An algebraic formulation of surgery

by Andrew Ranicki

Introduction

A systematic attempt at the homotopy classification of compact manifolds led C.T.C. Wall to develop a surgery obstruction theory, which reduces the geometry to the $K$-theory of quadratic forms, that is $L$-theory. Here, we present a reformulation of the algebra, in terms of the hamiltonian formalism due to S.P. Novikov.

The paper is divided into three parts:

Part I "Foundations of $L$-theory"

Functors

$U_{\ast}, V_{\ast}, W_{\ast} :$ rings with involution $\rightarrow \mathbb{Z}_4$-graded abelian groups are defined, such that
Part II "Algebraic L-theory"

Denoting the Laurent extension of $A$ by $A_\mathbb{Z}$, natural direct sum decompositions

$$V_n(A_\mathbb{Z}) = V_n(A) \oplus U_{n-1}(A)$$

$$W_n(A_\mathbb{Z}) = W_n(A) \oplus V_{n-1}(A)$$

are established, and generalized to Laurent extension in several variables.

Part III "Geometric L-theory"

Functors

$L_f$: rings with involution $\rightarrow$ Kan $\Delta$-sets (***) such that

$$\Omega L_f(A) \cong L_{f+1}(A),$$

are defined for $f \mod 4$. There are then analogous versions for $V$, $W$-theories.

The exact sequences (***) and the direct sum decompositions (**) had been previously proved by geometrical methods (applicable for group rings of finitely presented groups) in

J. L. Shaneson "Wall's surgery obstruction groups for $G \times \mathbb{Z}$" Ann. of Math. 90 (1969), 254-351.

The neat formulation of Theorem 3.1 of Part I is due to Andrew Casson.

Our method of proof of (**) is a simplification - cum - generalization of
S.P.Novikov "Algebraic construction and properties of hermitian analogues of K - theories over rings with involution, from the point of view of hamiltonian formalism. Some applications to differential topology and the theory of characteristic classes".
II, 476-500
where a similar result is proved, assuming $kA$ and modulo 2-torsion.
The functors (**) are an algebraic version of the "surgery spaces" constructed geometrically (for group rings) in
F. Quinn "A geometric formulation of surgery"
Their study was suggested by Andrew Casson.

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I. Foundations of $L$-theory

Sec. Conventions

Let $A$ be an associative ring with 1, and with an involution, that is a function

$$- : A \to A; a \mapsto \bar{a}$$

such that

1. $\bar{1} = 1$
2. $(a+b) = \bar{a} + \bar{b}$
3. $\bar{ab} = \bar{b} \cdot \bar{a}$
4. $\bar{a} = a$

for all $a, b \in A$.

It is further required that finitely generated (f.g.) free $A$-modules have well-defined dimension.

Example 0.1 The group ring $\mathbb{Z}[\pi]$ of a multiplicative group $\pi$, with involution

$$- : \mathbb{Z}[\pi] \to \mathbb{Z}[\pi]; \sum_{g \in \pi} n_g g \mapsto \sum_{g \in \pi} w(g) n_g g^{-1}$$

defined by a morphism

$$w : \pi \to \mathbb{Z}; = \xi_{1}, - \xi_{3},$$

satisfies these conditions.

II. 0.2

(This is the group ring occurring in topology, with $\pi$ the fundamental group $\pi_1(M)$ of a compact manifold $M$, and $w : \pi_1(M) \to \mathbb{Z}$ the first Stiefel-Whitney class of $[w]$.)

We shall be dealing with left $A$-modules, $M, N, P, Q, \ldots$

Denote by $\text{Hom}_A(M, N)$ the additive group of $A$-module morphisms $f : M \to N$.

Given $M$, define the dual $A$-module, $M^*$, to be $\text{Hom}_A(M, A)$, with $A$ acting by

$$A \times \text{Hom}_A(M, A) \to \text{Hom}_A(M, A); (a, f) \mapsto (x \mapsto f(ax))$$

Accordingly, given $f \in \text{Hom}_A(M, N)$ define the dual morphism

$$f^* : N^* \to M^*; g \mapsto (x \mapsto g(f(x)))$$

in $\text{Hom}_A(N^*, M^*)$.

Morphisms in $\text{Hom}_A(M \oplus N, P \oplus Q)$ can be...
displayed as matrices
\[ f = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : M \otimes N \rightarrow P \otimes Q, \]
\[(x, y) \mapsto (\alpha x + \beta y, \gamma x + \delta y)\]
with \( \alpha \in \text{Hom}_A(M, P) \), \( \beta \in \text{Hom}_A(N, P) \), \( \gamma \in \text{Hom}_A(M, Q) \), and \( \delta \in \text{Hom}_A(N, Q) \). Composition of such morphisms corresponds to right multiplication of the matrices. The morphism dual to \( f \) (as above) has matrix
\[ f^* = \begin{pmatrix} \alpha^* & \beta^* \\ \gamma^* & \delta^* \end{pmatrix} : \left( M^* \otimes N^* \right) \rightarrow P^* \otimes Q^*, \]
identifying \((M \otimes N)^*\) with \(M^* \otimes N^*\) in the obvious way.

If \( Q \) is a f.g. free \( A \)-module, with base \( F = \{ f_1, f_2, \ldots, f_m \} \), then \( Q^* \) is a f.g. free \( A \)-module of the same dimension, with dual base \( F^* = \{ f_1^*, f_2^*, \ldots, f_m^* \} \) given by
\[ f_i^* (f_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \]

It follows that if \( P \) is a f.g. projective \( A \)-module, so is \( P^* \), though \( P \) and \( P^* \) are not in general isomorphic. However, the natural map
\[ P \rightarrow P^{**} ; x \mapsto (f \mapsto f(x)) \]
is an isomorphism. It is used to identify \( P^{**} \) with \( P \), whenever \( P \) is f.g. projective. In particular, given a morphism \( f \in \text{Hom}_A(Q, P^*) \) with \( P \) f.g. projective, we can write
\[ f^* : P \rightarrow Q^* ; x \mapsto (y \mapsto f(y)(x)) \]

We shall make much use of the contravariant functors
\[ \Pi, \Pi_+, \Pi_- : (\text{f.g. projective } A \text{-modules}) \rightarrow (\text{abelian groups}) \]
and natural transformations
\[ \Pi_+ \Rightarrow \Pi, \ \Pi_- \Rightarrow \Pi \]
defined on objects by
\[ \Pi(Q) = \text{Hom}_A(Q, Q^*) \]

\[ \Pi_\pm(Q) = \text{Hom}_A(Q, Q^*) \backslash \sum \chi \neq \chi^* | \chi \in \text{Hom}_A(Q, Q^*) \]

\[ \Pi_\pm(Q) \rightarrow \Pi(Q), \ [\phi] \mapsto (\phi \pm \phi^*) \]

and on morphisms \( f \in \text{Hom}_A(P, Q) \) by

\[ \Pi(f) : \Pi(Q) \rightarrow \Pi(P), \ \phi \mapsto f^* \phi f \]

\[ \Pi_\pm(f) : \Pi_\pm(Q) \rightarrow \Pi_\pm(P), \ [\phi] \mapsto [f^* \phi f] \]

\[ \S 1 \text{ Forms} \]

A \( \pm \)-form (over \( A \)), \( (Q, \phi) \), is a \( \pm \)-form projective \( A \)-module \( Q \), together with \( \phi \in \Pi(Q) = \text{Hom}_A(Q, Q^*) \), and a choice of sign.

A morphism of \( \pm \)-forms

\[ (f, \chi) : (P, \theta) \rightarrow (Q, \phi) \]

is defined by \( f \in \text{Hom}_A(P, Q) \), \( \chi \in \Pi_\pm(P) \) such that

\[ f^* \phi f - \theta = \chi \neq \chi^* \in \text{Hom}_A(P, P^*) \]

The composite of \( \pm \)-form morphisms

\[ (f, \chi) : (P, \theta) \rightarrow (Q, \phi), \ (g, \psi) : (Q, \phi) \rightarrow (R, \gamma) \]

is the morphism

\[ (g, \psi)(f, \chi) = (gf, \chi + f^* \psi f) : (P, \theta) \rightarrow (R, \gamma) \]

We thus have a category of \( \pm \)-forms (or, rather, two such, one for each choice of sign). There is a direct sum operation,

\[ (P, \theta) \oplus (Q, \phi) = (P \oplus Q, \theta \oplus \phi) \]

with \( (0, 0) \sim 0 \).
A morphism of ± forms

\((f_\star) : (\Omega^\star, \Theta) \rightarrow (\Omega^\star, \Phi)\)

is an isomorphism precisely when \(f \in \text{Hom}_A(P, Q)\) is an \(A\)-module isomorphism. For example,

\((1, \chi) : (Q^\star) \rightarrow (Q^\star, \Phi + \chi \Theta)\)

is a ± form isomorphism for all ± forms \((Q, \Phi)\),
and \(\chi \in \Pi_\pm(Q)\).

In general, we shall be interested in ± forms up to isomorphism only.

For every ± form \((Q, \Phi)\), \(\Phi + \chi \Theta \in \text{Hom}_A(Q^\star, Q^\star)\) defines a ± symmetric sesquilinear function,
the associated pairing of \((Q, \Phi)\),

\[ \langle \cdot, \cdot \rangle_\Phi : Q \times Q \rightarrow A ; \ (x, y) \mapsto \langle x, y \rangle_\Phi = \Phi(x \Theta)(y) + \Phi(y \Theta)(x) \]

such that

i) \( \langle x, y \rangle_\Phi = \pm \langle y, x \rangle_\Phi \in A \)

ii) \( \langle x, ay \rangle_\Phi = a \langle x, y \rangle_\Phi \in A \).

For all \( x, y \in Q \) and \( a \in A \).

A morphism of ± forms

\((f_\star) : (P, \Theta) \rightarrow (Q, \Phi)\)

 preserves the associated pairings, in that

\(f^\star \Phi \Theta \Phi^\star f = \Theta \pm \Theta \in \text{Hom}_A(P^\star, P^\star)\),

which can also be expressed as

\[ \langle f(x), f(y) \rangle_\Phi = \langle x, y \rangle_\Theta \in A \quad (x, y \in P) \]

An ± form \((Q, \Phi)\) is non-singular if \(\Phi + \chi \Theta \in \text{Hom}_A(Q^\star, Q^\star)\)
is an \(A\)-module isomorphism.

Define the Hamiltonian ± form on a f.g. projective \(A\)-module \(P\),

\[ H_\pm(P) = \langle p \Theta p^\star, (\Theta \pm) : p \Theta p^\star \rightarrow p^\star \Theta p \rangle \]

The associated pairing

\[ \langle \cdot, \cdot \rangle : p \Theta p^\star \times p \Theta p^\star \rightarrow A ; \]

\[ ((x_1, y_1), (x_2, y_2)) \mapsto y_1(x_2) \pm y_2(x_1) \]
is defined by the isomorphism \((\Theta \pm) \in \text{Hom}_A(P^\star, P^\star)\)
so that \(H_\pm(P)\) is non-singular.
A ±form is trivial if it is isomorphic to a Hamiltonian ±form.

The remainder of §1 is devoted to answering the question:
when is a ±form trivial, or more generally, when does it have a trivial summand?

Given a ±form \((Q,\phi)\) and a submodule \(L \subseteq Q\) define the annihilator of \(L\) in \((Q,\phi)\)
\[ L^\perp = \{ x \in Q | \langle L, x \rangle_\phi = 0 \text{ for all } L \} \]
also a submodule of \(Q\).

Submodules \(L, M\) of \(Q\) are orthogonal in the ±form \((Q,\phi)\) if \(\langle L, M \rangle_\phi = 0\) for all \(L\), that is if \(L \subseteq M^\perp\), or (equivalently) if \(M \subseteq L^\perp\).

A partial trivialization, or sublagrangian \([L,\lambda]\) of a ±form \((Q,\phi)\), is a self-orthogonal direct summand \(L\) of \(Q\), together with \(\lambda \in \mathfrak{H}_\phi(L)\)
such that

i) \(j^* (\phi \pm \delta) : Q \to L^*\) is onto

ii) \(j^* \phi j = \lambda + \lambda^* : L \to L^*\)

where \(j : L \to Q\) is the inclusion. (Note that i) is always satisfied if \((Q,\phi)\) is non-singular.)

Then
\[ L^\perp = \ker (j^* \phi \pm \delta) : Q \to L^* \]
is a direct summand of \(Q\) containing \(L\), with
\[ \langle \cdot, \cdot \rangle_\phi : Q/\perp L^\perp \to L^* : [x] \mapsto (y \mapsto \langle x, y \rangle_\phi) \]

an \(A\)-module isomorphism.

An isomorphism of ±forms
\[(f, \chi) : (P, \Theta) \to (Q, \phi)\]
sends a sublagrangian \([L, \lambda]\) of \((P, \Theta)\) to the sublagrangian
\[(f, \chi)[L, \lambda] = [f L, (f^* \phi)(\lambda + j^* \chi)(f^* \delta)]\]
of \((Q, \phi)\), with \(j : L \to P\) the inclusion.
A sublagrangian \([L, \lambda]\) of a \(\pm\)form \((Q, \phi)\) which is maximally self-orthogonal, in the sense that
\[ L^* = L, \]
is a trivialization, or lagrangian, of \((Q, \phi)\). Isomorphisms of \(\pm\)forms preserve lagrangians. Lagrangians are maximal sublagrangians:
if \([L, \lambda] \) is a lagrangian of \((Q, \phi)\), and \([M, \mu]\) is a sublagrangian of \((Q, \phi)\) with \(L \leq M\), then
\[ L \leq M \leq M^+ \leq L^* = L, \]
and \(L = M\).

Theorem 1.1 A \(\pm\)form is trivial iff it admits a trivialization.

Proof: It is clear that \([L, 0]\) is a lagrangian of \(H^\pm(L)\) for any \(f.g.\) projective \(L\).

Conversely, let \([L, \lambda]\) be a lagrangian of a \(\pm\)form \((Q, \phi)\). Choosing a direct complement \(L_0 \to L\) in \(Q\), express \(\phi\) as

\[ \phi = \left( \begin{array}{c} \lambda^* \lambda \end{array} \right) : \text{L} \otimes \text{L} \to \text{L} \otimes \text{L}, \]
so that
\[ \gamma^* \phi = [\Phi \Phi^*] : L_0 \to Q \]
is an \(A\)-module isomorphism, and \((Q, \phi)\) is non-singular.

Define
\[ \alpha = (\gamma^* \phi)' \in \text{Hom}_A(L^*, L), \quad \chi = \left( \begin{array}{c} \lambda^* \lambda \end{array} \right) \in \text{Hom}_A(L \otimes L). \]

Then
\[ (F, F^* \chi F) : H^\pm(L) \to (Q, \phi) \]
is an isomorphism of \(\pm\)forms (sending \([L, 0]\) to \([L, \lambda]\)).

A sublagrangian \([L, \lambda]\) of a \(\pm\)form \((Q, \phi)\) is given by a morphism of \(\pm\)forms
\[ (j, \lambda) : (L, 0) \to (Q, \phi) \]
with \(j \in \text{Hom}_A(L, Q)\) split mono, \(j^* (\Phi \Phi^*) : \text{Hom}_A(Q, L)\) onto. Theorem 1.1 shows that the inclusion of a lagrangian
\[ (j, \lambda) : (L, 0) \to (Q, \phi). \]
may be extended to an isomorphism of ±forms
\((f, \gamma) : H^\pm(L) \to (\mathbb{Q}, \phi)\)
recalling Witt's theorem in the classical theory
of quadratic forms. More generally:

**Corollary 1.2** A fg projective \(A\)-module \(L\) defines a
sublagrangian \([L, \delta] of H^\pm(L) \oplus (\mathbb{Q}, \phi)\), where
any ±form \((P, \delta)\).

Conversely, the inclusion of a
sublagrangian
\((f, \gamma) : (L, \delta) \to (\mathbb{Q}, \phi)\)
may be extended to an isomorphism of ±forms
\((f, \gamma) : (L^\perp, \hat{\phi}) \oplus H^\pm(L) \to (\mathbb{Q}, \phi)\),
where \((L^\perp, \hat{\phi})\) is the ±form to which \(\hat{\phi}\) restricts on
a direct complement to \(L\) in \(L^\perp\), a different
such choice leading to an isomorphic ±form.

**Proof:** The first part is obvious.
Conversely, let \([L, \lambda]\) be a sublagrangian
of \((L, \delta)\), so that
\[\phi = \begin{pmatrix} \phi & \lambda \bar{\lambda} \\ \lambda \bar{\lambda} & \delta \end{pmatrix} : L^\perp \oplus L \oplus \mathbb{Q} \to (L^\perp)^* \oplus \mathbb{Z} \oplus \mathbb{Q},\]
for some choice of direct complements to \(L, L^\perp\) in
\(L^\perp, \mathbb{Q}\) respectively.

**New**
\((4, (\mathbb{Q} \oplus \mathbb{Q})) : (L^\perp \oplus L \oplus \mathbb{Q}, (\frac{x}{\delta}, \bar{\delta})) \to (\mathbb{Q}, \delta)\)
is an isomorphism of ±forms, with \([L, \lambda]\) a
lagrangian of \((L \oplus \mathbb{Q}, (\frac{x}{\delta}, \bar{\delta}))\). Applying
Theorem 1.1, we have an isomorphism of ±forms
\((f, \gamma) : (L^\perp, \hat{\phi}) \oplus H^\pm(L) \to (\mathbb{Q}, \phi)\)
as required.

A different choice of direct complement
to \(L\) in \(L^\perp\) replaces \((L^\perp, \hat{\phi})\) by \((L^\perp, \hat{\phi} + \chi \bar{\chi} \chi^*),\)
where \(\chi = \alpha h + h^* \lambda h \in \Pi_{\hat{\phi}}(L^\perp),\) for some
\(h \in Hom_A(L^\perp, \mathbb{Q}),\) clearly an isomorph.

\[\square\]

A **subhamiltonian complement** to a
sublagrangian \([L, \lambda]\) of a ±form \((\mathbb{Q}, \phi)\) is a
sublagrangian \([M, \mu]\) such that
\[Q = L \oplus M^\perp = L^\perp \oplus M\]
for example \([L, \delta], [L^*, \delta]\) are subhamiltonian
complements in \(H^\pm(L) \oplus (\mathbb{Q}, \delta)\), for any fg projective
\(L\) and ±form \((P, \delta)\).

Isomorphisms of ±forms preserve...
subhamiltonian complements, so that by Corollary 1.2, every sublagrangian has a subhamiltonian complement.

Given subhamiltonian complements \([L, \lambda], [M, \mu]\) in a \(\pm\) form \((Q, \phi)\) we can identify \(M\) with \(L^*\) via the \(A\)-module isomorphism

\[ M \to L^* \; \; \alpha \mapsto (y \mapsto \langle x, y \rangle). \]

Then \(\phi : Q \to Q^*\) can be expressed as

\[ \phi = \begin{pmatrix} \hat{\alpha} & \hat{\beta} \\ \hat{\gamma} & \hat{\delta} \end{pmatrix} : L_\phi \otimes L_{\phi} \otimes L^* \to (L_\phi)^* \otimes L_{\phi} \otimes L \]

with \(\gamma \pm \delta^* = 1 : L^* \to L^*\).

The subhamiltonian complements of lagrangians are also lagrangians, in which case they are called hamiltonian complements.

Given a lagrangian \([L, \lambda]\) in a \(\pm\) form \((Q, \phi)\) we shall in general identify \(M = L^*\)

for any one hamiltonian complement \([M, \mu]\) to \([L, \lambda]\) in \((Q, \phi)\), but having chosen one such, reserve the notation \(L^*\) for it alone.

A choice of hamiltonian complement to \([L, \lambda]\) is given by a morphism of \(\pm\) forms

\[ \langle \cdot | j(g), \alpha \rangle_\phi = g(\alpha) e A \quad (x \in L, g \in L^*). \]

There is one such choice for every \(\mu \in \Pi_2(L^*)\), as is clear from:

**Lemma 1.3** The hamiltonian complements to \(L^*, \phi\) in \(H_\pm(P)\) are the graphs

\[ \Gamma_{(p, \theta)} = \text{im} \left( \left( \begin{pmatrix} 1 \\ \theta \end{pmatrix} \right) : (P, \theta) \to H_\pm(P) \right) \]

of \(\pm\) forms \((P, \theta)\).

**Proof.** The direct complements to \(P^*\) in \(P \otimes P^*\) are just the graphs

\[ \Gamma^h = \text{im} \left( \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) : P \to P \otimes P^* \right) \]

of morphisms \(h \in \text{Hom}_0(P, P^*).\) Now \([\Gamma^h, \lambda]\) defines a
The lagrangian of $H_\pm(p)$ precisely when

$$(1.12) \quad 0 \quad \theta (\frac{1}{h})(\frac{1}{h}) \theta = h = \lambda \pm \lambda^* : p \rightarrow \pi.$$ 

The next result, corresponding to Theorem 3 in (x), is used by Wall to justify the sort of definition of quadratic form adopted above. (x) C. T. C. Wall "On the axiomatic foundations of the theory of Hamilton forms." Proc. Camb. Phil. Soc. 67 (1970) 243-250.

Lemma 1.4 The diagonal of a non-singular $\pm$-form $(Q, \phi)$,

$$\Delta_{(Q, \phi)} = \text{im} \left( \left( \psi \right), (Q, \phi) \rightarrow (Q \otimes Q, \phi \otimes \phi) \right)$$

is a lagrangian of $(Q, \phi) \otimes (Q, -\phi)$, with hamiltonian complements

$$\Delta^*_{(Q, \phi)} = \text{im} \left( \left( \psi \right), (Q^*, \phi) \rightarrow (Q \otimes Q, \phi \otimes \phi) \right)$$

classified by $\pm$-forms $(Q^*, \psi)$ such that

$$\psi \pm \psi^* = (\phi \pm \phi^*)^{-1} : Q^* \rightarrow Q,$$

with

$$\psi - (\psi \pm \psi^*)^{-1} : (Q, \phi) \rightarrow (Q^*, \phi).$$

In particular, the diagonal of a trivial $\pm$-form $(Q, \phi)$ is a hamiltonian complement in $(Q \otimes Q, \phi \otimes \phi)$ to $(F, \lambda) \otimes (G, -\mu)$ for any hamiltonian complements $(F, \lambda), (G, \mu)$ in $(Q, \phi)$.
§ 2.1

A \textit{±-formation} (over $A$), $(Q, \phi; [F, X], [G, \mu])$

is a triple consisting of

i) a trivial ±-form $(Q, \phi)$

ii) a lagrangian $[F, X]$ of $(Q, \phi)$

iii) a sublagrangian $[G, \mu]$ of $(Q, \phi)$.

