matrix of L will be $-\det(V)P'(V' - V)^{-1}P$, for some matrix P unimodular over R. Thus if W is another non-singular Seifert matrix for K, there is some P such that $W'W^{-1} = P^{-1}V'V^{-1}P$ and $(W' - W)^{-1} = P'(V' - V)^{-1}P$. (Recall that $\det(W)$ must equal $\det(V)$

since both are equal to the leading coefficient of the Alexander polynomial.) W' can be expressed as $(E - W'W^{-1})^{-1}(W'W^{-1})(W - W')$, and by expressing $W'W^{-1}$ and $W - W'$ in terms of V and P this can be reduced to $P^{-1}V'(P^{-1})'$. Thus $V = PWP'$, and V and W are unimodularly congruent over R. This proves

THEOREM 2. *If K, $\bar{K}$ are tame knots such that $S^3 - K$ and $S^3 - \bar{K}$ are homeomorphic under an orientation-preserving map carrying the selected generator of $H_1(S^3 - K)$ onto the selected generator of $H_1(S^3 - \bar{K})$, and V, $\bar{V}$ are non-singular Seifert matrices for K and $\bar{K}$, then V and $\bar{V}$ are congruent over any integral domain in which the leading coefficient of the Alexander polynomial of K is a unit.*

As a special case we have

PROPOSITION 4.3. *If V is a non-singular linking matrix and* $\det(V) = \pm 1$, *then W is h-equivalent to V if and only if W and V are congruent over the integers.*

Since Seifert matrices are not symmetric the problem of classifying them by congruence is not the classical quadratic form problem. For any Seifert matrix V, $V - V'$ is unimodular, and by arguments similar to those following formula (4.3), there is a congruent matrix $\bar{V}$ with $\bar{V} - V = S$. We shall call such a $\bar{V}$ a *standard* Seifert matrix. A matrix P is *symplectic* if $PSP' = S$. Clearly two standard Seifert matrices V, W are congruent if and only if the symmetric matrices $V + V'$, $W + W'$ are congruent by a symplectic matrix. Thus the equivalence problem reduces to that of equivalence of quadratic forms under the symplectic group, a problem which does not appear to have been solved. Classical quadratic form theory yields invariants when applied to the symmetric matrix $V + V'$. Also, the similarity class of $(V + V')S^{-1}$ is an invariant of the symplectic congruence class of $V + V'$, and this yields additional invariants.

The preceding theorems have referred to a choice of generator for $H_1(S^3 - K)$; once an orientation for S^3 is fixed, the generator for $H_1(S^3 - K)$ is determined (by its linking number with K) in terms of the orientation of K. In our development of the formulas in terms of the Seifert matrix, t is the generator represented by a loop which goes from the "top" of the surface F around K to the "bottom" of F. Which side of F is the top depends of course on the orientation which F inherits from K, and the orientation of S^3. If we consider the *inverse* knot (i.e., the same knot with the opposite orientation) the roles of top and bottom, and hence the labels "sharp" and "flat", are interchanged. It follows that if V is a Seifert matrix for K, then V' is a Seifert matrix for the inverse of K.

On the other hand, $-V'$ is a Seifert matrix for the mirror image of K, for the configuration of sharp and flat cycles whose linking numbers give V is reflected into a configuration with linking numbers of opposite sign, with the sharp and flat labels interchanged. A knot is *invertible* if it is equivalent to its inverse and *amphicheiral* if it is equivalent to either orientation of its reflection. (This is the classical terminology [21]. For non-invertible knots one could also distinguish between direct and inverse amphicheirality.) Applying Theorem 2, we obtain

PROPOSITION 4.4. *Let V be a non-singular Seifert matrix for K, and R any integral domain which is an extension of the integers and in which* $\det(V)$ *is a unit. If K is invertible then V is congruent over R to V'. If K is amphicheiral, then V is congruent over R to $-V$ or $-V'$.*

I have been unable to determine whether every linking matrix is congruent to its transpose. It would be interesting to find a counter-example, since it would show the existence of non-invertible knots.

5. Finite coverings

A representation of a group G by permutations on n symbols gives a representation of G by permutation matrices which extends to a homomorphism of $Z(G)$ into the ring Λ of $n \times n$ integer matrices. Let R be an abelian group, and R_n and R^n the additive groups of n-component row and column vectors with entries in R. Then R_n is a right Λ-module, R^n a left Λ-module, and we may consider the homology (cohomology) of a G-complex with coefficients in $R_n(R^n)$.

We assume through the rest of this section that some fixed finite representation of G is given. If G is the fundamental group of a space, then such a representation is associated with a finite covering of the space, and if the space is aspherical the homology of G with coefficients in R_n can be identified with the homology of the covering with coefficients in R. We shall proceed, however, in a purely algebraic way. The calculation of relation matrices for the homology and cohomology groups of a free G-complex with these coefficients is straightforward, and we shall give details only for a special case of particular interest in knot theory.

The exact sequence

$$0 \longrightarrow Z^n \xrightarrow{\ i\ } Q^n \xrightarrow{\ j\ } M^n \longrightarrow 0$$

where Z is the integers, Q the rationals, and M the quotient group Q/Z, gives rise to an exact cohomology sequence

$$\longrightarrow H^p(W;\, Q^n) \xrightarrow{\ j^*\ } H^p(W;\, M^n) \xrightarrow{\ \delta^*\ } H^{p+1}(W;\, Z^n) \xrightarrow{\ i^*\ } H^{p+1}(W;\, Q^n) \longrightarrow$$

for any G-complex W. Kyle showed in [15] how a cohomological version of self-linking [24] could be defined by use of the coefficient sequence $0 \to Z \to Q \to M \to 0$, and what follows is modeled closely on his work.

Define $\lambda: M^n \otimes Z^n \to M$ by $\lambda(m \otimes z) = m'z$, where the prime denotes transposition, and the right side is a matrix product. Similarly define $\mu: Q^n \otimes Q^n \to Q$ by $\mu(q_1 \otimes q_2) = q_1' q_2$. For $g \in G$ let $\bar{g}$ be its image in Λ. Because $\bar{g}$ is a permutation matrix, $\bar{g}' = \bar{g}^{-1}$, and therefore $\lambda(\bar{g}m \otimes \bar{g}z) = (\bar{g}_m)'\bar{g}_z = m'\bar{g}'\bar{g}z = m'z = \lambda(m \otimes z)$. A similar result holds for μ, so that, with M and Q considered as G-modules under the trivial action of G, λ and μ are homomorphisms under the diagonal map of G and give cup-products

$$\smile_\lambda: H^p(W; M^n) \otimes H^q(W; Z^n) \to H^{p+q}(W; M)$$
$$\smile_\mu: H^p(W; Q^n) \otimes H^q(W; Q^n) \to H^{p+q}(W; Q) .$$

An element of $H^q(W; M^n)$ is represented by a cochain with values in Q^n whose coboundary has values in Z^n. If $\bar{u}, \bar{v}$ represent u, v in $H^p(W; M^n)$, $H^q(W; M^n)$ respectively, then $\bar{u} \smile_\mu \delta(\bar{v})$ represents $u \smile_\lambda \delta^*(v)$ in $H^{p+q+1}(W; M)$. Because of the relation $\delta(\bar{u} \smile_\mu \bar{v}) = \delta\bar{u} \smile_\mu \bar{v} + (-1)^p \bar{u} \smile_\mu \delta\bar{v}$, $\delta\bar{u} \smile_\mu \bar{v}$ and $(-1)^{p+1}\bar{u} \smile_\mu \delta\bar{v}$ are cohomologous. Since μ is symmetric, (1.3.1) implies that $\bar{u} \smile_\mu \delta\bar{v}$ and $(-1)^{p(q+1)}\delta\bar{v} \smile_\mu \bar{u}$ are cohomologous, and we have the commutation rule $u \smile_\lambda \delta^*(v) = (-1)^{(p+1)(q+1)} v \smile_\lambda \delta^*(u)$.

The image of δ^* in $H^{p+1}(W; Z^n)$ is the kernel of i^*, i.e., the torsion subgroup, which we shall denote by $N^{p+1}(W; Z^n)$. Because of the commutation rule, if either u or v is in the kernel of δ^* then $u \smile_\lambda \delta^*(v) = 0$, and consequently a function

$$\nu: N^{p+1}(W; Z^n) \otimes N^{q+1}(W; Z^n) \longrightarrow H^{p+q+1}(W; M)$$

is well-defined by the formula $\nu(x \otimes y) = \delta^{*^{-1}}(x) \smile_\lambda y$, $\delta^{*^{-1}}(x)$ being any w with $\delta^*(w) = x$.