An isomorphism of ±-formations

$$(h, \zeta) : (Q, \phi; [F, X], [G, \mu]) \rightarrow (Q', \phi'; [F', X'], [G', \mu'])$$

is an isomorphism of ±-forms

$$(h, \zeta) : (Q, \phi) \rightarrow (Q', \phi')$$

sending $[F, X], [G, \mu]$ to $[F', X'], [G', \mu']$ respectively.

We thus have a category of ±-formations, with every morphism an equivalence. A direct sum operation is defined by

$$(Q, \phi; [F, X], [G, \mu]) \oplus (Q', \phi'; [F', X'], [G', \mu']) = (Q \oplus Q', \phi \oplus \phi'; [F \oplus F', X \oplus X'], [G \oplus G', \mu \oplus \mu'])$$

with $(0, 0; 0, 0)$ as zero.

§ 2.2

By Theorem 1.1, every ±-formation $(Q, \phi; [F, X], [G, \mu])$ is isomorphic to one of the type $(H_{\pm}(F); [F, O], [G, \mu])$.

A ±-formation $(Q, \phi; [F, X], [G, \mu])$ is non-singular if $[G, \mu]$ is a lagrangian.

For any f.g. projective $A$-module $P$

define the Hamiltonian ±-formation on $P$, $(H_{\pm}(P); [P, O], [P', O])$, clearly non-singular.

A ±-formation is trivial if it is isomorphic to a Hamiltonian ±-formation.

Lemma 2.1 A ±-formation $(Q, \phi; [F, X], [G, \mu])$ is trivial iff it is non-singular and $[F, X], [G, \mu]$ are Hamiltonian complements in $(Q, \phi)$.

Proof: Given Hamiltonian complements $[F, X], [G, \mu]$ in a ±-form $(Q, \phi)$ express $\phi : Q \rightarrow Q^*$ as

$$\phi = \left( \begin{array}{cc} \lambda & \lambda^* \\ \delta & \mu \end{array} \right) : F \oplus G \rightarrow F^* \oplus G^*$$
Then
\[
\begin{pmatrix}
1 & 0 \\
0 & 0 \pm 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & 0 \pm 1
\end{pmatrix}^{-1}
\]

is an isomorphism of ±formations.

The converse is obvious. □

Given a ±formation \((P, \theta)\), define the graph ±formation on \((P, \theta)\), \((H_{\pm}(P), [P, \theta], \Gamma_{\pm}(P, \theta))\), where \(\Gamma_{\pm}(P, \theta)\) is as in Lemma 1.3.

A ±formation isomorphic to a graph ±formation is elementary.

Lemma 2.2 A ±formation \((Q, \phi, [F, \lambda], [G, \mu])\) is elementary iff it is non-singular and \([F, \lambda], [G, \mu]\) share a hamiltonian complement in \([Q, \phi]\).

Proof: For any ±formation \((P, \theta)\), \([P, \theta]\) is a hamiltonian complement in \(H_{\pm}(P)\) to both \([P, \theta]\) and \(\Gamma_{\pm}(P, \theta)\).

Conversely, let \([H, \xi]\) be a hamiltonian complement to \([F, \lambda], [G, \mu]\) in a ±form \((Q, \phi)\).

By Lemma 2.1 there exists an isomorphism of ±formations

\(f, \chi) : (H_{\pm}(F), [F, \lambda], [F', \lambda]) \rightarrow (Q, \phi, [F, \lambda], [G, \mu])\).

By Lemma 1.3 \((f, \chi)\) sends some \(\Gamma_{\pm}(F, \theta)\) to \([G, \mu]\), so that

\(f, \chi) : (H_{\pm}(F), [F, \lambda], [F', \lambda]) \rightarrow (Q, \phi; [F, \lambda], [G, \mu])\)

is an isomorphism of ±formations. □

Lemmas 2.1, 2.2 are the special cases \(P=0, L=0\) of

Theorem 2.3 A ±formation \((Q, \phi, [F, \lambda], [G, \mu])\)

is isomorphic to the direct sum

\((H_{\pm}(P), [P, \theta], \Gamma_{\pm}(P, \theta)) \oplus (H_{\pm}(L), [L, O], [L', L'])\)

of an elementary and a trivial ±formation.
If it is non-singular and \([F, \lambda] \) has a Hamiltonian complement \([F^*, \lambda^*]\) in \((Q, \phi)\) such that the projection on \(F\) along \(F^*\),

\[
\pi : Q = F \oplus F^* \twoheadrightarrow F,
\]
sends \(G\) onto a direct summand \(\pi(G) = P\) of \(F\).

The roles played by \([F, \lambda] , [G, \mu]\) may be reversed.

**Proof:** For any \(\pm\)-form \((P, \theta)\) and \(\pm\)-projective \(L\),

\([P^* \oplus L^*, \psi]\) is a Hamiltonian complement to

\([P \oplus L, \psi]\) in \(H_+ (P \oplus L)\) such that the projection on \(P \oplus L\) along \(P^* \oplus L^*\) sends \(\pi(P \oplus L)\) onto \(P\).

Conversely, let \((Q, \phi), [F, \chi], [G, \mu]\) be a non-singular \(\pm\)-formation, and \([F^*, \lambda^*]\) be a Hamiltonian complement to \([F, \lambda]\) in \((Q, \phi)\) such that the projection on \(F\) along \(F^*\),

\[
\pi : Q = F \oplus F^* \twoheadrightarrow F,
\]
sends \(G\) onto a direct summand \(P\) of \(F\),

with \(F = P \oplus L\) say.

Dualizing, we have a direct sum decomposition

\[
F^* = P^* \oplus L^* \subseteq Q
\]

and

\[
L^\perp = F \oplus P^* , \quad P^\perp = F \oplus L^*
\]

with

\[
P^* = F^* \cap L^\perp , \quad L^* = F^* \cap P^\perp.
\]

Hence

\[
\langle P \oplus P^*, L \oplus L^* \rangle_\phi = 0
\]

and there is defined an isomorphism of \(\pm\)-forms

\[
(1, \phi ) : (Q, \phi) \longrightarrow H_+ (P \oplus L),
\]

for some \(\chi \in \Pi^+ (\omega)\), sending \([F, \lambda]\) to \([P \oplus L, \psi]\).

Now

\[
G \subseteq \Pi^+ (\omega) \otimes (1 - \phi) (\omega) \subseteq P \oplus P^* \oplus L^* ,
\]

so that

\[
\langle G, L^* \rangle_\phi \subseteq \langle P \oplus P^*, L^* \rangle_\phi = 0
\]

and

\[
L^* \subseteq G^\perp = G.
\]

Defining \((P \oplus P^*) \cap G\) by \(M, \perp\) follows that

\[
G = M \oplus L^* ,
\]
and so there is defined an isomorphism of \(\pm\)-formations:
\[
(I, \nu) : (Q,\phi; [F, \lambda], [G, \mu]) \rightarrow (H^+(P); [P, \delta], [Q, \gamma], [G, \theta]),
\]
for some \(\mu_1 \in \Pi_1(M), \chi \in \Pi_0(Q)\).

As \(\pi\) sends \(M\) onto \(\pi(M) = P\), the projection on \(P\) along \(P^*\),
\[
(1o) : P \oplus P^* \rightarrow P,
\]
a restriction of \(\pi : Q \rightarrow F\), does the same.

Thus \([M, \mu_1]\) is a hamiltonian complement to \([P^*, 0]\) in \(H^+(P)\), necessarily the graph \(\Gamma(P, 0)\) of a \(\pm\)-form \((P, 0)\), by Lemma 1.3.

Symmetry with respect to \([F, \lambda], [G, \mu]\) follows from that of Lemmas 2.1, 2.2.

\[
\text{A stable isomorphism of } \pm\text{-formations}
\]

\[
[h, \nu] : (Q,\phi; [F, \lambda], [G, \mu]) \rightarrow (Q',\phi'; [F', \lambda'], [G', \mu']).
\]
is an equivalence of isomorphisms.

\[
(h, \nu) : (Q,\phi; [F, \lambda], [G, \mu]) \rightarrow (H^+(P); [P, \delta], [Q, \gamma], [G, \theta]),
\]

\[
\rightarrow (Q',\phi'; [F', \lambda'], [G', \mu']) \rightarrow (H^+(P); [P', \delta], [Q', \gamma], [G', \theta]),
\]
defined for \(F, G\) projective \(P, P'\) under the equivalence relation
\[
(h, \nu) \sim (h', \nu')
\]

\[
\Leftrightarrow \exists A\text{-module isomorphisms } \alpha : L \rightarrow L, \alpha' : L \rightarrow L' \text{ s.t.}
\]

\[
(h, \nu) \oplus ((\alpha \otimes \cdot \cdot \cdot), 0)
\]

\[
= (h', \nu') \oplus ((\alpha' \otimes \cdot \cdot \cdot), 0)
\]

\[
: (Q,\phi; [F, \lambda], [G, \mu]) \oplus (H^+(P); [P, \delta], [Q, \gamma], [G, \theta])
\]

\[
\rightarrow (Q',\phi'; [F', \lambda'], [G', \mu']) \oplus (H^+(P); [P', \delta], [Q', \gamma], [G', \theta]).
\]

In general, we shall be interested in \(\pm\)-formations up to stable isomorphism only.

Stable isomorphism is an equivalence relation on \(\pm\)-formations. A \(\pm\)-formation stably isomorphic to a trivial \(\pm\)-formation is itself trivial, by Lemma 2.1.

It should be noted that there is only one stable isomorphism \(0 \rightarrow 0\), because every morph-
of hamiltonian forms

\((f,x):(H^\pm(L);[L,\partial],[L^*,\partial]) \to (H^\pm(L);[L,\partial],[L^*,\partial])\)

is necessarily of the type

\((\Theta \circ \alpha),\theta):(H^\pm(L);[L,\partial],[L^*,\partial]) \to (H^\pm(L);[L,\partial],[L^*,\partial]),\)

for some isomorphism \(\alpha \in \text{Hom}_\partial(L,L)\).

§3 U-theory

Let \(I\) be an abelian monoid, \(J\) a submonoid. Call \(i_1,i_2 \in I\) \(J\)-stably equivalent, \(i_1 \sim_j i_2\), if

\[i_1 \oplus j = i_2 \oplus j, \quad \text{for some } j \in J,\]

where \(\oplus\) is the composition law in \(I\).

The quotient monoid \(I/\sim_j\) may be denoted by \(I/J\), because it depends only on the stabilization of \(J\) in \(I\), the submonoid

\[\overline{J} = \{i \in I | i \sim_0 j\}\]

of \(I\), containing \(J\).

Note that \(I/\overline{J}\) is an abelian group iff for every \(i \in I\) there exists \(j \in I\) such that \(i \oplus j \in J\).
Theorem 3.1 For $n (\text{mod} \ 4)$ let $X_n(A)$ be the abelian monoid of $\Sigma$ isomorphism classes of $\Sigma^\pm$-forms over $A$, under the direct sum $\oplus$, with $n = \{2i \mid i \leq \frac{n}{2} \}$. The morphisms

$$\vartheta: X_n(A) \rightarrow X_{n-1}(A); \begin{cases} (p, \Phi) \mapsto (H^i(p), [p, \partial], \Gamma_{(p, \partial)}) \\ (Q, \Phi) \mapsto (\mathfrak{g}, \varphi, \Gamma_{(Q, \Phi)}) \end{cases}$$

are well-defined and such that $\vartheta^2 = 0$.

The quotient monoids

$$U_n(A) = \frac{\ker (\vartheta: X_n(A) \rightarrow X_{n-1}(A))}{\text{im} (\vartheta: X_{n-1}(A) \rightarrow X_n(A))}$$

are groups.

Proof: i) $n$ even.

An isomorphism of $\Sigma^\pm$-forms

$$(f, \chi): (p, \Theta) \rightarrow (Q, \Phi)$$

defines an isomorphism of $\Sigma^\pm$-formations

$$\delta(f, \chi) = ((\delta \circ \delta), \chi): (H^i(p), [p, \partial], \Gamma_{(p, \partial)}) \rightarrow (H^i(Q), [Q, \partial], \Gamma_{(Q, \partial)})$$

By Lemma 2.1

$$\ker (\vartheta: X_2(A) \rightarrow X_{2i-1}(A)) = \{\varnothing, (Q, \Phi) \in X_2(A) \mid (Q, \Phi, [F, \partial], [\mathfrak{g}, \varphi]) \text{ trivial?} \}$$

By Corollary 1.2

$$\ker (\vartheta: X_{2i+1}(A) \rightarrow X_{2i}(A)) = \{\varnothing, (Q, \Phi) \in X_{2i+1}(A) \mid (Q, \Phi, [F, \partial], [\mathfrak{g}, \varphi]) \text{ trivial?} \}$$

By Theorem 1.1 and Lemma 1.4, for every $(Q, \Phi) \in \ker (\vartheta: X_2(A) \rightarrow X_{2i-1}(A))$,

$$(Q, \Phi) \oplus (Q, -\Phi) = 0 \in U_{2i}(A)$$

giving inverses for $U_{2i}(A)$. 

By Theorem 1.3.2 and Lemma 1.4, for every $(Q, \Phi) \in \ker (\vartheta: X_2(A) \rightarrow X_{2i-1}(A))$,
ii) n odd

A stable isomorphism of ±-forms

\[ [h, x] : (G, \phi ; [F, \lambda], [G, \mu]) \rightarrow (G', \phi' ; [F', \lambda'], [G', \mu']) \]

sends \( G \rightarrow G' \) and \( G^\perp \rightarrow G'^\perp \), so that there is defined an isomorphism of ±-forms

\[ \Theta[h, x] : (G, \phi, \psi) \rightarrow (G', \phi', \psi') . \]

Here,

\[ \ker(\Theta : X_{2i+1}(A) \rightarrow X_2(A)) \]

\[ = \{ (Q, \phi, [F, \lambda], [G, \mu]) \in X_{2i+1}(A) | (G, \phi) = 0 \in X_2(A) \} \]

\[ = \{ (Q, \phi, [F, \lambda], [G, \mu]) \in X_{2i+1}(A) | (Q, \phi, [F, \lambda], [G, \mu]) \text{ non-singular} \} \]

and by Lemma 2.2

\[ \im(\Theta : X_{2i+2}(A) \rightarrow X_{2i+1}(A)) \]

\[ = \{ (H, (P); [P, \delta], [\rho, \theta]) \in X_{2i+2}(A) | (P, \delta) \in X_{2i+1}(A) \} \]

\[ = \{ (Q, \phi, [F, \lambda], [G, \mu]) \in X_{2i+1}(A) | (Q, \phi, [F, \lambda], [G, \mu]) \text{ elementary} \}

\[ \leq \ker(\Theta : X_{2i+1}(A) \rightarrow X_2(A)) . \]

For every \( (Q, \phi, [F, \lambda], [G, \mu]) \in \ker(\Theta : X_{2i+1}(A) \rightarrow X_2(A)) , \)

\[ (Q, \phi, [F, \lambda], [G, \mu]) \in \{ (Q, -\phi, [F, \lambda], [G, -\mu]) \}

\[ = 0 \in U_{2i+1}(A) , \]

as the diagonal \( \Delta_{(Q, \phi)} \) is a hamiltonian complement in \( (G \oplus Q, \phi \oplus -\phi) \) to \( \{ F \oplus \delta, x \oplus -x \} \)

and \( \{ G \oplus G, \mu \oplus -\mu \} \) for any pair of hamiltonian complements \( \{ F, \lambda \}, [F', \lambda'], [G, \mu] \) of \( \{ G, \mu \} \) in \( \{ G, \mu \} \) by Lemma 1.4 , giving inverses for \( U_2(A) \).

Example 3.2 For the group ring \( \mathbb{Z}[\pi] \) of Example 0.1

\[ U_2(\mathbb{Z}[\pi]) = L^A_2(\pi) , \]

the surgery obstruction group in the category \( \mathbb{A} \) of \( S^1 D \) in \( [W] \), of Poincaré complexes up to homotopy.

The construction of the groups \( U_2(A) \)

is not unlike that of the groups \( \tilde{K}_2(A), \tilde{K}_1(A) \)

of algebraic K-theory.

The projective class group of \( A \), \( \tilde{K}_2(A) \), is the group of isomorphism classes \( [P] \) of f.g.

projective \( A \)-modules \( P \) modulo the stable \( \Phi \)-...
free $A$-modules, under the direct sum $\oplus$. Similarly, $U_{2i}(A)$ is the group of isomorphism classes of non-singular $\pm$-forms over $A$, modulo the stably trivial ones.

The Whitehead torsion group of $A$

$$\tilde{K}_1(A) = \frac{\text{GL}(A)}{\text{E}(A)}$$

is a quotient of the general linear group $\text{GL}(A)$ of $A$ by $-1$ and $\text{E}(A)$, the subgroup generated by the elementary matrices, those with $1$'s on the diagonal and at most one other non-zero entry. Whitehead's Lemma states that

$$\text{E}(A) = [\text{GL}(A), \text{GL}(A)]$$

the commutator subgroup of $\text{GL}(A)$. This allows $\tilde{K}_1(A)$ to be considered as the abelian group of isomorphism classes of triples $(Q, f, g)$ consisting of a $F_q$-free $A$-module $Q$ and bases $f = (f_1, \ldots, f_m)$, $g = (g_1, \ldots, g_n)$, under the direct sum.

Similarly, $U_{2i+1}(A)$ is the group of stable isomorphism classes of non-singular $\pm$-forms modulo the elementary ones. Although it is not possible to identify the elements of $U_{2i+1}(A)$ as the "torsions" of automorphisms of a trivial $\pm$-form, they have the formal properties of such. In particular, we have the sum-formula:

$$\text{Lemma 3.3} \quad (Q, \phi; \{F_i\}, [G, \mu]) \oplus (Q, \psi; [G, \nu], [H, \omega]) = (Q, \phi; \{F_i\}, [H, \omega]) \in U_{2i+1}(A).$$

**Proof:** Consider first the special case when $[F, \lambda], [G, \mu]$ have a common hamiltonian complement in $(Q, \delta)$, $[L, \xi]$ say. Then

$$(Q, \delta; [F, \lambda], [G, \mu]) \ominus 0 \in U_{2i+1}(A).$$
and
\[(Q,\phi;[F,\lambda],[H,\mu]) = -(Q,-\phi;[L,-\lambda],[H^*,-\mu])\]
\[= (Q,\phi; [G,\mu],[H,\nu]) \in U_{2i+1}(A),\]

for any hamiltonian complement \([H^*,\nu]\) to \([H,\nu]\) in \((Q,\phi)\).

For general \((Q,\phi;[F,\lambda],[G,\mu]) \in U_{2i+1}(A)\),
\[(Q,\phi;[F,\lambda],[G,\mu]) \Theta (Q,\phi;[G,\mu],[H,\nu]) = (Q,\phi;[F,\lambda],[G,\mu]) \Theta (Q,\phi;[G,\mu],(G,G^*,\mu,-\mu),[H,H^*,\mu,-\mu])\]
(for any hamiltonian complement \([G^*,\nu]\) to \([G,\mu]\) in \((Q,\phi)\))
\[= (Q,\phi;[F,\lambda],[G,\mu]) \Theta (Q,\phi;[G,\mu],(G,G^*,\mu,-\mu),[H,H^*,\mu,-\mu],(F,F^*,\lambda,-\lambda),[L,L^*,\lambda,-\lambda])\]
(by special case and Lemma 1.4, for any hamiltonian complement \([F^*,\lambda]\) to \([F,\lambda]\) in \((Q,\phi)\))
\[= (Q,\phi;[G,\mu],(G,G^*,\mu,-\mu),[H,H^*,\mu,-\mu])\]
\[\Theta (Q,\phi;[F,\lambda],[H,\nu])\]
\[= (Q,\phi;[F,\lambda],[H,\nu]) \in U_{2i+1}(A).\]

\[\square\]

Given lagrangians \([F,\lambda],[F',\lambda']\) in a \(\pm\) form \((Q,\phi)\) such that \(F=F'\), it is clear that the hamiltonian complements to \([F,\lambda]\) in \((Q,\phi)\) are just those to \([F,\lambda]\), and hence, by the sum formula, that
\[(Q,\phi;[F,\lambda],[G,\mu]) = (Q,\phi;[F,\lambda],[G,\mu]) \in U_{2i+1}(A),\]
for any lagrangian \([G,\mu]\) of \((Q,\phi)\).
Similarly,
\[(Q,\phi;[F,\lambda],[G,\mu]) = (Q,\phi;[F,\lambda],[G,\mu]) \in U_{2i+1}(A)\]
Therefore in dealing with elements of \(U_{2i+1}(A)\) it is sufficient to consider the triples \((Q,\phi;F,G)\), also termed \(\pm\) formations, in which \(F,G\) are submodules of \(Q\) carrying lagrangians of the \(\pm\) form \((Q,\phi)\), and which we shall also call lagrangians. The full force of the definitions will be exploited later on, in Part III.
§ 4. V-theory

V-theory is the analogue of U-theory obtained by considering ±-forms and ±-formations on stably f.g. free A-modules rather than f.g. projective ones. All the U-theory done above has an obvious V-theory version. In particular, we can define abelian monoids \( Y_n(A) \) for \( n \equiv 0 \mod 4 \), and morphisms \( \Theta: Y_n(A) \to Y_{n-1}(A) \), exactly as for \( X_n(A) \), to obtain quotient groups

\[
V_n(A) = \ker(\Theta: Y_n(A) \to Y_{n-1}(A)) / \text{im}(\Theta: Y_{n+1}(A) \to Y_n(A))
\]

Example 4.1 For the ground ring \( \mathbb{Z}[T] \) of Example 0.1

\[
V_n(\mathbb{Z}[T]) = B_n(\mathbb{T})
\]

the surgery obstruction group in the category \( B \) of §17 D in [W], of finite Poincaré complexes up to homotopy.

\[
\square
\]

The odd-dimensional groups \( V_{2i+1}(A) \) will now be identified as stable unitary groups.