These functions can be computed explicitly if W is a finitely generated free G-complex. For $\alpha \in Z(G)$ let $\bar{\alpha}$ denote its image in Λ under the given homomorphism. Let W_p be freely generated by $a_1^p, a_2^p, \cdots, a_{k_p}^p$. Identify $q \in \mathrm{Hom}_G(W_p, R^n)$ with the column vector of nk_p components in R obtained by writing the n-component columns $q(a_i^p)$ in order in a single column. Let the boundary operator in W be given by $d(a_i^{p+1}) = \sum_{ij} \alpha_{ij} a_j^p$. Then the coboundary operator from $\mathrm{Hom}_G(W_p, R^n)$ to $\mathrm{Hom}_G(W_{p+1}, R^n)$ is given by the $nk_{p+1} \times nk_p$ matrix Δ_p which, partitioned into $n \times n$ submatrices, has $\bar{\alpha}_{ij}$ in the ij^{th} cell.

For $u \in \mathrm{Hom}_G(W_p, M^n)$, $v \in \mathrm{Hom}_G(W_q, Z^n)$ the value of $u \smile_\lambda v$ on $c \in W_{p+q}$ is obtained as follows. The component of $D^\sharp(c)$ in $W_p \otimes W_q$ has the form $\sum_{ij} \alpha_{ij} a_i^p \otimes \beta_{ij} a_j^q$ with $\alpha_{ij}, \beta_{ij} \in Z(G)$. From the definition

$$\begin{aligned}(u \smile_\lambda v)(c) &= \lambda\left(\sum_{ij} u(\alpha_{ij}a_i^p) \otimes v(\beta_{ij}a_j^q)\right)\\ &= \sum_{ij}\lambda\left(\bar\alpha_{ij}u(a_i^p) \otimes \bar\beta_{ij}v(a_j^q)\right)\\ &= \sum_{ij}[u(a_i^p)]'\bar\alpha'_{ij}\bar\beta_{ij}v(a_j^q)\ .\end{aligned}$$

Identifying u and v with column vectors as before, this may be written as $u'\Omega_{pq}v$ where Ω_{pq} is the $nk_p \times nk_q$ matrix which, partitioned into $n \times n$ submatrices, has $\bar\alpha'_{ij}\bar\beta_{ij}$ in the ij^{th} cell.

If W is the relative complex of the group system of a knot, $H^3(W; M) \approx M$ by an isomorphism under which a cocycle corresponds to its value on the fundamental cycle c. Hence there is a pairing of $N^2(W; Z^n)$ with itself to M, defined by $L(x, y) = \nu(x \otimes y)(c)$.

Let Δ_1 be the $nk_2 \times nk_1$ coboundary matrix and Ω_{12} the $nk_1 \times nk_2$ matrix describing $D^\sharp(c)$. An element x of $N^2(W; Z^n)$ is represented by an nk_2-dimensional integral column vector ξ which is a rational combination of the columns of Δ_1, and any rational vector ζ such that $\Delta_1\zeta = \xi$ represents an element of $H^1(W; M^n)$ in $\delta^{*-1}(x)$. (Note that any rational combination of the columns of Δ_1 is a rational coboundary and hence, if integral, an integral cocycle. Provided we are concerned only with $N^2(W; Z^n)$ and not with $H^2(W; Z^n)$, we need not determine which vectors represent cocycles.) Thus if ξ, η represent $x, y \in N^2(W; Z^n)$ and $\Delta_1\zeta = \xi$, then $L(x, y) = (\zeta \smile_\lambda \eta)(c) \equiv \zeta'\Omega_{12}\eta \pmod 1$.

It is convenient to introduce a formal definition. An *L-group*[13] is a finite abelian group on which is defined a bilinear function, called the *linking*, with values in $M = Q/Z$. Two L-groups are considered isomorphic if there is a group isomorphism between them which is compatible with the linkings. A pair of $p \times q$ integer matrices (A, B) such that for every rational row vector ξ, if ξA is integral, then so is $\xi BA'$, determines an L-group as follows.[14] The group is the quotient of the group of integer q-component row vectors which are rational combinations of the rows of A by the subgroup of integer combinations of the rows of A. For x, y in the L-group, represented by vectors ξ, η, the linking is defined by $L(x, y) = \zeta B\eta' \pmod 1$ where ζ is any rational vector such that $\zeta A = \xi$. (L is easily seen to be well-defined if and only if A and B satisfy the condition imposed above.)

In terms of this definition the result above may be restated as $N^2(W; Z^n)$ *with* $L(x, y) = \nu(x \otimes y)(c)$ *is an L-group isomorphic to that determined by the pair of matrices* (Δ_1', Ω_{12}). We have given explicit rules for the

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[13] L-groups have much in common with the *V-groups* defined in [12], but we do not require the linking to be symmetric or primitive.

[14] The definition is motivated by the topological situation (see, e.g., [23]) in which A is a matrix of boundary relations and B a matrix of intersection numbers.

calculation of these matrices, and their application in any specific example is mechanical. In case G is represented by a cyclic permutation group, the result can be expressed very compactly in terms of a Seifert matrix for K, and we shall examine this case in detail. The algebraic manipulations involved are essentially the the same as those used by Seifert [23].

The required representation of $Z(G)$ by $n \times n$ matrices is obtained by composing the canonical map $Z(G) \to Z(H)$ htiw the representation taking t into the matrix

$$T = \begin{bmatrix} 0 & 1 & 0 & \cdot & 0 \\ 0 & 0 & 1 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & 1 \\ 1 & 0 & 0 & \cdot & 0 \end{bmatrix}.$$

(Here H and t are as in §4.) Since the coefficients are abelian, we may compute with a $Z(H)$-resolution as in §4. Let V be a $2h \times 2h$ Seifert matrix of the knot K and consider the $Z(H)$-resolution described in Proposition 4.1. The relative complex $W = X/Y$ is generated by the a_i in dimension 1 and b_y and the b_i in dimension 2, with a_y replaced by 0 in the formula for $d(b_y)$ and in (4.2). The $(2h+1)n \times 2hn$ coboundary matrix Δ_1 has n rows of zeros above a matrix which partitions into a $2h \times 2h$ matrix of $n \times n$ cells with $(V)_{ij}T - (V'_{ij})E$ in the ij^{th} cell. (E is the $n \times n$ identity matrix.) The matrix Ω_{12} has n columns of zeros followed by a matrix which, partitioned as above, has $[V(V' - V)^{-1}]_{ij}E$ in the ij^{th} cell. (Of course, a different choice of orientation for S^3 would change the sign of Ω_{12}.)

Let us say that two pairs of matrices (A, B), $(\bar{A}, \bar{B})$ are L-equivalent if they determine isomorphic L-groups. We shall need the following properties of L-equivalence.

If $(A, B), (\bar{A}, \bar{B})$ are pairs of $p \times q$ matrices which determine L-groups, then in each of the following cases (A, B) and $(\bar{A}, \bar{B})$ are L-equivalent.

(5.1) $\bar{A} = PA, \quad \bar{B} = PB \qquad$ *P an integer unimodular $p \times p$ matrix*

(5.2) $\bar{A} = AQ^{-1}, \quad \bar{B} = BQ' \qquad$ *Q an integer unimodular $q \times q$ matrix*

(5.3) $\bar{A} = \begin{bmatrix} 0 \\ A \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} * \\ B \end{bmatrix} \qquad$ *or* $\bar{A} = [0\ A], \quad \bar{B} = [*\ B]$

(5.4) $\bar{A} = \begin{bmatrix} 1 & * \\ 0 & A \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} * & * \\ * & B \end{bmatrix}$

(5.5) $\bar{A} = A, \quad \bar{B} = B + AC \qquad$ *C any integer $q \times q$ matrix .*

(In (5.3) and (5.4) $\bar{A}$ and $\bar{B}$ are supposed to be similarly partitioned, zeros represent zero submatrices of appropriate dimensions, and stars represent arbitrary submatrices.)