Define: for \( m > 1 \) the unitary group \( U_{2}(A, m) \) of automorphisms of \( H_{\pm}(mA) \), where \( mA \) is the free \( A \)-module on \( m \) generators.

The function

\[
\tau_m: U_{2}(A, m) \to V_{2i+1}(A);
\]

\[
(f, x) \mapsto (H_{\pm}(mA); mA, f(mA))
\]

is a group morphism: given \( (f, x), (g, y) \in U_{2}(A, m) \) we have that

\[
\tau_m((f, x) \cdot (g, y)) = (H_{\pm}(mA); mA, g(f(mA))
\]

\[
= (H_{\pm}(mA); mA, g(mA)) \oplus (H_{\pm}(mA), g(mA), f(mA))
\]

(by the sum formula of Lemma 3.3 on V-theory)

\[
= (H_{\pm}(mA); mA, g(mA)) \oplus (H_{\pm}(mA), mA, f(mA))
\]

\[
= \tau_m(g, y) \oplus \tau_m(f, x) \in V_{2i+1}(A)
\]

Defining inclusions

\[
U_{2i}(A, m) \to U_{2i+1}(A, m + 1);
\]

\[
(f, x) \mapsto (f, x) \oplus (1, 0),
\]
let
\[ \mathcal{U}_\pm(A) = \bigoplus_{m=1}^\infty \mathcal{U}_\pm(A,m) \]
with the obvious multiplicative group structure.

There is induced a group morphism
\[ \pi: \mathcal{U}_\pm(A) \to \mathcal{V}_2(A) \]
which agrees with \( \pi_m \) on each \( \mathcal{U}_\pm(A,m) \).

Denote the kernel of \( \pi \) by \( \mathcal{H}_\pm(A) \), calling its elements the Hamiltonian transformations.

**Theorem 4.2** The morphism
\[ \pi: \mathcal{U}_\pm(A) \to \mathcal{V}_2(A) \]
is onto, inducing an isomorphism
\[ \mathcal{U}_\pm(A)/\mathcal{H}_\pm(A) \cong \mathcal{V}_2(A). \]

\( \mathcal{H}_\pm(A) \) contains the commutator subgroup
\[ [\mathcal{U}_\pm(A), \mathcal{U}_\pm(A)] \] of \( \mathcal{U}_\pm(A) \), and it is generated by the elementary Hamiltonian transformations:
\[ i) \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right), \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \in \mathcal{U}_\pm(A,m) \text{ for any } \pm \text{ form } (mA, \xi) \]

\[ ii) \left( \begin{array}{cc} 0 & \xi^{-1} \\ \xi & 0 \end{array} \right), \left( \begin{array}{cc} 0 & \xi \\ \xi & 0 \end{array} \right) \in \mathcal{U}_\pm(A,m), \]
for any automorphism \( \alpha \in \text{Hom}_A(mA; mA) \)

\[ iii) \sigma \Theta \sigma^{-1} \cdots \Theta \sigma \in \mathcal{U}_\pm(A,m), \]

involving \( m \) copies of
\[ \sigma = \left( \begin{array}{cc} \phi & 0 \\ 0 & \phi^{-1} \end{array} \right), \left( \begin{array}{cc} 0 & \phi \\ \phi & 0 \end{array} \right) \in \mathcal{U}_\pm(A,1) \]

where
\[ \gamma: A \to A^*, \alpha \mapsto (b \mapsto b\alpha) \]

**Proof:** It is sufficient to verify that \( \pi \) is onto.

Every \((Q, \phi, \mathcal{G}) \in \mathcal{V}_2(A)\) may be represented by
a non-singular \( \pm \)-formation \((H_\pm(mA); mA, \mathcal{G})\) with \( \mathcal{G} \) free, of dimension
\[ \dim A \mathcal{G} = \frac{1}{2} \dim_A (mA \mathcal{G}) = \frac{1}{2} \dim_A (mA \mathcal{G}^*) = m. \]

Choosing a Hamiltonian complement \( \mathcal{G}^* \) of \( \mathcal{G} \)
in \( H_\pm(mA) \), and an isomorphism \( h \in \text{Hom}_A(mA, \mathcal{G}) \), note that the isomorphism
\[ \left( \begin{array}{cc} h & 0 \\ 0 & h^{-1} \end{array} \right): mA \mathcal{G}(mA)^* \to \mathcal{G} \mathcal{G}^* \]
defines an automorphism of \( H_\pm(mA) \) taking \( mA \) to \( \mathcal{G} \).

Thus
\[ \pi \left( \begin{array}{cc} h & 0 \\ 0 & h^{-1} \end{array} \right) = (Q, \phi, \mathcal{G}) \in \mathcal{V}_2(A). \]
V-theory differs from U-theory in 2-torsion only.

**Theorem 4.3** There is an exact sequence

\[ \cdots \to \Sigma_+^+(A) \to V_{2i+1}(A) \to U_{2i+2}(A) \to \Sigma_+^-(A) \to \cdots \]

where \( \Sigma_+^+(A) \) and \( \Sigma_+^-(A) \) are defined by

\[ \Sigma_+^+(A) = \{ [P] \in \text{K}_0(A) | [P^*] = \pm [P] \}/\{[P] = [P^*] | \omega \in \text{K}_0(A)\} \]

and are of exponent 2.

The groups \( \Sigma_+^+(A) \) are of abelian groups and morphisms, defined for \( i(\bmod 2) \).

The morphisms

\[ V_n(A) \to U_n(A) \]

are induced by the obvious inclusions of \( V_n(A) \) in \( X_n(A) \).

The others are given by:

\[ U_{2i}(A) \to \Sigma_+^+(A); (P, \theta) \mapsto [P] \]

\[ U_{2i+1}(A) \to \Sigma_+^-(A); (P, \phi, \psi, \xi) \mapsto [G] - [F^*] \]

\[ \Sigma_+^-(A) \to V_{2i}(A); [P] \mapsto H_+(P) \]

\[ \Sigma_+^+(A) \to V_{2i-1}(A); [P] \mapsto (H_+(P \oplus P^*), P \oplus P^* \oplus P^* \oplus P^*) \]

for any representative \(-P \oplus [-P] \in \text{K}_0(A)\).

**Proof:** It is easy to verify that the given morphisms are well-defined, except perhaps \( \Sigma_+^+(A) \to V_{2i-1}(A) \). This sends \([P] \in \Sigma_+^+(A)\) to

\[ (H_+(P \oplus P^* \oplus P^* \oplus P^*), P \oplus P^* \oplus P^* \oplus P^*) \to \]

which vanishes in \( V_{2i-1}(A) \) because

\[ \Sigma (0, 0, x, y, z, w, \pm x, \pm y, \pm z, \pm w) \in P \oplus P^* \oplus P^* \oplus (P \oplus P^* \oplus P^*) \]

is a common Hamiltonian complement.

Further, it is not difficult to see that the composition of successive morphisms is 0, except perhaps at \((\text{III})\) and \((\text{VI})\).

at \((\text{III})\) note that every \((G, \psi, \phi, \xi) \in U_{2i+1}(A)\) has a representative \pm-torsion with \( G \) free, so that

\[ H_+(F) = (G, \phi) - H_+(G) = 0 \in V_{2i}(A) \]

at \((\text{VI})\) the composite \( U_{2i}(A) \to V_{2i-1}(A) \) sends \((G, \phi) \in U_{2i}(A)\) to

\[ (H_+(G \oplus G); G \oplus G, G \oplus G) \to H_+(G \oplus G) = (G, \phi) \in V_{2i-1}(A) \]

(by Lemma 3.3 for V-theory)
and \( \Gamma_Q \oplus -Q^* \) is a hamiltonian component to both \( Q \oplus -Q \) and \( Q^* \oplus -Q \) in \( H \oplus (Q \oplus -Q) \), as \( Q \oplus \Gamma_Q \) is non-singular, so that

\[
H \oplus (Q \oplus -Q); Q \oplus -Q, Q^* \oplus -Q \rangle = \text{im } V_{2i-1}(A).
\]

We now verify exactness at each point of the sequence:

(I) \( \Sigma_+(A) \rightarrow V_{2i+1}(A) \rightarrow U_{2i+1}(A) \)

Every \( (Q, \phi, F, G) \in \ker (V_{2i+1}(A) \rightarrow U_{2i+1}(A)) \) can be represented as

\[
(H \oplus (P \oplus L); P \oplus L, P^* \oplus L^*)
\]
for some \( f \)-form \( (P, \Theta) \) and \( f \)-projective \( L \), such that \( P \oplus L, P^* \oplus L^* \) are free.

Applying the sum formula of Lemma 3.3 for \( V \)-theory

\[
(Q, \phi; F, G) = (H \oplus (P \oplus L); P \oplus L, P^* \oplus L^*)
\]

\[
= (H \oplus (P \oplus L); P \oplus L, P^* \oplus L) \in \text{im } (\Sigma_+(A) \rightarrow V_{2i+1}(A))
\]

(II) \( V_{2i+1}(A) \rightarrow U_{2i+1}(A) \rightarrow \Sigma_-(A) \)

Let \( (Q, \phi; F, G) \in \ker (U_{2i+1}(A) \rightarrow \Sigma_-(A)) \), so that

\[
\phi - [Q^*] = [P^*] - [P] \in \mathcal{K}_0(A)
\]

for some \( f \)-projective \( P \).

Denote by \( M \) a \( f \)-projective \( A \)-module such that

\[
[M] = -[G^* \oplus P^*] = -[F \oplus P] \in \mathcal{K}_0(A)
\]

Then

\[
(Q, \phi, F, G) = (Q, \phi) \oplus H_\pm(P \oplus L^*); F \oplus P, G \oplus P \in \text{im } (V_{2i+1}(A) \rightarrow U_{2i+1}(A)).
\]

(III) \( U_{2i+1}(A) \rightarrow \Sigma_-(A) \rightarrow V_{2i}(A) \)

If \( [P] \in \ker (\Sigma_-(A) \rightarrow V_{2i}(A)) \), it may be assumed that \( H \oplus (P \oplus L) \) has a free lagrangian \( L, \) say, then

\( U_{2i+1}(A) \rightarrow \Sigma_-(A) \) sends \( (H \oplus (P \oplus L), P \) to \( [P] \in \Sigma_-(A) \).

(IV) \( \Sigma_-(A) \rightarrow V_{2i}(A) \rightarrow U_{2i}(A) \)

If \( (Q, \phi) \in \ker (V_{2i}(A) \rightarrow U_{2i}(A)) \), it may be assumed that \( Q \oplus \Gamma_Q \) has a \( (\text{projective}) \) lagrangian, \( P, \) say. Then

\[
[P] + [P^*] = [Q] = 0 \in \mathcal{K}_0(A)
\]

and \( (Q, \phi) = H \oplus (P \in \text{im } (\Sigma_-(A) \rightarrow V_{2i}(A)) \).

(V) \( V_{2i}(A) \rightarrow U_{2i}(A) \rightarrow \Sigma_+(A) \)

If \( (Q, \phi) \in \ker (U_{2i}(A) \rightarrow \Sigma_+(A)) \), then

\[
[Q] = [P] + [P^*] \in \mathcal{K}_0(A)
\]

for some \( f \)-projective \( P \) and

\[
(Q, \phi) = (Q, \phi) \oplus H \oplus (-P) \in \text{im } (V_{2i}(A) \rightarrow U_{2i}(A)).
\]
(V1) $U_{2i}(A) \rightarrow \Sigma_{+}(A) \rightarrow V_{2i-1}(A)$

Given $[P] \in \ker(\Sigma_{+}(A) \rightarrow V_{2i-1}(A))$, it may be assumed that up to isomorphism of formations

$$(H_{F}(P \Theta - P); P \Theta - P, P \Theta - P^*) \subseteq (H_{F}(L); L, L, L)$$

$$= (H_{F}(M); M, M, M)$$

for some $\pm$ forms $(L, \lambda), (M, \mu)$ defined on $F$; free $L, M$.

Now $(Q, \Phi; F, G) = (H_{F}(M); M, M, M)$ is an elementary $F$ formation, with $H = M^* \Phi$ a hamiltonian complement to both $F$ and $G$ in $(Q, \Phi)$. Moreover,

$F^* = P \Theta - P \Theta L$ is a hamiltonian complement to $F = P \Theta - P \Theta L$ in $(Q, \Phi)$ such that the projection on $F$ along $F^*$,

$$\pi: Q = F \Theta F^* \rightarrow F,$$

sends $G = P \Theta - P \Theta L$ onto the direct summand $\pi(G) = P \Theta L$ of $F$. Thus $F \Theta F^*$ is a hamiltonian complement to $\Delta_{(Q, \Phi)}$ in $G \Theta Q, \Phi \Theta - \Phi$ such that the projection on $\Delta_{(Q, \Phi)}$ along $F \Theta F^*$,

$$Q \Theta Q \rightarrow \Delta_{(Q, \Phi)};$$

$$(x, y) \mapsto (\pi(x) + (1-\pi)(y), \pi(x) + (1-\pi)(y)),$$

sends $G \Theta H$ onto the submodule

$$N = \{ (x, y) \in \Delta_{(Q, \Phi)} | x \in \pi(G) \Theta (1-\pi)(H)^{3} \} \subseteq \Delta_{(Q, \Phi)}.$$

As $H$ is a hamiltonian complement to $F$ in $(Q, \Phi)$,

$$(1-\pi)(H) = F^*,$$

and $N$ is a direct summand of $\Delta_{(Q, \Phi)}$ isomorphic to $P \Theta L \Theta F^*$, with direct complement isomorphic to $-P$.

Applying Theorem 2.3, we have that up to isomorphism

$$(Q \Theta Q, \Phi \Theta - \Phi, \Delta_{(Q, \Phi)}, G \Theta H) = (H_{F}(N), N, P_{(N, \mu)} \subseteq (H_{F}(P), -P, F^*))$$

for some $\pm$ form $(N, \lambda)$, which must be non-singular, as

$$(Q \Theta Q, \Phi \Theta - \Phi, \Delta_{(Q, \Phi)}, G \Theta H)$$

is a trivial $\Theta$ formation

($H$ being a hamiltonian complement to $G$ in $(Q, \Phi)$). Thus

$$[N] = [P \Theta L \Theta F^*] = [P]$$

$\in \text{im}(U_{2i}(A) \rightarrow \Sigma_{+}(A)).$
§5 W-theory

A based $A$-module $Q$, is a f.g. free $A$-module $Q$ together with a base $b=(b_1,b_2,\ldots,b_m)$. The dual based $A$-module $Q^*$ is defined, with base $b^*=(b_1^*,b_2^*,\ldots,b_m^*)$, where

$$b_j^*(b_k) = \begin{cases} 1 & \text{if } j=k \\ 0 & \text{if } j\neq k \end{cases}$$

and $Q^{**}$ is identified with $Q$.

The matrix representation in $GL(A)$ of an isomorphism $f \in \text{Hom}_A(P,Q)$ of based $A$-modules $P,Q$ defines the torsion of $f$, $\tau(f) \in K_1(A)$. The isomorphism is simple if $\tau(f)=0 \in K_1(A)$.

A based $\pm$-form (over $A$), $(Q,\phi)$, is a $\pm$-form $(Q,\phi)$ defined on a based $A$-module $Q$. An isomorphism of based $\pm$-forms $(f,\chi):(P,\theta) \rightarrow (Q,\phi)$ is simple if the isomorphism $f:P \rightarrow Q$ is simple.

W-theory deals with the simple isomorphism properties of based $\pm$-forms, just as $U$-theory considers the isomorphism of $\pm$-forms and $V$-theory that of $\pm$-forms on stably f.g. free modules.

Define the hamiltonian based $\pm$-form on a based $A$-module $P$, $H_+(P)$, to be $H(P)$ with base $P \oplus P^*$, A based $\pm$-form is trivial if it is simple isomorphic to a based hamiltonian one.

Let $L$ be a free lagrangian of a trivial $\pm$-form $(Q,\phi)$. A base $L$ of $L$, together with the dual $L^*$ on a hamiltonian complement $L^*$ defines a hamiltonian base of $(Q,\phi)$, $Q=L \oplus L^*$. A different choice of hamiltonian complement $L^*$ alters this hamiltonian base by a simple automorphism

$$\left( \begin{array}{cc} 1 & \phi \phi^* \\ 0 & 1 \end{array} \right): L \oplus L^* \rightarrow L \oplus L^*$$

of $Q$ for some $\pm$-form $(L,\theta)$, by Lemma 1.8. Thus every base $L$ can be extended to a hamiltonian base of $(Q,\phi)$ uniquely up to simple isomorphism.
A based lagrangian $L$ of a based $\pm$-form $(Q,\Phi)$ is a lagrangian $L$ of $(Q,\Phi)$ (or, more properly, $L(L)$ together with a base $L$ such that $L$ differs from a hamiltonian base extending $L$ by a simple automorphism.

By analogy with Theorem 1.1 we have:

**Theorem 5.1** A based $\pm$-form is trivial iff it admits a based lagrangian. $\Box$

Given a non-singular based $\pm$-form $(Q,\Phi)$ over $A$, define its torsion to be

$$\tau(Q,\Phi) = \tau(Q^*;Q \to Q^*) \in \mathbb{K}_1(A).$$

Torsion is a simple $\pm$-form isomorphism invariant, and as based hamiltonian $\pm$-forms have zero torsion, so do all trivial based $\pm$-forms.

A based sublagrangian $L$, of a based $\pm$-form $(Q,\Phi)$ is a free sublagrangian $L$ of $(Q,\Phi)$ such that $L/L$ is free, together with bases $L_L$, $L_L$ such that the subhamiltonian base

these determine on $(Q,\Phi)$ agrees with $Q$ up to simple isomorphism.

By analogy with Corollary 1.2 we have:

**Corollary 5.2** The inclusion of a based sublagrangian

$$(f,\lambda): (L,\Phi) \to (Q,\Phi)$$

may be extended to a simple isomorphism

$$(f,\lambda): (L,\Phi) \oplus H^*(L) \to (Q,\Phi).$$

In particular, if $(Q,\Phi)$ is non-singular, then

$$\tau(Q,\Phi) = \tau(L,\Phi) \in \mathbb{K}_1(A).$$ $\Box$

A based hamiltonian complement to a based lagrangian $E$ of a $\pm$-form $(Q,\Phi)$ is a based lagrangian $G$ such that

$$G \to E^*; \alpha \mapsto (y \mapsto <x, y>_{\Phi})$$

defines a simple isomorphism (of $A$-modules), in which case $G$ may be identified with $E^*$. Lemma 1.3 has based version:
Lemma 5.3 The based hamiltonian complements to $P^*$ in $H^\pm(R)$ are the graphs

$$P^*_{(\phi, \theta)} = \text{im}((\phi^* \theta, \theta) : (P, \phi) \to H^\pm(P))$$

of based $\pm$-forms $(P, \phi)$, up to simple changes of base.

Lemma 1.4 has based version:

Lemma 5.4 Let $(Q, \phi)$ be a non-singular based $\pm$-form such that $z(Q, \phi) = 0 \in K(A)$. Then

$$\Delta_{(Q, \phi)} = \text{im}((\phi^* \theta, \theta) : (Q, \phi) \to (\text{Sym} Q, \phi \pm \phi^*))$$

is a based lagrangian of $(Q, \phi \pm (Q, -\phi))$, with based hamiltonian complements

$$\Delta^*_{(Q^*, \psi)} = \text{im}((\mu, \psi \psi^* \theta) : (Q^*, \psi) \to (\text{Sym} Q^*, \psi \psi^*))$$

with

$$\psi = (\phi \pm \phi^*)^* \phi (\phi \pm \phi^*)^* + \chi \star \chi^* \in \text{Hom}_A(Q^* , Q)$$

for some $\chi \in \Pi_{\pm}(Q^*)$.

In particular, the diagonal of a trivial based $\pm$-form $(Q, \phi)$ is a based hamiltonian complement to $\text{Sym}^* Q$ in $(\text{Sym} Q, \phi \pm \phi^*)$ for any based hamiltonian complement $E^*$ in $(Q, \phi)$.

A based $\pm$-formation $(Q, \phi; [E, \lambda], [G, \mu])$ is defined by:

i) a trivial based $\pm$-form $(Q, \phi)$

ii) a based lagrangian $[E, \lambda]$ of $(Q, \phi)$

iii) a based sublagrangian $[G, \mu]$ of $(Q, \phi)$.

An isomorphism of based $\pm$-formations

$$(h, \psi) : (Q, \phi; [E, \lambda], [G, \mu]) \to (Q', \phi'; [E', \lambda'], [G', \mu'])$$

is simple if it is defined by a simple isomorphism

$$(h, \psi) : (Q, \phi) \to (Q', \phi')$$

which restricts to simple isomorphisms

$$(E \to E', G \to G', \phi \to \phi').$$

The definitions and propositions of §5.2 have obvious based analogues in particular:

Theorem 5.5 For $n \equiv \pm 1 (\text{mod} 4)$ let $Z_n(A)$ be the abelian monoid of simple isomorphism classes of based $\pm$-formations over $A$, under the direct sum operation, where

$$z \pm \text{forms} \text{ stable simple isomorphism}$$

$$\pm = (-)^i \text{ if } n = \frac{2^i}{2i+1}.$$
The monoid morphisms
\[ \Theta : \mathbb{Z}_n(A) \to \mathbb{Z}_{n-1}(A) ; \]
\[ \left\{ \begin{array}{l}
(\mathcal{P}, \theta) \mapsto (\mathcal{H} \circ \mathcal{Q}, \mathcal{P} \circ \mathcal{Q}, \mathcal{Q}, \phi, \epsilon_1, \epsilon_2, \epsilon_3) \\
(\mathcal{Q}, \phi, \epsilon_1, \epsilon_2, \epsilon_3) \mapsto (\mathcal{Q}, \phi, \epsilon_1, \epsilon_3)
\end{array} \right. \quad n = \ell \geq 1 \]
are well-defined and such that \( \Theta^2 = 0 \), with the quotient monoids
\[ W_n(A) = \ker(\Theta : \mathbb{Z}_n(A) \to \mathbb{Z}_{n-1}(A)) / \text{im}(\Theta : \mathbb{Z}_{n-1}(A) \to \mathbb{Z}_n(A)) \]
are the quotient groups.

As in the proof of Theorem 3.1 we can identify:
\[ \ker(\Theta : \mathbb{Z}_{2i}(A) \to \mathbb{Z}_{2i+1}(A)) \]
\[ = \{ (\mathcal{P}, \theta) \in \mathbb{Z}_{2i}(A) \mid (\mathcal{P}, \theta) \text{ non-singular, } c(\mathcal{P}, \theta) = 0 \in \mathcal{K}_1(A) \} \]
\[ \text{im}(\Theta : \mathbb{Z}_{2i+1}(A) \to \mathbb{Z}_{2i}(A)) \]
\[ = \{ (\mathcal{Q}, \phi) \in \mathbb{Z}_{2i}(A) \mid (\mathcal{Q}, \phi) \text{ trivial} \} \]
\[ \ker(\Theta : \mathbb{Z}_{2i+1}(A) \to \mathbb{Z}_{2i}(A)) \]
\[ = \{ (\mathcal{Q}, \phi, [\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3]) \in \mathbb{Z}_{2i+1}(A) \mid (\mathcal{Q}, \phi, [\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3]) \text{ non-singular} \}
\[ \text{im}(\Theta : \mathbb{Z}_{2i+2}(A) \to \mathbb{Z}_{2i+1}(A)) \]
\[ = \{ (\mathcal{Q}, \phi, [\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3]) \in \mathbb{Z}_{2i+1}(A) \mid (\mathcal{Q}, \phi, [\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3]) \text{ trivial} \} \]

Example 5.6 For the ground ring \( \mathbb{Z}[\mathfrak{B}] \) of Example 2.4:
\[ W_n(\mathbb{Z}[\mathfrak{B}]) = L_n^\mathfrak{B}(\pi) \]
The surgery obstruction group in the category \( \mathcal{E} \) of \( \mathcal{E} \)-modules of simple Poincaré complexes up to simple homotopy.

(This is slightly broadened: in the geometrical case, equivalence of bases is measured not in \( \mathcal{E}(\mathbb{Z}[\mathfrak{B}]) \), but in the Whitehead group of \( \pi \)
\[ \text{Wh}(\pi) = \mathcal{E}(\mathbb{Z}[\mathfrak{B}]) / \text{im}(\mathcal{E}(\mathbb{Z}[\mathfrak{B}]) \to U(\mathbb{Z}[\mathfrak{B}]) \to \mathcal{E}(\mathbb{Z}[\mathfrak{B}])) \]
where \( U(\mathbb{Z}[\mathfrak{B}]) \) is the multiplicative group of units of \( \mathbb{Z}[\mathfrak{B}] \), regarded as a subgroup of \( \mathcal{E}(\mathbb{Z}[\mathfrak{B}]) \) in the obvious way).

The odd-dimensional groups \( W_{2i+1}(A) \) will now be identified as stable special unitary groups, by analogy with Theorem 4.2 for \( U \)-theory.

Define for \( m \geq 1 \) the special unitary group \( SU^+(A, m) \) of simple automorphisms of \( H^*_A(\mathfrak{b}) \) where \( m \mathbb{A} \) is the based \( A \)-module on \( m \)-generators.
The functions

$$\pi'_m: \mathcal{S}U_\pm(A, m) \to W_{2i+1}(A);$$

$$(f, \lambda) \mapsto (H_\pm(mA); mA, f(mA))$$

are group morphisms (by Lemma 3.3 for W-theory).

Defining inclusions

$$\mathcal{S}U_\pm(A, m) \to \mathcal{S}U_\pm(A, m+1); (f, \lambda) \mapsto (f, \lambda) \Theta(1, 0)$$

there is induced a group morphism

$$\pi': \mathcal{S}U_\pm(A) = \bigoplus_{m=1}^{\infty} \mathcal{S}U_\pm(A, m) \to W_{2i+1}(A)$$

agreeing with $$\pi'_m$$ on each $$\mathcal{S}U_\pm(A, m).$$

Denote the kernel of $$\pi'$$ by $$\mathcal{H}C_\pm(A)$$, calling its elements the special hamiltonian transformations.

**Theorem 5.6** The morphism

$$\pi': \mathcal{H}C_\pm(A) \to W_{2i+1}(A)$$

is onto, inducing an isomorphism

$$\mathcal{S}U_\pm(A)/\mathcal{H}C_\pm(A) \cong W_{2i+1}(A)$$

of abelian groups.

Moreover, $$\mathcal{H}C_\pm(A)$$ contains the commutator subgroup $$[\mathcal{S}U_\pm(A), \mathcal{S}U_\pm(A)]$$ of $$\mathcal{S}U_\pm(A)$$ and it is generated by the elementary special hamiltonian transformations.

i) $$\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & \alpha \\ \beta & 0 \end{smallmatrix}\right) \in \mathcal{S}U_\pm(A, m)$$

for any basis form $$(mA, \beta)$$

ii) $$\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha^* \end{smallmatrix}\right), 0 \in \mathcal{S}U_\pm(A, m)$$

for any simple automorphism $$\alpha: mA \to mA$$

iii) $$\sigma \Theta \Theta: \Theta \sigma \in \mathcal{S}U_\pm(A, m)$$

involving $$m$$ copies of

$$\sigma = \left(\begin{smallmatrix} 0 & \pm \bar{\gamma} \\ \bar{\gamma} & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & \gamma \\ \gamma & 0 \end{smallmatrix}\right) \in \mathcal{S}U_\pm(A, 1)$$

where

$$\gamma: A \to A^*; a \mapsto (b \mapsto b \alpha).$$

Then Theorem 6.3 in [W] gives

**Corollary** The quotient $$\mathcal{S}C_\pm(A)/[\mathcal{S}U_\pm(A), \mathcal{S}U_\pm(A)]$$

is generated by $$\sigma$$, so has order at most 2.

W-theory is related to V-theory by
**Theorem 5.7** There is an exact sequence
\[ \ldots \to \Omega_{\pm}(A) \to W_*(A) \to V_*(A) \to \Omega_{\pm}(A) \to \ldots \]
of abelian groups and morphisms defined for \( n \equiv 1 \pmod{2} \). The groups
\[ \Omega_{\pm}(A) = \{ \omega \in \Omega_{\pm}(A) \mid \omega^* = \pm \omega \in \Omega_{\pm}(A) \} \]
are of exponent 2.

The morphisms \( W_*(A) \to V_*(A) \) are induced by the monoid morphisms \( \mathbb{Z}_n(A) \to V_n(A) \) which forget bases. The others are given by:
\[ V_{2i}(A) \to \Omega_{+}(A) ; (P, \phi) \mapsto \tau(P, \phi) \]
for any basis of \( P \), assumed free.

\[ V_{2i+1}(A) \to \Omega_{-}(A) ; (Q, \phi ; F, \theta) \mapsto \tau(F; \phi, F, \theta) \]
\[ \Omega_{-}(A) \to W_{2i}(A) ; \tau(x : P \to P) \mapsto \tau(x : P \to P) \]
\[ \Omega_{+}(A) \to W_{2i+1}(A) ; \tau(x : P \to P) \mapsto \tau(x : P \to P) \]

**Proof:** By analogy with that of Theorem 4.3, with torsions of automorphisms in \( \Omega_{\pm}(A) \), replacing projective classes in \( \Omega_{\pm}(A) \).

\[ \square \]

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**§6 Functionality**

All our constructions are functorial on the ground ring \( A \). Let
\[ f : A \to B \]
be a morphism of ground rings (preserving the \( \mathcal{L} \) and the involutions). Give \( B \) an \((A,B)\)-bimodule structure by
\[ B \times B \times A \to B ; (b, x, a) \mapsto b \cdot x \cdot f(a) \]

Given a \( F \)-projective left \( A \)-module \( P \),
let \( fP \) denote the \( F \)-projective left \( B \)-module.
\( B \otimes_A P \), identifying \((fP)^*\) with \( f(P)^*\). A morphism
\[ \phi \in \text{Hom}_A(P,Q) \]
induces
\[ f\phi = (1 \otimes \phi : B \otimes_A P \to B \otimes_A Q) \in \text{Hom}_B(fP,fQ) \]

Given a \[ \sum \pm \text{form } (P, \phi) \]
with torsions of automorphisms in \( \Omega_{\pm}(A) \), replacing projective classes in \( \Omega_{\pm}(A) \).

over \( A \), there is defined a \[ \sum \pm \text{form } f(P, \phi) = (fP, f\phi) \]
over \( B \) (with \( f[F, X] = [FF, FX] \) etc.), and similar \( \square \)
for morphisms.
The induced monoid morphisms
\[ f : X_n(A) \longrightarrow X_n(B) \]
are such that the squares
\[ \begin{array}{ccc}
X_n(A) & \xrightarrow{f} & X_n(B) \\
\downarrow & & \downarrow \\
X_{n-1}(A) & \xrightarrow{f} & X_{n-1}(B)
\end{array} \]
commute, inducing abelian group morphisms
\[ f : U_n(A) \longrightarrow U_n(B) \quad (n \mod 4). \]
Similarly for \( V - \), \( W - \) theories.

The isomorphisms of Theorem 4.2, 5.6 and the exact sequences of Theorems 4.3, 5.7
are natural on \( A \).

\[ I 6.2 \]

\[ \text{II Algebraic L-theory} \]

\[ \text{§ 1 Laurent extensions} \]

Let \( A \) be a ring with involution
such as is considered in I, and let \( z \) be
an invertible indeterminate over \( A \), which
commutes with every element of \( A \). The
Laurent extension of \( A \) by \( z \), \( A_z \), is the ring
of polynomials \( \sum_{j=-\infty}^{\infty} a_jz^j \) in \( z, z^{-1} \) with only a
finite number of the coefficients \( a_j \in A \) non-zero.

Then \( A_z \) is an associative ring with a 1, under
the usual addition and multiplication of
polynomials. The function
\[ \iota : A_z \longrightarrow A_z, \quad a = \sum_{j=-\infty}^{\infty} a_jz^j \mapsto \bar{a} = \sum_{j=-\infty}^{\infty} \bar{a}_jz^{-j} \]
is an involution of \( A_z \). The projection
\[ \pi : A_z \longrightarrow A, \quad \sum_{j=-\infty}^{\infty} a_jz^j \mapsto \sum_{j=-\infty}^{\infty} a_j \]
is a ring morphism which preserves 1's (and the
involutions), so every \( F_g \)-free \( A_z \)-module \( Q \) has a
well-defined dimension, namely that of the f.g. free $A$-module $\mathcal{E}$. Thus $A_\mathcal{E}$ satisfies all the conditions imposed on $A$.

For example, if $A = \mathbb{Z}[\pi]$ (as in Example 0.4 of I), with $\pi = \pi_1(M)$ for some compact manifold $M$, then $A_\mathcal{E} = \mathbb{Z}[\pi \times \mathcal{E}]$, with $\pi \times \mathcal{E} = \pi_1(M \times S^1)$.

Denote (left) $A_\mathcal{E}$-modules by $\mathcal{M}, \mathcal{N}, \mathcal{P}, \mathcal{Q}$.

In general, the injection $\mathcal{E} : A \rightarrow A_\mathcal{E} ; a \mapsto a$ splits $\mathcal{E}$, that is $\mathcal{E} \mathcal{E} = 1$, and $\mathcal{E} A$ is identified with $A$.

Every $A_\mathcal{E}$-module $\mathcal{Q}$ can be regarded as an $A$-module by restricting the action of $A_\mathcal{E}$ to one of $A$.

A modular $A$-base of an $A_\mathcal{E}$-module $\mathcal{Q}$ is an $A$-submodule $\mathcal{Q}_1$ of $\mathcal{Q}$ such that every $x \in \mathcal{Q}$ has a unique expression as

$$x = \sum_{j=-\infty}^{\infty} z^j x_j \in \mathcal{Q} \ (x_j \in \mathcal{Q})$$

with $\sum x_j \in \mathcal{Q} \mid x_j \neq 0$ finite, corresponding to an infinite direct sum

$$\mathcal{Q} = \sum_{j=-\infty}^{\infty} \mathcal{E} z^j \mathcal{Q}$$

of $A$-modules isomorphic to $\mathcal{Q}$. Hence there is an $A$-module isomorphism

$$(\mathcal{E} \mathcal{Q}) / (\mathcal{E} - 1) \mathcal{Q} \cong \mathcal{Q}$$

and modular $A$-bases of isomorphic $A_\mathcal{E}$-modules are isomorphic.

Given an $A$-module $\mathcal{Q}$, define the $A_\mathcal{E}$-module freely generated by $\mathcal{Q}, \mathcal{Q}_1$, to be the direct sum

$$\mathcal{Q}_1 = \sum_{j=-\infty}^{\infty} \mathcal{E} z^j \mathcal{Q}$$

of a countably infinity of copies of $\mathcal{Q}$ with the action of $A_\mathcal{E}$ indicated — that is, $\mathcal{Q}_1 = \mathcal{E} \mathcal{Q}$.

Then $\mathcal{Q}$ is a modular $A$-base of $\mathcal{Q}_1$.

It is convenient to list here several properties of modular $A$-bases:
i) Every modular A-base $Q$ of an $A_\mathbb{Z}$-module $Q$ determines a dual modular A-base $Q^*$ of $Q^*$, with
\[
(z^k g)(z^j x) = g(x) \cdot z^{-k} \in A_\mathbb{Z}, \quad (g \in Q^*, x \in Q, j, k \in \mathbb{Z})
\]
i) For any $A$-modules $P, Q$ give $\text{Hom}_A(P, Q)$ a (left) $A$-module structure by
\[
A \times \text{Hom}_A(P, Q) \rightarrow \text{Hom}_A(P, Q); (a, f) \mapsto (x \mapsto a \cdot f(x))
\]
and similarly for $A_\mathbb{Z}$-modules.

Every $f \in \text{Hom}_{A_\mathbb{Z}}(P_\mathbb{Z}, Q_\mathbb{Z})$ defines $\sum_{j = -\infty}^{\infty} z^j f_j \in (\text{Hom}_A(P, Q))_z$ by
\[
f(x) = \sum_{j = -\infty}^{\infty} z^j f_j(x) \in A_\mathbb{Z}, \quad (x \in P, f(x) \in Q)
\]
and conversely, so that we may identify
\[
\text{Hom}_{A_\mathbb{Z}}(P_\mathbb{Z}, Q_\mathbb{Z}) = (\text{Hom}_A(P, Q))_z.
\]
Given $f \in \text{Hom}_A(P, Q)$, let $f$ also denote the element of $\text{Hom}_{A_\mathbb{Z}}(P_\mathbb{Z}, Q_\mathbb{Z})$ defined by
\[
f : P_\mathbb{Z} \rightarrow Q_\mathbb{Z};
\]
\[
\sum_{j = -\infty}^{\infty} z^j x_j \mapsto \sum_{j = -\infty}^{\infty} z^j f(x_j) \quad (x_j \in P)
\]

ii) Let $F, G$ be two modular $A_\mathbb{Z}$-bases of a $fg.$ free $A_\mathbb{Z}$-module $Q.$ Then $FG$ are $fg.$ free $A$-modules and $z^N F^+ \subseteq G^+$
for sufficiently large integers $N > 0.$ For such $N$
define the $A$-module
\[
B^+_N(F, G) = z^N F^- \oplus z^N G^+
\]
a direct summand of $Q$ (regarded as an $A_\mathbb{N}$-module with $G^+ = z^N F^+ \oplus B^+_N(F, G)$,
If $H$ is another modular $A$-base of $G$, and $M > 0$ is so large that $z^{M}G^{+} \subset H^{+}$, then

$$B_{M+N}^{+}(F, H) = z^{M}B_{N}^{+}(FG) \oplus B_{M}^{+}(GH).$$

In particular, for $N > 0$ so large that $z^{N}G^{+} \subset H^{+}$,

$$z^{N}B_{N}^{+}(G,F) \oplus B_{N}^{+}(FG) = B_{N+N}^{+}(G,G) = \sum_{j=0}^{N+N-1} z^{j} G,$$

so that, as $G$ is f.g. free, $B_{N}^{+}(FG)$ is a f.g. projective $A$-module.

Moreover, as

$$B_{N+1}^{+}(F,G) = B_{N}^{+}(FG) \oplus z^{N}F,$$

and $F$ is f.g. free, the projective class $[B_{N}^{+}(FG)] \in \mathcal{K}_{1}(A)$ does not depend on $N$.

The $A$-module isomorphism

$$B_{N}^{+}(F^{*},G^{*}) \rightarrow B_{N}^{+}(FG)^{*},$$

$$g \mapsto (a \mapsto [g(a)]_{0})$$

is used as an identification, where $[a]_{0} = a_{0} \in A$

$$a = \sum_{j=-\infty}^{\infty} a_{j}z^{j} \in A_{A}.$$

We now quote a principal result of algebraic $K$-theory (Chapter XII of Bass' "Algebraic $K$-theory")

**Theorem** There exists a natural direct sum decomposition

$$\mathcal{K}_{1}(A_{2}) = \mathcal{K}_{1}(A) \oplus \mathcal{K}_{1}(A) \oplus \text{Nil}^{+}(A) \oplus \text{Nil}^{-}(A)$$

where $\text{Nil}^{+}(A)$ is the subgroup of $\mathcal{K}_{1}(A_{2})$ generated by

$$\{ \tau((1+z^{2}a): \mathcal{F}_{z} \rightarrow \mathcal{F}_{z}) \in \mathcal{K}_{1}(A_{2}) \mid \text{det}(A_{z} \sigma) \text{ nilpotent} \}$$

The splitting is by injections

$$E: \mathcal{K}_{1}(A) \rightarrow \mathcal{K}_{1}(A_{2});\tau(\alpha: \mathcal{F}_{z} \rightarrow \mathcal{F}_{z}) \mapsto \tau(\alpha: \mathcal{F}_{z} \rightarrow \mathcal{F}_{z})$$

$$B: \mathcal{K}_{0}(A) \rightarrow \mathcal{K}_{1}(A_{2});\tau(\alpha: \mathcal{F}_{z} \rightarrow \mathcal{F}_{z}) \mapsto \tau(\alpha: \mathcal{F}_{z} \rightarrow \mathcal{F}_{z})$$

and by projections

$$E: \mathcal{K}_{1}(A_{2}) \rightarrow \mathcal{K}_{1}(A);\tau(\sum_{j=0}^{\infty} a_{j}z^{j}: \mathcal{F}_{z} \rightarrow \mathcal{F}_{z}) \mapsto \tau(\sum_{j=0}^{\infty} a_{j}z^{j}: \mathcal{F}_{z} \rightarrow \mathcal{F}_{z})$$

$$B: \mathcal{K}_{1}(A_{2}) \rightarrow \mathcal{K}_{0}(A);\tau(\alpha: \mathcal{F}_{z} \rightarrow \mathcal{F}_{z}) \mapsto [B_{N}^{+}(F^{*},\sigma(F))].$$

**Corollary** The diagram of abelian groups and morphisms

$$\begin{array}{ccc}
\mathcal{K}_{1}(A_{2}) & \xrightarrow{\pi} & \mathcal{K}_{1}(A) \\
E \downarrow & & E \downarrow \\
\mathcal{K}_{1}(A) & \xrightarrow{\pi} & \mathcal{K}_{1}(A)
\end{array}$$

commutes (in the sense that $E \circ \pi = \pi \circ E$, $\pi \circ E = E \circ \pi$).
We wish to establish an analogous result in algebraic L-theory.

**Theorem 1.1** There exists a natural diagram

\[
\begin{array}{cccc}
\ldots & \Omega(A) & \rightarrow & \Omega(A) \\
\ldots & \Omega(A) & \rightarrow & \Omega(A) \\
\ldots & \Omega(A) & \rightarrow & \Omega(A) \\
\ldots & \Omega(A) & \rightarrow & \Omega(A) \\
\end{array}
\]

of abelian groups and morphisms, defined for \( n \equiv a \mod 4 \), with the squares of shape \( \square \), \( \sqcup \), \( \sqcup \) commuting. The rows are the exact sequences of Theorems 4.2, 5.7, 11.1. The columns are split short exact, with \( \varepsilon \varepsilon = 1, \beta \beta = 1 \) whenever defined, corresponding to direct sums

\[
W_n(A) = W_n(A) \oplus V_{n-1}(A) \\
V_n(A) = V_n(A) \oplus U_{n-1}(A).
\]
§ 2. Proof of Theorem 1.1 (n odd)

Given $A_2$-modules $P, Q$ and $\Theta \in \text{Hom}_{A_2}(P, Q^*)$

define $[\Theta]_0 \in \text{Hom}_{A}(P, \text{Hom}_{A}(Q, A))$ by

$[\Theta]_0(x)(y) = \Theta(x(y)) \in A \quad (x \in P, y \in Q)$

where $[a]_0 = a_0 \cdot A$ if $a = \sum_{j=-\infty}^{\infty} a_j z^j \in A_2$.