PROOF. In (5.1) the groups are actually the same, since the sets of rational and integral multiples of the rows of A are the same as those of PA; the linking is easily seen to be the same. In (5.2) there is an isomorphism between the groups under which the element represented by ξ, a rational combination of the rows of AQ^{-1}, corresponds to the element represented by ξQ, which is necessarily a rational combination of the rows of A. Case (5.3) is obvious. In (5.4) any rational combination of the rows of $\bar{A}$ which is an integral vector, must involve an integral multiple of the first row. Hence any element of the group can be represented without making use of the first row. Therefore it can be dropped, and the first column omitted by (5.3). Let ξ, η represent x, y and let ζ be a rational vector such that $\zeta A = \xi$. Then in case (5.5), $\zeta\bar{B}\eta' = \zeta B\eta' + \xi C\eta' \equiv \zeta B\eta'$ (mod 1), so that B and $\bar{B}$ determine the same linking.

The transposed matrix Δ_1' and the matrix Ω_{12}, which determine the L-group $N^2(W; Z^n)$, both begin with n columns of zeros; by (5.3), omitting the zero columns gives an L-equivalent pair. By (5.1) and (5.2), the rows and columns of these matrices may be rearranged to give an L-equivalent pair (A, B) which partition into $n \times n$ arrays of $2h \times 2h$ cells as follows:

$$A = \begin{bmatrix} -V & 0 & \cdot & 0 & V' \\ V' & -V & \cdot & 0 & 0 \\ 0 & V' & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & V' & -V \end{bmatrix},$$

$$B = \begin{bmatrix} V(V'-V)^{-1} & 0 & \cdot & \cdot & 0 \\ 0 & V(V'-V)^{-1} & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & V(V'-V)^{-1} \end{bmatrix}.$$

$V' - V$ is unimodular; by (5.2) we may multiply each column in the partition of A on the right by $(V' - V)^{-1}$, and each column of B on the right by $(V' - V)'$. Writing Γ for $-V(V' - V)^{-1}$ (as in § 4) we have $V'(V' - V)^{-1} = E - \Gamma$ and obtain the pair (A_1, B_1),

$$A_1 = \begin{bmatrix} \Gamma & 0 & 0 & \cdot & E-\Gamma \\ E-\Gamma & \Gamma & 0 & \cdot & 0 \\ 0 & E-\Gamma & \Gamma & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \Gamma \end{bmatrix},$$

$$B_1 = \begin{bmatrix} \Gamma(V'-V) & 0 & \cdot & \cdot & 0 \\ 0 & \Gamma(V'-V) & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \Gamma(V'-V) \end{bmatrix},$$

which is L-equivalent to (A, B).

For $i = 1, 2, \cdots, n-1$ we define pairs of square matrices (A_i, B_i) of dimension $2h(n-i+1)$ as follows:

$$A_i = \begin{bmatrix} \Gamma^i & 0 & 0 & \cdot & -(\Gamma-E)^i \\ E-\Gamma & \Gamma & 0 & \cdot & 0 \\ 0 & E-\Gamma & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \Gamma \end{bmatrix},$$

$$B_i = \begin{bmatrix} \Gamma^i(V'-V) & 0 & \cdot & \cdot & 0 \\ 0 & \Gamma(V'-V) & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \Gamma(V'-V) \end{bmatrix}.$$

Note that for $i = 1$ there is agreement with the previous definition of (A_1, B_1). For $i = n$ we put $A_n = \Gamma^n - (\Gamma - E)^n$, $B_n = \Gamma^n(V' - V)$.

We claim that (A_i, B_i) and (A_{i+1}, B_{i+1}) are L-equivalent for $i = 1, 2, \cdots, n-1$. To show this, define $C_i = E + \Gamma + \cdots + \Gamma^{i-1}$ so that $C_i(E - \Gamma) = E - \Gamma^i$. By (5.1) we may add C_i times the second row in the partition of A_i and B_i to the first row, and then add $\Gamma - E$ times the first row of the result to the second. The first two rows of the resulting matrices are

$$\begin{bmatrix} E & C_i\Gamma & 0 & \cdot & -(\Gamma-E)^i \\ 0 & \Gamma^{i+1} & 0 & \cdot & -(\Gamma-E)^{i+1} \end{bmatrix}, \begin{bmatrix} \Gamma^i(V'-V) & C_i\Gamma(V'-V) & 0 & \cdot & 0 \\ (\Gamma-E)\Gamma^i(V'-V) & \Gamma^{i+1}(V'-V) & 0 & \cdot & 0 \end{bmatrix}$$

and the other rows are unchanged. By repeated use of (5.4), the first $2h$

rows and columns may be dropped, and (A_{i+1}, B_{i+1}) is the result. (Strictly speaking, the case $i = n - 1$ is slightly different from the others, but the same manipulations produce the desired result.)