Given $A$-modules $P, Q$ and $\Theta = \sum_{j=-\infty}^{\infty} z^j \Theta_j \in \text{Hom}_{A}(P_2, Q_{2^*})$
(with $\Theta_j \in \text{Hom}_{A}(P, Q^*)$), $[\Theta]_0 \in \text{Hom}_{A}(P_2, Q_{2^*})$ is
given by

$[\Theta]_0(z^jx)(z^k y) = \Theta_{k-j}(x(y)) \in A \quad (x \in P, y \in Q, j, k \in \mathbb{Z})$

and

$[\Theta]_0(x)(y) = \sum_{j=-\infty}^{\infty} z^j ([\Theta]_0(z^j x))(y) \in A_2 \quad (x \in P, y \in Q)$.

Define

$B : \mathcal{V}_{2i+1}(A_2) \longrightarrow U_{2i}(A)$;

$(\Theta, \Xi, \mathcal{F}, \mathcal{G}) \mapsto (B_0^*(F \circ F^*), \mathcal{G} \circ \mathcal{G}^*), [\Xi]_0)$

where $\mathcal{F}$ and $\mathcal{G}$ are free, with modular $A$-bases
$F, G$ respectively and $N > 0$ so large that

$z^N(F \circ F^*) \subset \mathcal{G} \circ \mathcal{G}^*$

for some choice of Hamiltonian complements $F^*, G^*$ to $F, G$ in $(\Xi, \mathcal{G})$ with dual modular
$A$-bases $F^*, G^*$. Now

$[B_0^*(F \circ F^*, \mathcal{G} \circ \mathcal{G}^*)]_{\Xi} = \Xi \otimes A$

so that the hypotheses of Lemma 2.1 are satisfied.

Lemma 2.1 Let $(\Xi, \mathcal{G})$ be a non-singular $\pm$-form

over $A_2$, and let $C, D$ be complementary

$A$-submodules of $Q$ such that $C$ is finitely generated and

$[\langle C, D \rangle_{\Xi}]_0 = \Xi \otimes A$.

Then $(C, e^* [\Pi]_0, C)$ is a non-singular $\pm$-form over $A$, 
where $i : C \longrightarrow Q$ is the inclusion.

In short, $(C, e^* [\Pi]_0, C)$ will be denoted by $(C, e^* [\Pi], C)$

So does the choice of $F^*$ matter: for $N > 0$

so large that

$z^N F^* \subset \mathcal{G} \circ \mathcal{G}^*$
define the $A$-module
\[ E^+_N(F, G; G; G^*) = \sum x \in (G; G; G^*)^+ \mid [z_{NF}, x]\}_{A_0} = \emptyset \] and observe that the $\pm$-form defined over $A$ by
\[ (E^+_N(F, G; G; G^*)/z_{NF}^+, [\oplus]) \]
coincides with $(B^+_N(F; F; G; G^*), [\oplus])$ when $N$ is so large that $z_N(F; F; G; G^*)^+ \subseteq (G; G; G^*)^+$, as then
\[ E^+_N(F, G; G^*) = (F; z_{NF}^*) \cap (G; G; G^*)^+ 
= z_{NF}^+ \oplus B^+_N(F; F; G; G^*)^+.

The choice of $F^*$ did not enter in this new definition. The choice of $G^*$ may be dealt with similarly.

Next, suppose $(Q; \{A, F, G\} = 0 \in V_{2i+1}(A)$, and consider the generic cases:

1) $F$ and $G$ are hamiltonian complements in $(Q; \{A, F, G\})$.
Set $F^* = G$, $G^* = F$, $N = 0$ to obtain $B^+_N(F; F; G; G^*) = 0$, and so $B(Q; \{A, F, G\} = 0 \in U_{2i}(A)$.

2) $F$ and $G$ share a hamiltonian complement in $(Q; \{A, F, G\})$.
Set $F^* = G^*$ to obtain
\[ B(Q, \{A, F, G\} = B(Q, \{A, F, G^*\}^*) \text{ (by symmetry of definition) } = 0 \in U_{2i}(A) \text{ (taking } N = 0)\]

It now only remains to verify that the choice of modular $A$-bases $F, G$ of $F, G$ is immaterial to $B(Q; \{A, F, G\} \in U_{2i}(A)$.

Let $\hat{F}$ be another modular $A$-base of $F$, with dual modular $A$-base $\hat{F}^*$ of $F^*$, and let $\hat{N} \geq 0$ be so large that
\[ z_N^*(\hat{F}; \hat{F}^*) \subseteq (F; F; G; G^*)^+.

Then
\[ (B^+_N(\hat{F}; \hat{F}^*, G; G^*)^+ , [\oplus]) = (z_N^*(\hat{F}; \hat{F}^*; F; F), [\oplus]) \oplus (B^+_N(F; F; G; G^*), [\oplus]) = H_+(z_N^*(\hat{F}; \hat{F}^*; F; F)) \oplus (B^+_N(F; F; G; G^*), [\oplus]) \]
\[ = (B^+_N(F; F; G; G^*), [\oplus]) \subseteq U_{2i}(A),
\]
so that $\hat{F}$ will do as well as $F$.

Similarly for choice of $G$.

Hence
\[ B : V_{2i+1}(A) \longrightarrow U_{2i}(A) \]

is well-defined.

The composite
\[ V_{2i+1}(A) \xrightarrow{\bar{c}} V_{2i+1}(A) \xrightarrow{B} U_{2i}(A) \]
is 0 because
\[ B \in (Q; \{A, F, G\}) \in \bar{B}(Q; \{A, F, G\}) \]

is 0 because
\[ B \in (Q; \{A, F, G\}) = \bar{B}(Q; \{A, F, G\}) = (B^+_0(F; F; G; G^*; A) = 0 \text{ in } U_{2i}(A). \]
The diagram

\[ V_{2i+1}(A_2) \longrightarrow \Omega_-(A_2) \]

\[ \text{commutes, because given } (Q, \phi, \mathcal{F}, \mathcal{G}) \in V_{2i+1}(A_2), \]

with

\[ \pi^-(Q, \mathcal{F}, \mathcal{G}) = (Q, \phi, \mathcal{G} \circ (Q, \phi) \to (Q, \phi)) \in U_+(A_2)/\text{oc}_\pm(A_2) \]

such that \( \alpha(\mathcal{G}) = \mathcal{G} \) (in the notation of Theorem 4.2 of I)

Then

\[ B(\pi(Q)) = [B^+_N(F \otimes F^*, \alpha(F \otimes F^*))] \]

\[ = [B^+_N(F \otimes F^*, G \otimes G^*)] \in \Sigma_+(A), \]

for any modular base \( F \otimes F^* \), with \( G = \alpha(F) \).

Define

\[ \overline{B} : U_{2i}(A) \longrightarrow V_{2i+1}(A_2); \]

\[ (Q, \phi) \longmapsto (Q_2 \otimes Q_z, \phi \otimes \phi) \oplus H_\pm(-Q_2); \]

\[ \Delta(Q_2, \phi) \oplus -Q_z, \xi \Delta(Q_2, \phi) \oplus -Q_z) \]

(\text{where } -Q \text{ is any f.g. projective } A-\text{module such that } Q \oplus -Q \text{ is free}) \]

with

\[ \xi = \begin{pmatrix} 1 & 0 \\ 0 & \Xi \end{pmatrix} : Q_2 \otimes Q_z \longrightarrow Q_2 \otimes Q_z. \]

This is well-defined because

\[ \xi((\alpha, Q, \phi, Q_y)) \in (F \otimes F^*, \alpha(F \otimes F^*)) \oplus (-Q_2, \xi \Delta(Q_2, \phi) \oplus -Q_z) \]

\[ x \in P_2, \ g \in P^*, \ y \in (P \otimes P^*) \cdot \Xi \]

is a hamiltonian complement in \( H_\pm(A_2) \oplus -P \otimes P^* \),

to both \( \Delta H_\pm(A_2) \oplus -P \otimes P^* \) and \( \xi \Delta(A_2) \oplus -P \otimes P^* \),

so that

\[ \overline{B}(H_\pm(P)) = 0 \in V_{2i+1}(A_2). \]

For any f.g. projective \( A-\)module \( P \).

The composite

\[ U_{2i}(A) \mathrel{\mathclap{\overset{\overline{B}}{\longrightarrow}}} V_{2i+1}(A_2) \mathrel{\mathclap{\overset{\xi}{\longrightarrow}}} V_{2i+1}(A) \]

is 0 because

\[ \xi \overline{B}(Q_2) = ((Q_2, \phi \otimes Q_z) \oplus H_\pm(-Q_2), \Delta Q_2, \phi \otimes -Q_z, \Delta(A_2) \oplus -Q_z) \]

\[ = 0 \in V_{2i+1}(A). \]

The diagram

\[ U_{2i}(A) \longrightarrow \Sigma_+(A) \]

\[ \text{commutes, because given } (Q, \phi) \in U_{2i}(A) \]

\[ \tau(\pi^+(Q, \phi)) = \tau(\xi Q_2 \phi \otimes Q_z \oplus (-Q_z, \phi \otimes Q_z)) \longrightarrow \xi \Delta(A_2) \oplus -Q_z, \xi \Delta(A_2) \oplus -Q_z) \]

\[ = B((Q, \phi)) \in \Omega_-(A_2). \]
The composite

\[ U_{2i}(A) \xrightarrow{\varphi} V_{2i+1}(A) \xrightarrow{\text{B}} U_{2i}(A) \]

is the identity, because for each \((Q, \Phi) \in U_{2i}(A)\)

\[ \text{B}(Q, \Phi) = (Q \otimes Q, \Phi \otimes \Phi \otimes H_{\pm}(Q), \Delta \otimes -Q, \epsilon \otimes \Phi \otimes H_{\pm}(Q)) \]

\[ = (Q \otimes Q, Q \otimes Q, \Phi \otimes \Phi \otimes H_{\pm}(Q)) \]

where \(\Delta^*(Q, \Phi)\) is any hamiltonian complement to \(\Delta(Q, \Phi)\) in \((Q \otimes Q, \Phi \otimes \Phi)\), as given by Lemma 1.4 of I.

It now only remains to verify the exactness of the sequence

\[ V_{2i+1}(A) \xrightarrow{\varphi} V_{2i+1}(A) \xrightarrow{\text{B}} U_{2i}(A) \]

This will be done by characterizing the \(\pm\)-formations over \(A_{2}\) which are isomorphic to the ones obtained from \(\pm\)-formations over \(A\) via \(\varphi: A \to A_{2}\) (in Lemma 2.2, below), and then using the hamiltonian transformation of Lemma 2.3 to show that every element of \(\ker(\text{B} : V_{2i+1}(A) \to U_{2i}(A))\) has a representative satisfying the criterion.

**Lemma 2.2** A \(\pm\)-formation \((Q, \Phi; F, G)\) over \(A_{2}\) is isomorphic to \(\Xi(Q, \Phi; F, G)\) for some \(\pm\)-formation \((Q, \Phi; F, G)\) over \(A\) if \(\Xi\) has a modular \(A\)-base \(\mathcal{F}\) such that for some hamiltonian complement \(\Xi^*\) to \(\Xi\) in \((Q, \Phi; F, G)\) the positive projection on \(\mathcal{F}\)

\[ \nu: Q = F \oplus F^* \longrightarrow (\mathcal{F} \oplus \mathcal{F}^*)^+ \]

preserves \(G\), that is \(\nu(G) \subseteq G\).

**Proof:** It is clear that \(\Xi(Q, \Phi; F, G)\) satisfies the condition, for any \(\pm\)-formation \((Q, \Phi; F, G)\) over \(A\).

Conversely, assume the condition holds for \((Q, \Phi; F, G)\). The \(A\)-module morphism

\[ \epsilon = z(1-z) z' \nu: Q \longrightarrow Q \]

sends \(Q\) onto \(\mathcal{F} \oplus \mathcal{F}^*\), and has the property that

\[ \alpha = \sum_{j=1}^{\infty} z^j \epsilon z^j \alpha (\mathcal{F} \oplus \mathcal{F}^*) \subset Q \]

for every \(\alpha \in Q\).

Now \(\nu(G) \subseteq G\), so that

\[ \Xi(G) = G \cap (\mathcal{F} \oplus \mathcal{F}^*) \]

and is therefore a modular \(A\)-base \(\mathcal{F}\) of \(G\) contained in \(\mathcal{F} \oplus \mathcal{F}^*\). Thus, up to isomorphism of \(\pm\)-formations over \(A_{2}\),

\[ (Q, \Xi; F, G) = (H_{\pm}(F), F, G) = \Xi(H_{\pm}(F); F, G) \]

\[ \square \]
Lemma 2.3: Given a morphism of $\varphi$-forms over $A$

$$(f, \gamma) : (p, \theta) \rightarrow (q, \phi)$$

define the automorphism

$$H(f, \gamma) = \begin{pmatrix}
1 & f & 0 \\
0 & 1 & 0 \\
-f(\phi \circ \gamma) & 0 & 1
\end{pmatrix},$$

$$(q, \phi) \oplus H_\pm(p) \rightarrow (q, \phi) \oplus H_\pm(p).$$

If $(q, \phi)$ is non-singular, the automorphism $h' = H(f) \otimes (1, 0)$

$$(q, \phi') = (q, \phi \oplus H_\pm(p)) \otimes (q, \phi) \oplus H_\pm(-p \otimes -q)$$

is a Hamiltonian transformation, that is

$$(q', \phi', L', h(L')) = 0 \in V_{2i+1}(A)$$

for any free Lagrangian $L$ of $(q', \phi')$.

Proof: The automorphism $h' : (q, \phi) \rightarrow (q', \phi')$ preserves the
free Lagrangian

$$L = \sum_{x \in Q} \sum_{y \in Q} (p \otimes p \otimes p) \otimes x \otimes y \otimes \varphi \otimes \gamma \otimes (p \otimes -p \otimes -q)$$

so that it is necessarily the composite

$$(q, \otimes) \oplus (1, 0) \otimes (q, \varphi \otimes \gamma) : L \otimes L^* \rightarrow L \otimes L^*$$

can be expressed as

$$f^{*} \otimes f \in \text{Hom}_{A_\mathbb{Z}}(P, Ap)$$

with

$$[\sum_{j=1}^{\infty} \sum_{j=1}^{\infty} z_j p_j, \sum_{j=1}^{\infty} z_j p_j] \in \text{Hom}_{A_\mathbb{Z}}(P, Ap).$$

Choose Hamiltonian complements $L, L^*$ in $(P, \mathbb{Z}_2)$.

Denote $H_\pm(L_\mathbb{Z})$ by $(P, \mathbb{Z}_2)$, so that

$$[\sum_{j=1}^{\infty} \sum_{j=1}^{\infty} z_j p_j, \sum_{j=1}^{\infty} z_j p_j] : P \rightarrow P^*$$

for some $\varphi$-form $(P, \gamma)$ over $A_\mathbb{Z}$. 

Consider now the automorphism

\[
(h, \rho) = \begin{pmatrix}
(1 & 0 & 0) & (0 & 0 & 1) \\
(0 & 1 & 0) & (0 & 0 & \xi) \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\begin{pmatrix}
1 & -\xi \\
0 & 1
\end{pmatrix} & \begin{pmatrix}
0 & \xi \\
0 & 0
\end{pmatrix} \\
(\xi^* \oplus \xi) & (\xi^* \oplus \xi)
\end{pmatrix}
\]

\[\varphi : (\mathbb{R}, \mathbb{Z}) \oplus H_\pm(P) \longrightarrow (\mathbb{R}, \mathbb{Z}) \oplus H_\pm(P)\]

where

\[\eta = \begin{pmatrix}
0 & -z \\
z & 0
\end{pmatrix} : P^* = L_2 \oplus L_z \longrightarrow L_2 \oplus L_z = P,\]

\[\xi = \begin{pmatrix}
1 & 0 \\
0 & z
\end{pmatrix} : \mathfrak{P} = L_2 \oplus L_z^* \longrightarrow L_2 \oplus L_z^* = \mathfrak{P}\]

and

\[\varpi = [\overline{\mathbb{R}}]_\pm \pm [\overline{\mathbb{R}}]_\pm^* + \overline{\mathfrak{P}} \in \text{Hom}_{\mathbb{A}_\mathbb{R}}(\mathfrak{P}, \mathfrak{P}^*).\]

Defining the positive projection

\[\nu : Q \oplus (P \oplus P^*) \longrightarrow ((\mathfrak{P} \oplus \mathfrak{P}^*)^* \oplus \mathfrak{P} \oplus \mathfrak{P}^*)^+\]

and the \(A\)-module projection

\[\beta : Q = P \oplus (\mathfrak{Z} \oplus \mathfrak{Z}^*) \longrightarrow P,\]

note that

\[\nu h(x, y) = \begin{cases} h(x, y) & x \in \mathbb{R}^+ \setminus \{0\}, yeP \setminus \{0\} \\
h(0, \xi^* \beta(fS(y) - x)) & x \in \mathbb{R}^-, yeP \setminus \{0\} \end{cases}\]

where

\[\nu h (\mathfrak{S} \oplus \mathfrak{P}) \subseteq h (\mathfrak{S} \oplus \mathfrak{P}).\]

The product decomposition used to define \((h, \rho)\) shows that the automorphism \((h', \rho') = (h, \rho) \oplus (1, 0)\) of \((\mathbb{R}, \mathbb{Z}) = (\mathbb{R}, \mathbb{Z}) \oplus H_\pm(P) \oplus H_\pm(P)\)

is a hamiltonian transformation over \(A_\mathbb{R}\) : the matrix involving \(\eta\) is an elementary hamiltonian transformation, while the other corresponds to the hamiltonian transformation generated by the morphism of \(\pm\) -forms over \(A_\mathbb{R}\)

\[(fS, \xi^* (\overline{\mathbb{R}}_\pm + \mathfrak{P} \overline{\mathbb{R}}_\pm)) : (P, \xi^* \overline{\mathfrak{P}}) \longrightarrow (\mathbb{R}, \mathbb{Z})\]

in the sense of Lemma 2.3.

The lagrangians \(\mathcal{F} = A \mathfrak{S} \oplus -\mathcal{P}, \mathcal{Q} = A \mathfrak{P} \oplus -\mathcal{P}\)

of \((\mathbb{R}, \mathbb{Z})\) are such that

\[(\mathbb{R}, \mathbb{Z}, \mathcal{F}, \mathcal{Q}) = (\mathbb{R}, \mathbb{Z}, \mathcal{F}, \mathcal{Q}) = (A, \mathbb{Z}', \mathcal{F}', \mathcal{Q}') \in V_{\mathbb{A}_\mathbb{R}}(\mathbb{R}).\]
II.2.13

Using the V-theory sum formula of Lemma 3.3

of \( \Sigma \). The last representative formation satisfies the hypothesis of Lemma 2.2 with

the roles played by \( F \) and \( G \) reversed - this is clearly all right for non-singular formations.

Thus

\[ (Q, \Xi; F, G) \in \text{im}(\Xi : V_{2i+1}(A) \to V_{2i}(A)) \]

completing the proof of the part of Theorem 1.1 relating to \( V_n(A) \) with \( n \) odd.

We now give the analogous constructions for W-theory.

Define

\[ B : W_{2i+1}(A) \to V_{2i}(A) ; \]

\[ (Q, \Xi; F, G) \mapsto (B_n(F, G, \Xi), [\Xi]) \]

where \( F \) is the modular \( A \)-base of \( \Xi \) generated by the given \( A_z \)-base, and similarly for \( G, G \).

Then

\[ [B_n(F, G, \Xi, \Xi)] = 0 \in K_n(A) \]

as required for V-theory, because it is the unique under \( B : \bar{K}_1(A) \to K_0(A) \) of the automorphism

of \( Q \), taking \( P_0^{F \Xi} \) to \( G \in G^* \), which is simple by construction (cf. \( \Sigma \) of I.), so that

\[ B : W_{2i+1}(A) \to V_{2i}(A) \]

is so, as for V-theory,

The square

\[ \begin{array}{ccc}
  \Sigma_{2i}(A) & \longrightarrow & W_{2i+1}(A) \\
  B \downarrow & & \downarrow B \\
  \Sigma_-(A) & \longrightarrow & V_{2i}(A)
\end{array} \]

commutes, sending \( \tau(\alpha : F \to F) \in \Sigma_{2i}(A) \) to

\[ H_+(B_n(F, \alpha(F))) \in V_{2i}(A) \]

both ways.

Define

\[ B : V_{2i}(A) \to W_{2i+1}(A) ; \]

\[ (Q, \phi) \mapsto (Q \otimes \phi, \phi, \phi - \phi) \Delta \Delta (\phi \otimes \phi), \xi \Delta (\phi \otimes \phi) \]

where

\[ \xi = (\xi \phi \phi) : Q \otimes \phi \to Q \otimes \phi \]

with \( Q \) any base of \( Q \) (assumed free), and

\( \phi \in \mathbb{C} \) any hamiltonian base extending \( \phi \).
Then \( B(q, p) \) is just
\[
\pi'(\xi, \varphi) : (a \otimes G \otimes \xi \otimes \varphi) \to (a \otimes G \otimes \xi \otimes \varphi),
\]
in the terminology of Theorem 5.6 of I., as
\[
\tau(\xi, \varphi) = B(\xi, [\varphi]) = 0 \in K_1(A_z),
\]
so that we are dealing with an element of the special unitary group \( SU_1(A_z) \).

The composites
\[
\begin{align*}
V_{2i}(A) \xrightarrow{\overline{B}} W_{2i+1}(A) & \xrightarrow{\varepsilon} W_{2i+2}(A) \\
V_{2i}(A) \xrightarrow{\overline{B}} W_{2i+1}(A) & \xrightarrow{B} V_{2i}(A)
\end{align*}
\]
are \( 0, 1 \) as before.

The square
\[
\begin{array}{ccc}
\Sigma_+(A) & \xrightarrow{\overline{B}} & V_{2i}(A) \\
\downarrow & & \downarrow \\
\Omega_+(A_z) & \xrightarrow{\varepsilon} & W_{2i+1}(A)
\end{array}
\]
commutes, sending \([p] \in \Sigma_+(A)\) to
\[
((Q \otimes G)_z, \xi \otimes \varphi, p \otimes \phi)_z, \xi(p \otimes \phi)_z \in W_{2i+1}(A_z)
\]
both ways round, where \((Q, \xi) = H_+(\xi)\), and
\((p \otimes \phi)\) is any base.

The proof that
\[
0 \to W_{2i+1}(A) \xrightarrow{\varepsilon} W_{2i+1}(A_z) \xrightarrow{B} V_{2i}(A) \to 0
\]
is a split short exact sequence, is as for \( V \)-theory, using the following versions of Lemmas 2.2, 2.3:

Lemma 2.4. Given \((Q, \xi, \varphi, \pi) \in \ker(\mathbb{B} : W_{2i+1}(A) \to W_{2i}(A))\) and a positive projection
\[
\mathcal{V} : Q = G \otimes G^* \to (G \otimes G^*)^+
\]
such that \( \mathcal{V}(\varphi) \subseteq \mathcal{F} \), for some modular \( A \)-base \( G \) of \( G \) and Hamiltonian complement \( G^* \), then
\[
(Q, \xi, \varphi, \pi) \in \text{im } (\bar{\varepsilon} : W_{2i+1}(A) \to W_{2i+1}(A_z))
\]
Proof. As in the proof of Lemma 2.2
\[
F' = F \cap (G \otimes G^*) = Z(1-z)z^{-1} \mathcal{V}(\varphi)
\]
is a modular \( A \)-base of \( F \). It is by no means clear, however, that we can choose an \( A \)-base \( \mathcal{F}' \) such that
\[
(Q, \xi, \varphi, \pi, \mathcal{F}') = 0 \in W_{2i+1}(A_z),
\]
but if such an \( A \)-base exists it is immediate from the \( W \)-theory analogue of the sum formula of Lemma 2.3 of I., that
A judicious choice of $\mathbf{F}'$ ensures that $\varepsilon \mathbf{c}' = a = 0 \in \mathcal{K}_1(A_2)$ at least (allowing stabilization, if necessary).

Let $\mathbf{F}^*$ be a hamiltonian complement to $\mathbf{F}$ in $(Q, \Phi)$, with $\mathbf{F}^*, \mathbf{F}'$ the $A_2$-bases of $\mathbf{F}^*$ dual to $\mathbf{F}, \mathbf{F}'$ respectively. Now $\mathbf{F}' \in \mathcal{G} \otimes G^*$, so

$$\langle z(1 - \omega) \mathbf{z}^* \mathbf{f}_j', \mathbf{f}_k' \rangle_{\mathcal{G}^*} = \langle \mathbf{f}_j', \mathbf{f}_k' \rangle_{\mathcal{G}^*} = \sum_{j} \left\{ \begin{array} {c} 1 \text{ if } j = k \text{ or } \omega = 0 \text{ otherwise} \end{array} \right\}$$

Thus $z(1 - \omega) \mathbf{z} \mathbf{F}^*$ is the $A_2$-base of a direct complement to $\mathbf{F}'$ in $\mathcal{G} \otimes G^*$ (not a lagrangian, in general), and $\mathbf{F}'$ is a lagrangian of $(\mathcal{G} \otimes G^*, [\mathcal{M}])$.

Let $\mathbf{F}^{**}$ be the $A_2$-base dual to $\mathbf{F}'$ of some hamiltonian complement to $\mathbf{F}'$ in $(\mathcal{G} \otimes G^*, [\mathcal{M}])$. Now $\mathbf{F} \otimes \mathbf{F}^*$ and $\mathbf{F} \otimes \mathbf{F}^{**}$ are both hamiltonian $A_2$-bases of $(Q, \Phi)$ extending $\mathbf{F}'$, so that

$$(Q, \mathbf{F}' \otimes \mathbf{F}, \mathbf{F} \otimes \mathbf{F}^{**}) = 0 \in \mathcal{K}_1(A_2)$$
Let $g$ be the given $A_2$-base of $G$, with $g^*$ the dual $A_2$-base of $G^*$, the given hamiltonian complement to $g$ in $(\mathbb{Q}, \Phi)$. Then

$$(Q, f \otimes f^*, g \otimes g^*) = 0 \in \bar{K}_1(A_2)$$

(by construction of $\mathbb{Z}_{2i+1}(A_2)$ - c.f. §5 of I.)

Hence

$$\tau' - \tau'^* = (Q, f \otimes f^*, f^* \otimes f^*)$$

$$= (Q, g \otimes g^*, f^* \otimes f^*)$$

$$\in \ker (\varepsilon : \bar{K}_1(A_2) \to \bar{K}_0(A_2)) \cap \ker (\xi : \bar{K}_1(A_2) \to \bar{K}_0(A_2))$$

$$= \xi \circ \delta$$

and

$$\tau'^* = \tau' \in \bar{K}_1(A_2).$$

We can now express $\tau'$ as

$$\tau' = \overline{B}(b) \otimes c^t \otimes (c^t)^* \in \overline{B} \bar{K}_0(A_2) \otimes \text{Nil}^*(A) \otimes \text{Nil}(A)$$

where $b^* = -b \in \bar{K}_0(A)$ (using all the information given by the corollary quoted in §1).

Calculating directly,

$$b = B \tau' = [B_0^*(F,F^*), [z_0^*(F,F^*)] = [z_0^*(F,F^*), \Phi]_0 \in \bar{K}_0(A)$$

Note that $\omega(z_0^*(F,F^*))$ is a lagrangian of $B_0^*(F,F^*,G \otimes G^*)$ which vanishes in $V_0(A)$ (by hypothesis) and so may be assumed to have a free lagrangian, $L$ say.

Finally, consider the diagram

$$U_{2i+1}(A) \to \Sigma_-(A) \to V_{2i}(A)$$

$$\overline{\varepsilon} \downarrow \quad \quad \downarrow \overline{\varepsilon}$$

$$\Omega_+(A_2) \to W_{2i+1}(A_2)$$

in which the square commutes, and the $(\xi \circ \delta)$ is exact (by Theorem 4.3 of I), so that the composite $U_{2i+1}(A) \to W_{2i+1}(A_2)$ is 0.

Now $U_{2i+1}(A) \to \Sigma_-(A)$ sends

$$(B_0^*(F,F^*,G \otimes G^*), [\Phi]_0 ; L, \omega(z_0^*(F,F^*))) \in U_{2i+1}(A)$$

to

$$[\omega(z_0^*(F,F^*))] = b \in \Sigma_-(A),$$

and $\overline{B}b = \tau' \in \Omega_+(A_2)$ is sent by $\Omega_+(A_2) \to W_{2i+1}(A_2)$

to

$$(\overline{\varepsilon}, \overline{\Phi}; \overline{\Phi}, \overline{F}) = 0 \in W_{2i+1}(A_2)$$

as required.
Lemma 2.5 Let 
\[(f, \lambda) : (P, \Theta) \rightarrow (Q, \Phi)\]
be a morphism of based ±forms over A, with
\((Q, \Phi)\) non-singular and such that \(\tau(Q, \Phi) = 0 \in K_1(A)\).

Then the automorphism
\[H(f) \Theta (1, 0) : ((Q, \Phi) \Theta H_+(P)) \Theta (Q, \Phi) \rightarrow ((Q, \Phi) \Theta H_+(P)) \Theta (Q, \Phi)\]
is a special hamiltonian transformation, where \(H(f)\) is as in Lemma 2.3.

\[\square\]

This completes the proof of Theorem 1.1 for \(n\) odd.

§3. Proof of Theorem 1.1 (n even)

We define \(B : V_{2i}(A) \rightarrow U_{2i-1}(A)\) using

Lemma 3.1 Given a non-singular ±form \((Q, \Theta)\) over \(A_\infty\), with \(Q\) free, and a modular \(A\)-base \(Q\) of \(Q\), let
\[\nu : Q \Theta Q^* \rightarrow (Q \Theta Q^*)^+\]
be the positive projection, and let \(N > 0\) be so large that
\[\sum_{j=-N}^{N} z_i Q^* \leq N, (\Theta \pm \Theta^*)^+(Q^*) \leq \sum_{j=-N}^{N} z_i Q.\]

Then the \(A\)-submodule
\[B_N(Q, \Theta) = \{\left(\sum_{j=-N}^{N} z_i Q^* x, \nu(\Theta \pm \Theta^*) x\right) \in Q \Theta Q^* | x \in B_N(\Theta \pm \Theta^*)^+(Q^*), \Theta^*\}\]
of \(Q \Theta Q^*\) is a lagrangian of \(H_+(\sum_{j=0}^{N-1} z_i Q)\) such that
\[(H_+(\sum_{j=0}^{N-1} z_i Q), \sum_{j=0}^{N-1} z_i Q, B_N(Q, \Theta)) \in U_{2i-1}(A)\]
does not depend on the choices made of \(N\) and \(Q\).
Proof: The restriction of
\[(0 \ 1) \in \text{Hom}_\mathbb{A}(\sum_{j=0}^{N-1} z^j Q, \sum_{j=0}^{N-1} z^j Q^\ast), (\sum_{j=0}^{N-1} z^j Q) \oplus (\sum_{j=0}^{N-1} z^j Q)\]
to \(B_N(Q, \overline{\omega})\) is given by
\[B_N(Q, \overline{\omega}) \rightarrow B_N(Q, \overline{\omega})^\ast,\]
\[(z^{N(1-\overline{\omega})} z^N x, \nu(\overline{\omega} \pm 1\overline{\omega}) y) \mapsto \langle x, y \rangle_{[\overline{1}], 0},\]
which is of the type \(\lambda \pm \chi \in \text{Hom}_\mathbb{A}(B_N(Q, \overline{\omega}), B_N(Q, \overline{\omega})^\ast)\), so that we are dealing with a sublagrangian of \(H_\tau(\sum_{j=0}^{N-1} z^j Q)\). In fact, it is a lagrangian, a hamiltonian complement being defined by
\[B_N^\ast(Q, \overline{\omega}) = \{ (-\overline{\omega} y, \nu(\overline{\omega} \pm 1\overline{\omega}) (1-\overline{\omega}) y) \in \mathbb{C} \oplus \mathbb{C}^\ast \mid y \in B_N(Q, \overline{\omega})^\ast \} \].

Every \((s, t) \in (\sum_{j=0}^{N-1} z^j Q) \oplus (\sum_{j=0}^{N-1} z^j Q^\ast)\) can be expressed as
\[(s, t) = (z^{N(1-\overline{\omega})} z^N x, \nu(\overline{\omega} \pm 1\overline{\omega}) x) + (-\overline{\omega} y, \nu(\overline{\omega} \pm 1\overline{\omega})(1-\overline{\omega}) y) \in B_N(Q, \overline{\omega}) + B_N^\ast(Q, \overline{\omega})\]
with
\[x = \nu(\overline{\omega} \pm 1\overline{\omega})^{-1} (1-\overline{\omega}) (\overline{\omega} \pm 1\overline{\omega}) s + t \in B_N^+(\overline{\omega} \pm 1\overline{\omega})^\ast Q^\ast, Q,\]
\[y = (1-\overline{\omega}) (\overline{\omega} \pm 1\overline{\omega}) s + z^{N(1-\overline{\omega})} z^N (\overline{\omega} \pm 1\overline{\omega})^{-1} t \in B_N^+(Q, (\overline{\omega} \pm 1\overline{\omega})^{-1} \overline{\omega}).\]

and the associated pairing of \(H_\tau(\sum_{j=0}^{N-1} z^j Q),\)
\[(0 \ 1) \in \text{Hom}_\mathbb{A}(\sum_{j=0}^{N-1} z^j Q \oplus (\sum_{j=0}^{N-1} z^j Q) \oplus (\sum_{j=0}^{N-1} z^j Q^\ast), (\sum_{j=0}^{N-1} z^j Q) \oplus (\sum_{j=0}^{N-1} z^j Q) \oplus (\sum_{j=0}^{N-1} z^j Q^\ast))\]
restricts to the \(A\)-module isomorphism
\[B_N^\ast(Q, \overline{\omega}) \rightarrow B_N(Q, \overline{\omega})^\ast,\]
\[(-\nu(\overline{\omega} \pm 1\overline{\omega}) (1-\overline{\omega}) y) \mapsto \langle y, x \rangle_{[\overline{1}], 0},\]
so that we are dealing with hamiltonian complements.

Increasing \(N\) by 1 we have
\[B_{N+1}(Q, \overline{\omega}) = B_N(Q, \overline{\omega}) \oplus \xi((\overline{\omega} \pm 1\overline{\omega}) \nu(\overline{\omega} \pm 1\overline{\omega})(1-\overline{\omega}) y) \in B_N(Q, \overline{\omega}) \oplus (\sum_{j=0}^{N+1} z^j Q^\ast)\]
\[\mathbb{C} \oplus (\sum_{j=0}^{N+1} z^j Q), \nu(\overline{\omega} \pm 1\overline{\omega}) (\overline{\omega} \pm 1\overline{\omega})^{-1}(\overline{\omega} \pm 1\overline{\omega} \nu(\overline{\omega} \pm 1\overline{\omega})).\]

Now \(B_N^\ast(Q, \overline{\omega}) \oplus z^{Nq} Q\) is a hamiltonian complement in \(H_\tau(\sum_{j=0}^{N-1} z^j Q)\) to both \(B_{N+1}(Q, \overline{\omega})\) and \(B_N(Q, \overline{\omega}) \oplus z^{Nq} Q\) so that
\[(H_\tau(\sum_{j=0}^{N-1} z^j Q), \sum_{j=0}^{N-1} z^j Q, B_N(Q, \overline{\omega}))\]
\[= (H_\tau(\sum_{j=0}^{N-1} z^j Q), \sum_{j=0}^{N-1} z^j Q, B_N^\ast(Q, \overline{\omega}) \oplus z^{Nq} Q^\ast)\]
\[= (H_\tau(\sum_{j=0}^{N-1} z^j Q), \sum_{j=0}^{N-1} z^j Q, B_{N+1}(Q, \overline{\omega})) \in \mathcal{L}(H_\tau(A)).\]

Hence choice of \(N\) immaterial.
II 3.4

Let \( \hat{Q} \) be another modular \( A \)-base of \( Q \), with

\[ \hat{Q} : Q \oplus Q^* \to (\hat{Q} \oplus \hat{Q}^*)^+ \]

de the new positive projection. Let \( M > 0 \) be so large that

\[ \hat{Q} \subseteq \sum_{j=-M}^{M} z^{j} Q, \quad Q \subseteq \sum_{j=-M}^{M} z^{j} \hat{Q} . \]

Then \( \hat{N} = N + 2M \) is large enough for \( B_N(\hat{Q}, \Xi) \) to be defined, and

\[ B^+_N(\hat{Q}, \hat{Q}^*) = (\Xi \oplus \Xi^*) (z^{MN} B^+_N(Q, Q^*)) \oplus z^M B^+_N(\hat{Q}^*, Q) \]

so that

\[ B_N(\hat{Q}, \Xi) = \left( z^{N+2M} x \right) \Xi \oplus z^{2M} B^+_N(\Xi \oplus \Xi^*, Q) \]

and

\[ z^{MN} B^+_N(\hat{Q}, Q) \oplus z^M B^+_N(Q, \Xi) \oplus B_N(\hat{Q}, \hat{Q}^*) \]

is a Hamiltonian complement in \( H_{\Xi}(\sum_{j=0}^{N} z^{j} \hat{Q}) \)

to both \( B_N(\hat{Q}, \Xi) \) and \( z^{MN} B^+_N(\hat{Q}^*, Q) \oplus z^M B_N(\Xi \oplus \Xi^*, Q) \).

Thus

\[ (H_{\Xi}(\sum_{j=0}^{N} z^{j} \hat{Q}), \sum_{j=0}^{N} z^{j} \hat{Q}, B_N(\hat{Q}, \Xi)) \]

\[ = (H_{\Xi}(\sum_{j=0}^{N-M} z^{j} \hat{Q}), \sum_{j=0}^{N-M} z^{j} \hat{Q}, z^{MN} B^+_N(\hat{Q}^*, Q) \oplus z^M B_N(\Xi \oplus \Xi^*, Q)) \]

\[ = (H_{\Xi}(\sum_{j=0}^{N-M} z^{j} \hat{Q}), \sum_{j=0}^{N-M} z^{j} \hat{Q}, B_N(\hat{Q}, \Xi)) \in U_{2i-1}(A). \]

Hence independence of choice of \( Q \).

\[ \square \]

Define

\[ B : V_{2i}(A_2) \to U_{2i-1}(A), \quad (Q, \Xi) \mapsto (H_{\Xi}(\sum_{j=0}^{N-M} z^{j} Q), \sum_{j=0}^{N-M} z^{j} Q, B_N(\hat{Q}, \Xi)) \]

for any modular \( A \)-base \( Q \) of \( Q \) (which may be assumed free) and sufficiently large \( N > 0 \).

As shown in lemma 3.1 this does not depend on the choice of \( N \) and \( Q \). Given a \( \Xi \)-free

\[ A_2 \]-module \( F \), with modular \( A \)-base \( F \), we have

\[ B(H_{\Xi}(F)) = (H_{\Xi}(\emptyset), \emptyset, B_0(\emptyset \oplus \emptyset, (\emptyset, \emptyset))) \in U_{2i-1}(A) \]
Hence $B(Q, \Xi) = 0 \in U_2i-1(A)$ whenever $(Q, \Xi) = 0 \in V_2i(A_z)$, and $B : V_2i(A_z) \to U_2i-1(A)$ is well-defined.

The composite
$$V_2i(A) \xrightarrow{\overline{\varepsilon}} V_2i(A_z) \xrightarrow{B} U_2i-1(A)$$
is 0 because it sends $(Q, \phi) \in V_2i(A)$ to $B \overline{\varepsilon}(Q, \phi) = (H(Q), 0, B_0(Q, \phi)) = 0 \in U_2i-1(A)$.

The square
$$\begin{array}{ccc}
V_2i(A_z) & \to & \Omega_2(A_z) \\
\downarrow & & \downarrow B \\
U_2i-1(A) & \to & \Sigma_{-}(A)
\end{array}$$

commutes; for given $(Q_z, \Xi) \in V_2i(A_z)$, with $Q$ free
$$[B_0(Q, \Xi)] = [B_0^+(Q^*, (\Xi \pm \Xi^*)Q)]$$
$$= B \varepsilon (Q_z, \Xi) \in \Sigma_{-}(A)$$
for any $A$-base $Q$.

II 3.6

We define $\overline{B} : U_2i-1(A) \to V_2i(A_z)$ using $\overline{B}$. Let $(Q, \phi)$ be a trivial $\pm$-form over $A$, with lagrangian $L$, and hamiltonian complement $L^*$, so that
$$\phi = \left( \begin{array}{cc}
\lambda \pm \lambda^* \\
\delta
\end{array} \right) : L \overset{\phi}{\to} L^* \Theta L$$
with $\gamma \equiv \delta^* = 1 : L^* \to L^*$.

Then the isomorphism class of the $\pm$-form over $A_z$ defined by
$$(Q_z, \Xi) = (L_z \Theta L^*_{-z}, \left( \begin{array}{c}
\lambda \equiv \gamma \\
\delta \\
\end{array} \right)(Q, \phi))$$
does not depend on the choice of $L^*$.

If $(Q, \phi) = H_u(p)$, then $(Q_z, \Xi)$ is a non-singular $\pm$-form over $A_z$ such that
$$(Q_z, \Xi) \Theta H_u(-L_z) \in V_2i(A_z)$$
is sent to 0 by $\overline{\varepsilon} : V_2i(A_z) \to V_2i(A)$, and has torsion
$$\overline{B}([L] - [P^*]) \in \Omega_2(A_z).$$
Moreover,
$$(Q_z, \Xi) \Theta H_u(-L_z) = 0 \in V_2i(A_z)$$
is a hamiltonian complement in $(Q, \phi)$ to either $P$.
Proof: Change of hamiltonian complement $L^*$ corresponds to an automorphism

$$\alpha = \begin{pmatrix} 1 & k \pm k^* \\ 0 & 1 \end{pmatrix} \in \text{Hom}_A(L \oplus L^*, L \oplus L^*)$$

for some $\pm$-form $(L^*, \alpha^*)$. The $\pm$-form over $A_z = (Q_z, \alpha^*)$ determined by this new choice of hamiltonian complement to $L$ is given by

$$\omega' = \begin{pmatrix} \lambda & -z' y' \\ 0 & 1 \end{pmatrix} \in \text{Hom}_A(L_z \otimes L_z^*, L^*_z \otimes L_z^*)$$

where $y', s', s_1'$ are such that

$$\alpha^* \phi \alpha = \begin{pmatrix} \lambda + k \pm k^* \\ 0 \\ 0 \end{pmatrix} \in \text{Hom}_A(L \oplus L^*, L \oplus L^*).$$

Now

$$\begin{pmatrix} 1 & (1-z)(k \pm k^*) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -(\lambda \pm \lambda^*) (k \pm k^*) \\ 0 & (1-z)(k^* \pm k) (\lambda \pm \lambda^*) \end{pmatrix}$$

defines an isomorphism of $\pm$-forms over $A_z$, so that choice of $L^*$ immaterial.