We have shown that (Δ_1', Ω_{12}) and (A_n, B_n) are L-equivalent, which gives the following theorem.

THEOREM 3. *Let K be a tame knot with group system $\{G, G', i\}$, and let G act on Z^n by cyclic permutation of coordinates. Then there is a canonically defined linking on $N^2(\{i\}; Z^n)$, the torsion subgroup of $H^2(\{i\}; Z^n)$. If V is a Seifert matrix for K, the L-group $N^2(\{i\}; Z^n)$ is isomorphic to that determined by the pair of matrices $\Gamma^n - (\Gamma - E)^n$, $\pm\Gamma^n(V' - V)$, where $\Gamma = -V(V' - V)^{-1}$, and the sign depends on the choice of orientation of S^3.*

By applying (5.5) with $C = \pm(V' - V)$, we see that the second matrix of the pair may be replaced by $\pm(\Gamma - E)^n(V' - V)$. This puts our result in complete agreement with Satz I of [23], since if V is a standard Seifert matrix $V' - V$ coincides with Seifert's matrix Δ, and shows that the L-group $N^2(\{i\}; Z^n)$ is isomorphic to the L-group of the self-linking of the 1-dimensional torsion cycles of the n^{th} branched cyclic covering of K.

The case $n = 2$, corresponding to the second cyclic covering, is of special interest because of its connection with the quadratic form of a knot. $A_2 = 2\Gamma - E$, $B_2 = \Gamma^2(V' - V)$. Since A_2 reduces to E modulo 2, its determinant is odd, and it follows that the L-group is of odd order. Hence the map $x \to 2x$, which has the effect of multiplying the linking by 4, is an automorphism of the group structure. Consequently $(2\Gamma - E, 4\Gamma^2(V' - V))$ is L-equivalent to (A_2, B_2). Applying (5.5) with $C = -(2\Gamma + E)(V' - V)$ changes the second matrix to $(V' - V)$, and applying (5.2) with $Q = -(V' - V)^{-1}$ gives the pair $(-V - V', E)$. The opposite choice of orientation would give $-E$ and the pair would be L-equivalent to $(V + V', E)$. Now if A is symmetric and non-singular, the L-group determined by (A, E) is isomorphic to the V-group associated with A^{-1}, in the terminology of [12]. Thus we can say the L-group $N^2(\{i\}; Z^2)$ and the L-group of self-linking in the second cyclic covering are isomorphic to the V-group associated with the matrix $\pm(V + V')^{-1}$. By the results of [23], this means that $V + V'$ (with the appropriate sign) determines a quadratic form belonging to the family [14] of the classical quadratic form of the knot K.

It follows from the remarks after Theorem 2, that for any *non-singular* Seifert matrix V of K, the rank and signature of $V + V'$ are invariants of K. This, together with the results of [23] and the preceding paragraph, implies that the congruence class of $V + V'$ over the p-adic integers for any p is also an invariant of K.[15] In other words,

PROPOSITION 5.1. *If V is a non-singular Seifert matrix for a knot K, the genus of the quadratic form determined by $V + V'$ is an invariant of K and the orientation of S^3.*

We have shown indirectly that the quadratic form of a knot is determined by the external group system, and hence by the homeomorphism type of the complementary space. It follows, for example, that if K_1 is the product of two right-handed cloverleaf knots, and K_2 the product of a right-handed cloverleaf knot with a left-handed one, then the complementary spaces of K_1 and K_2 are not homeomorphic. (Compare [9], where the same result is obtained by considering representations of the group systems by finite permutations.)

QUEEN'S UNIVERSITY, CANADA, AND PRINCETON UNIVERSITY

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[15] For p not dividing $\det(V)$ this follows directly from Theorem 2.

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