Defining $\omega \in \text{Hom}_{A_z}(Q_z, Q_z)$ by

$$\omega = \begin{pmatrix} 1 & 0 \\ 0 & 1-z \end{pmatrix} : L_z \oplus L_z^* \longrightarrow L_z \oplus L_z^*$$

and $\hat{\omega} \in \text{Hom}_{A_z}(Q_z, Q_z^*)$ by

$$\hat{\omega} = \begin{pmatrix} 1-z^* & 0 \\ 0 & 1 \end{pmatrix} : L_z^* \oplus L_z \longrightarrow L_z^* \oplus L_z$$

note that there is an identity

$$\hat{\omega} (\hat{\omega} \pm \hat{\omega}^*) = (\phi \pm z \phi^*) \omega \in \text{Hom}_{A_z}(Q_z, Q_z^*)$$

Similarly, defining

$$\hat{\phi} = (\phi \pm \phi^*)' \phi (\phi \pm \phi^*') = \begin{pmatrix} \mp (\mu \pm \mu^*) & \delta \\ \gamma & \mp (\lambda \pm \lambda^*) \end{pmatrix} \in \text{Hom}(L \oplus L^*, L \oplus L^*)$$

and

$$\hat{\phi} = \begin{pmatrix} \mp (1-z^*) (\mu \pm \mu^*) & \delta \\ -z' \gamma & \pm \lambda \end{pmatrix} \in \text{Hom}_{A_z}(L_z \oplus L_z^*, L_z \oplus L_z^*),$$

note that

$$\omega (\hat{\phi} \pm \hat{\phi}^*) = (\hat{\phi} \mp z \hat{\phi}^*) \omega : Q_z \longrightarrow Q_z^*.$$

If $(Q, \phi) = H_{=}(P)$, then

$$\phi \pm z' \phi^* = \begin{pmatrix} \phi & 0 \\ z' \phi^* & \phi \end{pmatrix} \in \text{Hom}_{A_z}(P_z \oplus P_z^*, P_z \oplus P_z^*),$$

and combining the two identities above, we obtain
\(\omega (\tilde{\omega} \pm \tilde{\omega}) (\tilde{\omega} \pm \tilde{\omega})\)
\[= (\tilde{\phi} \mp z \tilde{\phi}) \tilde{\omega} (\tilde{\omega} \pm \tilde{\omega}) = (\tilde{\phi} \mp z \tilde{\phi})(\tilde{\phi} \mp z \tilde{\phi}) \omega\]
\[= \omega \in \text{Hom}_{A_2}(Q_2, Q_2),\]
and similarly
\[\tilde{\omega} (\tilde{\omega} \pm \tilde{\omega}) (\tilde{\omega} \pm \tilde{\omega}) = \tilde{\omega} \in \text{Hom}_{A_2}(Q_2^*, Q_2^*)\]

Both \(\omega \in \text{Hom}_{A_2}(Q_2, Q_2)\) and \(\tilde{\omega} \in \text{Hom}_{A_2}(Q_2^*, Q_2^*)\)
can be monomorphisms, so
\[\tilde{\omega} \pm \tilde{\omega} = (\tilde{\omega} \pm \tilde{\omega})^{-1} \in \text{Hom}_{A_2}(Q_2, Q_2)\]
and \((\tilde{\omega} \pm \tilde{\omega})\) is a non-singular \(\pm\) form over \(A_2\).

The projection \(\varepsilon: V_{2i}(A_2) \rightarrow V_{2i}(A)\) onto \((Q_2, \Theta) \oplus H_{\pm}(-L_2))\)
and
\[(L \oplus L^\perp, (\begin{smallmatrix} 1 & -z \\ 0 & \delta \end{smallmatrix})) \oplus H_{\pm}(-L_2) \in V_{2i}(A)\]
which vanishes in \(V_{2i}(A)\) because \(L \oplus L^\perp\) is a
free lagrangian.

Thus the component of \(\omega (Q_2, \Theta) \oplus H_{\pm}(-L_2) \in \Omega_+(A_2)\)
in \(\tilde{\Omega}_+(A)\) is 0, and
\[\omega (Q_2, \Theta) \oplus H_{\pm}(-L_2) = B \mathcal{B} \omega (Q_2, \Theta) \oplus H_{\pm}(-L_2)\]
\[= B \left[ \mathcal{B}_i^* (\tilde{\omega} \pm \tilde{\omega}) (\tilde{\omega} \pm \tilde{\omega}) \right] \in \Omega_+(A_2)\]

Computing directly,
\[B(m(x, y), Q^*, Q)\]
\[\mathcal{B} \left[ (\begin{smallmatrix} (x + z) \xi + (z - y) \delta \\ z + z \xi + (z - y) \delta \end{smallmatrix}) \right] = \mathcal{B} \left[ (\begin{smallmatrix} x \delta + \xi + y \delta \\ z + z \xi + (z - y) \delta \end{smallmatrix}) \right]\]
\[\mathcal{B} \left[ (\begin{smallmatrix} x \delta + \xi + y \delta \\ z + z \xi + (z - y) \delta \end{smallmatrix}) \right] = \mathcal{B} \left[ (\begin{smallmatrix} x \delta + \xi + y \delta \\ z + z \xi + (z - y) \delta \end{smallmatrix}) \right]\]
\[\mathcal{B} \left[ (\begin{smallmatrix} x \delta + \xi + y \delta \\ z + z \xi + (z - y) \delta \end{smallmatrix}) \right] = \mathcal{B} \left[ (\begin{smallmatrix} x \delta + \xi + y \delta \\ z + z \xi + (z - y) \delta \end{smallmatrix}) \right]\]
Now \(\ker(\phi: Q \rightarrow Q^*) = P\), and \(Q = L \oplus L^\perp = P \oplus P^*, \)
so
\[\tau ((Q_2, \Theta) \oplus H_{\pm}(-L_2))\]
\[= B([L] \oplus [\mathcal{B}] + [L \oplus L^\perp]) = B([L] \oplus [\mathcal{B}] \in \Omega^2(A_2)\]

Finally, suppose that \(L\) is a hamiltonian complement to either \(P\) or \(P^*\), choosing \(L^\perp\) accordingly.

Then \(\lambda = 0\) and the annihilator \(\mathcal{A} L^\perp\) in
\[(Q_2, \Theta) = (L \oplus L^\perp, (\begin{smallmatrix} 1 & -z \\ 0 & \delta \end{smallmatrix}))\]
is given by
\[\mathcal{A} (L^\perp) = \mathcal{A} \left( \begin{smallmatrix} x, y \\ z \end{smallmatrix} \right) \in L \oplus L^\perp \mid (\begin{smallmatrix} x + z \xi + y \delta \\ z + z \xi + (z - y) \delta \end{smallmatrix}) \]
\[= L \oplus \ker((\begin{smallmatrix} y \delta + \xi + x \delta \\ z + z \xi + (z - y) \delta \end{smallmatrix}): L \rightarrow L_2)\]
As \(x \neq 0\), \(\mathcal{A} \left( \begin{smallmatrix} x, y \\ z \end{smallmatrix} \right) \in \ker((\begin{smallmatrix} y \delta + \xi + x \delta \\ z + z \xi + (z - y) \delta \end{smallmatrix}): L \rightarrow L_2)\),
\[x \in \ker((\begin{smallmatrix} y \delta + \xi + x \delta \\ z + z \xi + (z - y) \delta \end{smallmatrix}): L \rightarrow L_2)\]
then \( x = (z-1)(\pm \delta x) \in (z-1)L_z \) and \((\pm \delta x) \in \ker ((\gamma^* \pm \delta^*): L_z \to L_z)\) as well.

By induction on \( N, x \in (z-1)^N L_z \) for every \( N \geq 1 \), which is impossible unless \( x = 0 \).

Thus \( L_z^* = L_z^* \) and \( L_z^* \oplus -L_z^* \) is a free lagrangian of \( (L_z \oplus L_z^*; (\delta \ 0), (0 \ -\delta)) \oplus \mathbb{H}_z (L_z) \), making it vanish in \( V_{2i}(A_z) \).

\( \square \)

Define
\[
\mathcal{B} : U_{2i-1}(A) \to V_{2i}(A_z)
\]
\[
(\mathcal{Q}, \phi; F, G) \mapsto (G \oplus \phi \oplus \mathbb{G}^*, \begin{pmatrix} 0 & \mathcal{Q} \\ \mathcal{Q}^* & 0 \end{pmatrix} \oplus \mathcal{H}_z(G))
\]

by choosing hamiltonian complements \( F^*, G^* \) to \( F, G \) in \( \mathcal{Q}, \phi \), and expressing
\[
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : \mathcal{F} \oplus F^* \to F^* \oplus \mathcal{F}
\]
as
\[
\begin{pmatrix} \lambda & \pm z \gamma \\ 0 & \lambda \pm z \gamma \end{pmatrix} : G \oplus \mathbb{G}^* \to \mathbb{G}^* \oplus G.
\]

We have already shown, in Lemma 3.2 above, that this does not depend on the choice of \( \gamma^* \), and that
\[
\mathcal{B}(\mathcal{Q}, \phi; F, G) = 0 \in V_{2i}(A_z)
\]
if \( (\mathcal{Q}, \phi; F, G) \in \mathcal{C}_U V_{2i}(A_z) \).

Hence the choice of \( \lambda \in \mathbb{H}_z(G) \) is also immaterial: for
\[
\mathcal{B}(\mathcal{Q} \oplus \mathcal{Q}, \phi \oplus \phi, \mathcal{F} \oplus \mathcal{F}, \mathbb{G} \oplus \mathbb{G}) = 0 \in V_{2i}(A_z),
\]
so that
\[
\mathcal{B}(\mathcal{Q}, \phi; F, G) = -\mathcal{B}(\mathcal{Q}, \phi; F^*, G^*) \in V_{2i}(A_z)
\]
and \(-\mathcal{B}(\mathcal{Q}, \phi; F^*, G^*)\) can be defined without making a choice of \( \lambda \in \mathbb{H}_z(G) \).

It remains to verify that the definition is invariant under changes of \( F^* \). For this, it is convenient to have available a more intrinsic characterization of \( 0 \pm \delta^* \in \text{Hom}_{A_z}(Q_z, Q^*_z) \) (as defined in Lemma 3.2):

given a lagrangian \( L \) of a \( \pm \delta \) form \( (\mathcal{Q}, \phi) \)
over \( A_z \), let \( \mathcal{Q} \in \text{Hom}_{A_z}(L \oplus Q_z, L^* \oplus Q^*_z) \) be the unique \( A_z \)-linear extension of
\[
(1-z) \phi \pm (1-z^\ast) \phi^* : Q_z \to Q^*_z
\]
such that
\[
\mathcal{Q}(\mathcal{R}) = 0,
\]
where
\[
\mathcal{R} = \frac{1}{2} \sum (z-1)(e,e) \in L_z \oplus Q_z, e \in L_z^3.
\]
Let 
\[ \overline{\phi} : L_2 \otimes Q_z \otimes R \to (L_2 \otimes Q_z \otimes R)^*, \]

\[ [e, x] \mapsto ([f, y] \mapsto \Phi(e, x)(f, y)) \]
be the induced $A_z$-module morphism, writing $[e, x]$ for the residue class mod $R$ of $(e, x) \in L_2 \otimes Q_z$.

A choice of hamiltonian complement $L^x$ to $L$ in $(Q, \phi)$ determines an $A_z$-module isomorphism

\[ \eta : L_2 \otimes L^x_z \to L_2 \otimes Q_z \otimes R, \]

\[ (e, u) \mapsto [e, u] \]
such that

\[ \eta^* \overline{\eta} = \hat{\otimes} \otimes \otimes^* \in \text{Hom}_{A_z}(Q_z, Q_z^*) \).

Now let $(Q, \phi) = H_{\phi}(P)$, and let $P^x$ be any hamiltonian complement to $P$ in $H_{\phi}(P)$, so that $P^x = F_{\phi}(P)$ or some $P^x$ (by Lemma 1.3 of I).

Let $(Q, \hat{\phi})$ be the $\pm$ form over $A_z$ defined as $(Q, \phi)$, but with $P^x$ in place of $P$. Let

\[ \beta : Q = P \otimes P^x \to P^x \]
be the projection on $P^x$ along $P$. Then

\[ L_2 \otimes Q_z \to L_2 \otimes Q_z, \]

\[ (e, x) \mapsto (e, x + \mu(\beta(-e + x - 0_2)) \]
is an $A_z$-module isomorphism which induces, via $\eta$, an isomorphism

\[ (Q_z, \otimes) \to (Q_z, \hat{\otimes}) \]
of $\pm$ forms over $A_z$.

Thus $\overline{B}(Q, \phi, F, \bar{G}) \in V_{2i}(A_z)$ does not depend on the representative $\pm$-formulation of $(Q, \phi, F, \bar{G})$ and

\[ \overline{B} : U_{2i-1}(A) \to V_{2i}(A_z) \]
is well-defined.

It should be noted that we can also give a more symmetric definition

\[ \overline{B} : U_{2i-1}(A) \to V_{2i}(A_z), \]

\[ (Q, \phi, F, \bar{G}) \mapsto \left( \hat{\otimes} \otimes \otimes (\phi, \beta_{\phi}(0 - z_2, z_x)), (\phi, \beta_{\phi}(0 - z_2, z_x)), (\phi, \beta_{\phi}(0 - z_2, z_x)) \right) \in H_{\phi}(F, \bar{G}) \]

\[ \otimes - (F_2 \otimes P^x, (\phi, \beta_{\phi}(0 - z_2, z_x))) \in H_{\phi}(F_z, F_2) \]
where $(Q, \phi) = H_{\phi}(P)$ and
\( \phi = \begin{pmatrix} \lambda \pm \lambda^* & \varsigma \\ \delta & \lambda_i \pm \lambda_i^* \end{pmatrix} : \mathcal{G} \oplus \mathcal{G}^* \to \mathcal{G}^* \oplus \mathcal{G} \)

\( \phi = \begin{pmatrix} \mu \pm \mu^* & \alpha \\ \beta & \mu_i \pm \mu_i^* \end{pmatrix} : F \oplus F^* \to F^* \oplus F \)

For some hamiltonian complements \( F_i^* \) to \( F \) in \( (\mathcal{G}, \phi) \).

The two definitions agree because

\[
(H_{\pm}(P); F, G) = (H_{\pm}(P); P, G) \oplus (H_{\pm}(P); F, P) \\
= (H_{\pm}(P), P, G) \oplus (H_{\pm}(P); P, F) \in U_{2i-1}(A)
\]

by the \( U \)-theory norm formula of Lemma 3.3 of \( I \).

It is immediate from Lemma 3.2 that the composite

\[
U_{2i-1}(A) \xrightarrow{\bar{B}} V_{2i}(A) \xrightarrow{\bar{E}} V_{2i}(A)
\]

is 0, and that the square

\[
\begin{array}{c}
U_{2i-1}(A) \xrightarrow{\bar{B}} V_{2i}(A) \\
\downarrow \quad \downarrow \\
V_{2i}(A) \to \Omega^+(A)
\end{array}
\]

commutes.

Lemma 3.3 The composite

\[
U_{2i-1}(A) \xrightarrow{\bar{B}} V_{2i}(A) \xrightarrow{\bar{E}} U_{2i-1}(A)
\]

is the identity.

Proof: Given \((Q, \phi; F, G) \in U_{2i-1}(A)\) we may assume \( (Q, \phi) = H_{\pm}(F) \), so that

\[
\bar{B}(Q, \phi; F, G) = (Q, \phi; \Theta H_{\pm}(G_0)) \in V_{2i}(A)
\]

where

\[
\Theta = \begin{pmatrix} \lambda & -\varsigma \\ \delta & (\lambda \pm \lambda_i^*) \end{pmatrix} \in \text{Hom}_{\mathcal{A}}(G_0 \oplus G_0^*, G_0 \oplus G_0^*)
\]

and

\[
\phi = \begin{pmatrix} \lambda \pm \lambda^* & \varsigma \\ \delta & \lambda_i \pm \lambda_i^* \end{pmatrix} \in \text{Hom}_{\mathcal{A}}(\mathcal{G} \oplus \mathcal{G}^*, \mathcal{G}^* \oplus \mathcal{G})
\]

for some hamiltonian complement \( \mathcal{G}^* \) to \( \mathcal{G} \) in \( (\mathcal{G}, \phi) \).

Thus

\[
\bar{B}(Q, \phi; F, G) = B(Q_0, \Theta) \oplus H_{\pm}(F_0)
\]

\[
= (H_{\pm}(\Theta); Q, B(Q, G)) \oplus (H_{\pm}(\Theta - F_0); -F_0 \oplus F_0, \Gamma_{\pm}(F_0))
\]

\[
= (H_{\pm}(\Theta); Q, B(Q, G)) \in U_{2i-1}(A),
\]

where

\[
B(Q, G) = \Sigma_{(\xi(1, 2) = \{
\alpha \oplus \Theta \xi)(G_0 \oplus G_0^*) | \alpha \in B^*_+(G_0 \oplus G_0^*, \Theta) \})
\]
with \( \omega : (\mathfrak{g} \oplus \mathfrak{g}^*)^+ \to (\mathfrak{g} \oplus \mathfrak{g}^*)^+ \)
the positive projection. As in the proof of Lemma 3.2
\[ B_1^+((\mathfrak{g} \oplus \mathfrak{g}^*)^+, Q^*, Q) = \mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}^* \times \mathfrak{g} \oplus \mathfrak{g}^* \]
so that
\[ B_1(Q, \mathbb{Q}) = \mathbb{F} \{ x, y \in \mathfrak{g} \oplus \mathfrak{g}^* \mid x \in \mathfrak{g}^* \}
\oplus \mathbb{F} \{ y, \pm \psi y \in \mathfrak{g} \oplus \mathfrak{g}^* \mid y \in \mathfrak{g}^* \}.

The isomorphism of \( \tau \) forms over \( A \)
\[ \left( \begin{array}{cc} 1 & 1 \\ \phi & \phi \end{array} \right), \left( \begin{array}{cc} 0 & 0 \\ \phi & \phi \end{array} \right) : (\mathfrak{g} \oplus \mathfrak{g}^*, \phi \otimes \phi) \to H_1(Q) \]
sends \( \mathbb{F} \oplus \mathbb{F}^* \) onto \( Q \), and \( \mathbb{G} \oplus \mathbb{F} \) onto \( B_1(Q, \mathbb{Q}) \).
Thus
\[ B_1(Q, \mathbb{Q}) = (H_1(Q), Q, B_1(Q, \mathbb{Q})) \]
\[ = (\mathfrak{g} \oplus \mathfrak{g}^*, \phi \otimes \phi; \mathbb{F} \oplus \mathbb{F}^*, \mathbb{G} \oplus \mathbb{F}) \]
\[ = (Q, \phi; F, G) \in U_2(A) \).

We need just one more result to prove that
\[ 0 \to V_{2i}(A_{2i}) \xrightarrow{\bar{c}} V_{2i}(A_{2i}) \xrightarrow{\delta} U_{2i-1}(A) \to 0 \]
is a split short exact sequence.

Let \( z_1, z_2 \) be independent commuting indeterminates over \( A \). The double extension of \( A \) by \((z_1, z_2), A = A_{z_2} \), is the ring of polynomials in \( z_1, z_1', z_2, z_2' \) with involution by \( z_1 \mapsto z_1', z_2 \mapsto z_2' \). It is clear that \( A_{z_1} \) may be regarded as either \( A_{z_1} \) or \( A_{z_2} \), and satisfies all the conditions imposed on \( A \).

Lemma 3.4 The diagram
\[ V_{2i}(A_{2i}) \xrightarrow{\delta} U_{2i-1}(A) \]
\[ \xrightarrow{\bar{c}} V_{2i}(A_{2i}) \xrightarrow{\delta} U_{2i-1}(A) \]
\[ \xrightarrow{\delta} V_{2i}(A_{2i}) \]

commutes.

Proof: Given \((z_1, z_2) \in V_{2i}(A_{2i})\) we may assume that \( Q \) is a \( F \)-free \( A_{2i} \)-module, as usual. Choose a modular \( A \)-base \( Q \) of \( Q \), so that
\[ \Delta = \mathbb{F} \{ (x, z) \times (z, z) \mid x \in \mathfrak{g} \} \]
is a modular \( A \)-base of \( \Delta(Q, z) \).
Let $(\mathbb{A}^k, \mathbb{A})$ be a form over $\mathbb{A}_z$ such that there is an isomorphism of forms

$$(\mathbb{C}^k, \mathbb{C}) : (\mathbb{A}^k, \mathbb{A}) \to (\mathbb{C}^k, \mathbb{C})$$

(cf. Lemma 1.4 of I). Then

$$\Delta^* = \sum (z^1, z^2) \in \mathbb{A} \otimes \mathbb{A}^* \mid t \in \mathbb{A}^*$$

is the modular $A$-base dual to $\Delta$ of the hamiltonian complement $\Delta^*(\mathbb{A}^* \otimes \mathbb{A})$ in $(\mathbb{A} \otimes \mathbb{A}^*, \mathbb{A} \otimes \mathbb{A}^*)$.

Let $N > 0$ be an integer so large that

$$(\mathbb{C}^k, \mathbb{C}) (\mathbb{A}^k, \mathbb{A}) \subseteq \sum_{j=-N}^{N} \mathbb{C}^k \otimes \mathbb{C}^* \subseteq \sum_{j=-N}^{N} \mathbb{Q}^* .$$

Adding on some $\mathbb{A}^* \otimes \mathbb{A}^* \in \mathbb{A} \otimes \mathbb{A}^*$ to $\Delta^*$ if necessary, it may be assumed that

$$\Delta^*(\mathbb{A}^* \otimes \mathbb{A}) \subseteq \sum_{j=0}^{N} \mathbb{Q}^* .$$

This ensures that

$$\mathbb{Z}^N \Delta_i^* \subseteq \mathbb{Z}_2 (\mathbb{A} \otimes \mathbb{A}^*)_{\mathbb{Z}_2} ,$$

where

$$\mathbb{Z}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \in \mathbb{A}_z (\mathbb{Q}_2 \otimes \mathbb{Q}_2, \mathbb{Q}_2 \otimes \mathbb{Q}_2) .$$

because every $(\mathbb{Q}, \mathbb{Q}) \in \mathbb{Z}^N \Delta_i^* \subseteq \mathbb{Z}_2 (\mathbb{A} \otimes \mathbb{A}^*)_{\mathbb{Z}_2}$ can be expressed as

$$(\mathbb{Q}, \mathbb{Q}) = (a, z_2 a) + (\mathbb{Q}_z, \mathbb{Q}_z) \in \mathbb{Z}_2 (\mathbb{A} \otimes \mathbb{A}^*)_{\mathbb{Z}_2}$$

with

$$y = (1 - z_2^2) (\mathbb{Q} \otimes \mathbb{Q}_z) \in \mathbb{Q}^*_z \setminus \mathbb{Q}_z .$$

For any $\mathbb{A}_z$-base $\mathbb{Q}$,

$$\mathbb{E}(\mathbb{Q}, \mathbb{Q}) = (\mathbb{Q} \otimes \mathbb{Q}_z, \mathbb{Q} \otimes \mathbb{Q}_z) \Delta (\mathbb{Q}, \mathbb{Q}) \subseteq \mathbb{E} (\mathbb{Q}, \mathbb{Q}) \subseteq \mathbb{Z}_2 (\mathbb{A} \otimes \mathbb{A}^*)_{\mathbb{Z}_2}$$

where $\mathbb{E} (\mathbb{Q}, \mathbb{Q}) = \Delta (\mathbb{Q}, \mathbb{Q}) \subseteq \mathbb{Z}_2 (\mathbb{A} \otimes \mathbb{A}^*)_{\mathbb{Z}_2}$, defining a hamiltonian base of $(\mathbb{Q} \otimes \mathbb{Q}_z, \mathbb{Q} \otimes \mathbb{Q}_z)$. Thus

$$\mathbb{E} (\mathbb{Q}, \mathbb{Q}) \subseteq \mathbb{E} (\mathbb{Q}, \mathbb{Q})$$

$$= \left( E(\mathbb{Q}, \mathbb{Q}) \otimes \mathbb{Q} \otimes \mathbb{Q}_z \otimes \mathbb{Q}_z \right) \Delta_i (\mathbb{Q}, \mathbb{Q}) \otimes \mathbb{Q} \otimes \mathbb{Q}_z \otimes \mathbb{Q}_z ,$$

where

$$E(\mathbb{Q}, \mathbb{Q}) = \mathbb{Z}_2 (\mathbb{A} \otimes \mathbb{A}^*)_{\mathbb{Z}_2}$$

$$= \{ (\mathbb{Q}, \mathbb{Q}) \otimes \mathbb{Q} \otimes \mathbb{Q}_z \otimes \mathbb{Q}_z \mid \mathbb{Q}_z \otimes \mathbb{Q}_z \} \subseteq \mathbb{E} (\mathbb{Q}, \mathbb{Q}) \subseteq \mathbb{A}_z \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2$$

$$= \mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2 .$$
(using the alternative definition of $B : W_{2i+1}(A_2) \to V_{2i}(A)$
given for $V$-theory in §2) with

\[ \psi : C \to \mathbb{Q} \oplus \mathbb{Q}^* \]

the positive projection.

Next, set $P = \sum_{j=0}^{N-1} z_j \mathbb{Q}$ (an $A$-module), and
define an $A_2$-module isomorphism

\[ f : P_2 \mathbb{Q} \oplus P_2^* \to E^+_2(A_2, \mathbb{Q}(\mathbb{Q} \oplus \mathbb{Q}^*), \Delta) ; \]

so that

\[ B(z) B(z) \in V_{2i}(A_2) \]

Defining $\odot \in \text{Hom}_{A_2}(P_2 \mathbb{Q} \oplus P_2^*, P_2 \mathbb{Q} \oplus P_2^*)$ by

\[ \odot(a, b) \cdot (a', b') = \sum \left( (a \oplus a') - (a \oplus a') \right) \cdot \left( b \oplus b' \right) \cdot a \cdot b' \]

\[ \in B(z) \mathbb{Q} \oplus B(z) \mathbb{Q}^* \in A_2 \]

\[ (a, a' \in P_2, b, b' \in P_2^*) \]

it is not difficult to verify that

i) $\odot$ differs from $f^* \left( \mathbb{Q} \oplus \mathbb{Q}^* \right)_2$ of

some $\chi \in \text{Hom}_{A_2}(P_2 \mathbb{Q} \oplus P_2^*, \mathbb{Q} \oplus \mathbb{Q}^*)$

ii) defining $\eta = (1 \mathbb{Q}^* \mathbb{Q}) \in \text{Hom}_{A_2}(P_2 \mathbb{Q} \oplus P_2^*, P_2 \mathbb{Q} \oplus P_2^*)$

we have that

\[ \eta^* \odot \eta = \left( \begin{array}{cc} (1 - z_2)(1, \lambda), \delta \end{array} \right) \]

\[ \in \text{Hom}_{A_2}(P_2 \mathbb{Q} \oplus P_2^*, P_2 \mathbb{Q} \oplus P_2^*) \]

where

\[ \left( \begin{array}{ll} \lambda, & \lambda^* \\ \delta, & \lambda^* \end{array} \right) \in \text{Hom}_{A_2}(P_2 \mathbb{Q} \oplus P_2^*, P_2 \mathbb{Q} \oplus P_2^*) \]

is an expression for

\[ \left( \begin{array}{ll} 0 & 0 \\ 1 & 0 \end{array} \right) \in \text{Hom}_{A_2}(B_2, \mathbb{Q} \oplus \mathbb{Q}^*, \mathbb{Q} \oplus \mathbb{Q}^*) \]

with $B_2, B_2^*$ the hamiltonian complements

in $H_2$ defined in Lemma 3.1.

Thus

\[ B(z) B(z) \in A_2 \]

\[ = B(z) (H_2(P), B_2, \mathbb{Q}, \mathbb{Q}^*) \]

\[ = B(z) (-H_2(P), P, B_2, \mathbb{Q}) \]

\[ = - B(z) B(z) (z, z) \in V_{2i}(A_2) \]

using the $V$-theory sum formula of Lemma 3.1.
Applying $B(z)$ to the decomposition

$$W_{2i}(A_z,A_z) = \epsilon(z) W_{2i}(A_z) \oplus \overline{B}(z) V_{2i}(A_z)$$

obtained in $S2$, it is now immediate that

$$V_{2i}(A_z) = \epsilon(z) V_{2i}(A_z) \oplus \overline{B}(z) V_{2i-1}(A_z).$$

This proves the part of Theorem 1.1 relating to $V_n(A_z)$, $n$ even.

To complete the proof, we give analogous constructions for $W$-theory.

Define

$$B : W_{2i}(A_z) \to V_{2i-1}(A_z),$$

$$(Q, \overline{P}) \mapsto (H_+(\frac{N-1}{2} z; Q), \frac{N-1}{2} z; Q, B_n(Q, \overline{P}))$$

with $Q$ the modular $A_z$-base of $Q$ generated by the given $A_z$-base, and $B_n(Q, \overline{P})$ as in Lemma 3.1.

Then

$$[B_n(Q, \overline{P})] = B \in (Q, \overline{P}) = \epsilon \in F_n(A_z),$$

as required for $V$-theory, since

$$\epsilon(Q, \overline{P}) = \epsilon \in F_n(A_z)$$

by construction of $W_{2i}(A_z)$ (cf $S5$ of I.)

The composite

$$W_{2i}(A_z) \xrightarrow{\epsilon} W_{2i}(A_z) \xrightarrow{B} V_{2i-1}(A_z)$$

is $0$, as for $V$-theory.

The square

$$\begin{array}{ccc}
\Omega_-(A_z) & \xrightarrow{B} & W_{2i}(A_z) \\
\downarrow & & \downarrow \\
\Sigma_+(A) & \xrightarrow{B} & V_{2i-1}(A_z)
\end{array}$$

commutes, and elements of $\Omega_-(A_z)$ are sent to $0$ by both ways round the square, while the composition

$$\Sigma_+(A) \xrightarrow{B} \Omega_-(A_z) \xrightarrow{B} W_{2i}(A_z) \xrightarrow{B} V_{2i-1}(A_z)$$

sends $[P] \in \Sigma_+(A)$ to

$$B([P] \otimes (\frac{P_0}{P_0} - P_0 P_0)^\times, \left(\begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array}\right))$$

$$= (H_+(P_0 P_0 - P_0 P_0), P_0 P_0 P_0 P_0 P_0, P_0 P_0 P_0 P_0 P_0, P_0 P_0 P_0 P_0 P_0, P_0 P_0 P_0 P_0 P_0) \in E_\infty(A_z)$$

agreeing with the map $\Sigma_+(A) \to V_{2i-1}(A_z)$ defined in $S3.4$ of I.
Next, define
\[ \overline{B} : V_{2i-1}(A) \to W_{2i}(A), \]
\[ (Q, \phi; F, G) \mapsto (Q_{x}, \phi) \otimes (F \otimes F^{*}, (F, G)) \]
where \( (Q, \phi) = H_{x}(E) \) for any base \( E \) of \( F \) (assumed free),
and \( (Q_{x}, \phi) \) is the \( \pm \) form over \( A_{x} \) defined in
Lemma 3.2 (with \( F, G \) replacing \( P_{j}L \) respectively),
so that
\[ \tau(Q_{x}, \phi) = \overline{B}([G] - [E]) = 0 \in \Omega^{+}(A_{x}), \]
and
\[ \overline{B} : \mathbb{R} \to \mathbb{R}^{*} \]
is an automorphism of a based \( A_{x} \)-module \( \mathbb{R} \)
such that
\[ \tau(Q_{x}, \phi) + \tau(E) + \tau(E^{*}) = 0 \in \overline{B}(A_{x}). \]

The composites
\[ V_{2i-1}(A) \xrightarrow{\overline{B}} W_{2i}(A) \xrightarrow{E} W_{2i}(A) \]
\[ V_{2i-1}(A) \xrightarrow{\overline{B}} W_{2i}(A) \xrightarrow{E} W_{2i-1}(A) \]
are 0,1 as for \( V \)-theory.

There is no need to prove directly
that the square
\[ \Omega^{-}(A) \xrightarrow{\Omega_{-}} V_{2i-1}(A) \]
\[ \Omega_{+}(A) \xrightarrow{\Omega_{+}} W_{2i}(A) \]
commutes: we have already shown that its commutator lies in
\[ \ker (E : W_{2i}(A) \to W_{2i}(A)) \cap \ker (B : W_{2i}(A) \to V_{2i-1}(A)). \]
This intersection will presently be shown to be null
(without using the commutativity of the square!)

The (split) exactness of
\[ 0 \to W_{2i}(A) \xrightarrow{E} W_{2i}(A) \xrightarrow{B} V_{2i-1}(A) \to 0 \]
follows from a diagram chase round:

\[ \Omega_{-}(A) \xrightarrow{\delta} W_{2i}(A) \xrightarrow{\bar{E}} V_{2i}(A) \xrightarrow{\bar{B}} \Omega_{+}(A) \]

\[ V_{2i}(A_{x}) \xrightarrow{\delta} \Omega^{-}(A_{x}) \xrightarrow{\bar{E}} W_{2i}(A_{x}) \xrightarrow{\bar{B}} V_{2i}(A_{x}) \xrightarrow{\bar{B}} \Omega_{+}(A_{x}) \]

in which all the squares commute, and the rows are parts of the exact sequences of Theorems 4.3, 5.7 of I. The inside left and right columns are exact - we wish to verify that the central column
commutes as well.
§ 4. Multiple Laurent extensions

Let \( T(p) \) be the free abelian group of rank \( p(\infty) \), written multiplicatively. The group ring \( A[T(p)] \), with involution

\[
\sum_{g \in T(p)} a_g g \mapsto \sum_{g \in T(p)} \bar{a}_g \bar{g}^{-1}, \quad (a_g \in A)
\]

is the \( p \)-fold Laurent extension of \( A \).

We may identify

\( A[T(0)] = A \), \( A[T(1)] = A_z \), \( A[T(\infty)] = A_{z,-z} \),

and also

\( (A[T(p)][T(q)] = A[T(p+q)] \) \((p, q \geq 0)\), so that each \( A[T(p)] \) satisfies the conditions imposed on the ground ring \( A \). Denoting some set of generators of \( T(p) \) by \( z_1, z_2, \ldots, z_p \) (for \( p \geq 0 \)), we can also write

\( A[T(p)] = A_{z_1, z_2, \ldots, z_p} \)

extending the previous notation.

This completes the proof of Theorem 1.1.
In order to give a complete description of the $L$-theory of $A_2, \ldots, z_p$ we recall first the "lower $K$-theory" of Chapter $\Xi \Pi$ of Bass' Algebraic $K$-Theory. Involving $K$-groups $\hat{K}_m(A)$ for $m < 0$, and subgroups $N^+_m(A), N^-_m(A)$ of $\hat{K}_{m+1}(A_2)$.

There are defined morphisms

$$\hat{K}_{m+1}(A_2) \xrightarrow{B} \hat{K}_m(A) \xleftarrow{B} \hat{K}_{m+1}(A_2) \tag{m < 0}$$

such that

$$B \circ B = 1 : \hat{K}_m(A) \rightarrow \hat{K}_m(A),$$

giving natural direct sum decompositions

$$\hat{K}_{m+1}(A_2) = \hat{K}_{m+1}(A) \oplus \hat{K}_m(A) \oplus N^+_m(A) \oplus N^-_m(A) \tag{m < 0}$$

Duality involutions

$$* : \hat{K}_m(A) \rightarrow \hat{K}_m(A)$$

are defined for all $m < 0$, with

$$\hat{K}_m(A_2) \xrightarrow{*} \hat{K}_m(A_2)$$

commuting, and with

$$\hat{K}_m(A) \xrightarrow{*} \hat{K}_m(A)$$

skewcommuting, and

$$*(N^+_m(A)) = N^+_m(A) \subseteq \hat{K}_{m+1}(A_2).$$

In short, $\hat{K}_{m+1}(A_2)$ is related to $\hat{K}_m(A)$ in exactly the same way for $m < 0$ as for $m = 0$.

Regarding $\hat{K}_m(A)$ as a $\mathbb{Z}_2$-module via $*$, there are defined Tate cohomology groups (for all $m \geq 0$)

$$H^{(m)}_n(A) = H_n(\hat{K}_m(A_2), \hat{K}_m(A)) = \{x \in \hat{K}_m(A_2) \mid x = (0)^\infty \} \bigcup \{x \in \hat{K}_m(A) \mid x = (0)^\infty \},$$

depending only on $n (\text{mod} 2)$, which are abelian groups of exponent 2. This generalizes to $m < 0$ the definitions of

$$\Omega_{2\infty}(A) = H^{(1)}_n(A) \quad \Sigma_{2\infty}(A) = H^{(1)}_n(A).$$

The induced maps

$$H^{(m)}_n(A) \xleftarrow{B} H^{(m)}_n(A_2) \xrightarrow{B} H^{(m+1)}_n(A)$$

give natural splittings

$$H^{(m)}_n(A_2) = H^{(m)}_n(A) \oplus H^{(m+1)}_n(A) \quad (n \text{ (mod} 2).$$

For each $m < 0$, (exactly as for $m = 0$).
We now define the "lower L-groups"

\[ L_n^{(m)}(A) = \ker \left( \varepsilon : L_{n+1}^{(m+1)}(A) \to L_{n}^{(m+1)}(A) \right) \]

inductively, for \( m \leq 1 \), where \( L_n^{(0)} = W_n \).

It is clear from Theorem 1.1 that we can identify

\[ L_n^{(1)} = V_n, \quad L_n^{(0)} = U_n \]

and that there is defined a natural exact sequence

\[ \ldots \to H_{n+2}^{(m+1)}(A) \to L_{n+1}^{(m+1)}(A) \to L_n^{(m+1)}(A) \to H_n^{(m+1)}(A) \to \ldots \]

of abelian groups and morphisms for \( m \leq 0,1 \).

Hence all the L-theories differ in 2-torsion only. More precisely:

**Theorem 4.1** There is defined a natural exact sequence

\[ \ldots \to H_{n+2}^{(m+1)}(A) \to L_{n+1}^{(m+1)}(A) \to L_n^{(m+1)}(A) \to H_n^{(m+1)}(A) \to \ldots \]

of abelian groups and morphisms, for all \( m \leq 1, n \equiv m (\text{mod } 4) \).

**Proof:** By induction on \( m \), downwards.

\[ \square \]

**Theorem 4.2** There are defined isomorphisms of graded abelian groups

\[ L_n^{(p)}(A[T(p)]) \cong L_n^{(p)}(A) \otimes_{\mathbb{Z}} \Lambda_x(p) \]

where \( \Lambda_x(p) \) is the graded exterior \( \mathbb{Z} \)-algebra on generators \( z_1, z_2, \ldots, z_p \) in degree 1, which are natural in \( A \) and \( T(p) \), with components

\[ L_n^{(m)}(A[z_1, z_2, \ldots, z_p]) \cong \sum_{r=0}^{p} \binom{p}{r} L_n^{(m-r)}(A) \]

\( (m \leq 2, n \equiv m (\text{mod } 4), p > 0) \)

**Proof:** It is sufficient to consider the case \( W_n(A[z_1, z_2]) \), the others following by induction on \( p \).

We need first the odd-dimensional counterpart of Lemma 3.4, that the diagram

\[ \begin{array}{ccc}
V_{2i+1}^{(m)}(A[z_1]) & \xrightarrow{\beta(z_1)} & U_{2i}(A) \\
\downarrow{\beta(z_1)} & & \downarrow{\beta(z_2)} \\
W_{2i+2}(A[z_1, z_2]) & \xrightarrow{\beta(z_2)} & V_{2i+1}(A[z_1])
\end{array} \]

squares commutes: the proof of this is left to the reader.

(Hint: it is known that

\[ V_{2i+1}^{(m)}(A[z_1]) = \varepsilon(z_1) V_{2i+1}(A[z_1]) \]

The elements of \( \varepsilon(z_1) V_{2i+1}(A[z_1]) \) are sent to \( 0 \) in \( V_{2i+1}(A[z_1]) \).
both ways round the square, so it is sufficient
to verify that the composite
\[ U_{2i}(A) \rightarrow V_{2i+1}(A) \rightarrow W_{2i+2}(A_2, z_2) \rightarrow V_{2i+1}(A_2) \]
coincides with
\[ B(\varepsilon) : U_{2i}(A) \rightarrow V_{2i+1}(A_2). \]
Thus
\[ B(\varepsilon) B(\varepsilon) = -B(\varepsilon), B(\varepsilon) : V_{n}(A_z) \rightarrow V_{n}(A_z) \]
for all \( n \equiv 0 \pmod{4} \), and as
\[ B B + \varepsilon \varepsilon = 1 : W_{n}(A_z) \rightarrow W_{n}(A_z) \]
it follows that
\[ B(\varepsilon) B(\varepsilon) = (B(\varepsilon) B(\varepsilon) + \varepsilon(\varepsilon) \varepsilon(\varepsilon)) B(\varepsilon) B(\varepsilon) \]
\[ = B(\varepsilon) (-B(\varepsilon) B(\varepsilon) B(\varepsilon) + \varepsilon(\varepsilon) B(\varepsilon)) \varepsilon(\varepsilon) B(\varepsilon) \]
\[ = -B(\varepsilon) B(\varepsilon) : U_{n-2}(A) \rightarrow W_{n}(A_z, z_2). \]
Accordingly, we have an isomorphism of abelian groups
\[ L^{(a)}_{n}(A_{z_2}, z_2) \cong \sum_{j=0}^{n} L^{(a-j)}_{n-j}(A) \otimes_{\mathbb{Z}} \otimes \mathbb{A}(\varepsilon)(n \equiv 0 \pmod{4}) \]
sending
\[ \varepsilon(\varepsilon) \varepsilon(\varepsilon) L^{(a)}_{n}(A) \text{ to } L^{(a)}_{n}(A) \otimes 1 \]
\[ B(\varepsilon) \varepsilon(\varepsilon) L^{(a)}_{n-1}(A) \text{ to } L^{(a)}_{n-1}(A) \otimes \mathbb{Z}, \]
\[ \varepsilon(\varepsilon) B(\varepsilon) L^{(a)}_{n-1}(A) \text{ to } L^{(a)}_{n-1}(A) \otimes \mathbb{Z}, \]
\[ B(\varepsilon) B(\varepsilon) L^{(a)}_{n-2}(A) \text{ to } L^{(a)}_{n-2}(A) \otimes (z_1, z_2). \]
Naturality with respect to \( A \) is obvious.

Naturality with respect to \( T(z) \), and more generally, \( T(p) \), follows on noting that a morphism
\[ f : T(p) \rightarrow T(q) \]
is determined by the \( p \times q \) integer matrix
\[ (f_{jk})_{1 \leq j \leq p, 1 \leq k \leq q}, \]
such that
\[ f(z_j) = \prod_{k=1}^{q} z_k f_{jk} \quad (1 \leq j \leq p, f_{jk} \in \mathbb{Z}), \]
the composition of such morphisms corresponding to multiplication of the matrices. Every such
matrix can be expressed as the product of elementary matrices, such as
\[ (0 1), (1 0), \mathbb{N}(\varepsilon \varepsilon), \ldots \]
and their enlargements
\[ (0 1), (1 0), (0 1 \mathbb{Z}), \ldots \]

It is easy to show directly that for \( p, q \leq 2 \) the elementary \( p \times q \) matrices
induce the corresponding morphisms.
\(1 \otimes f : \Lambda^*(A) \otimes \Lambda^*(p) \to \Lambda^*(A) \otimes \Lambda^*(q)\)

in the exterior algebra, where

\[ f : \Lambda^*_*(p) \to \Lambda^*_*(q) ; \]

\[ e_{j_1} \wedge \cdots \wedge e_{j_r} \longmapsto \bigwedge_{m=1}^{r} \left( \sum_{k=1}^{q} f_{j_m} e_{k} \right) \quad (1 \leq r \leq p). \]

\[ \square \]

\section*{Geometric L-theory}

\section*{§0 Conventions}

An inclusion of \(\pm\) forms is a morphism

\[(f, \chi) : (P, \theta) \to (Q, \phi)\]

such that \(f \in \text{Hom}_{\Lambda}(P, Q)\) is split mono.

A \underline{subform} of a \(\pm\) form \((Q, \phi)\) is an equivalence class of inclusions \((f, \chi) : (P, \theta) \to (Q, \phi)\), under the relation

\[(f, \chi) : (P, \theta) \to (Q, \phi) \sim (f', \chi') : (P', \theta') \to (Q, \phi')\]

iff there exists an isomorphism \(g \in \text{Hom}_{\Lambda}(P, P')\) such that

\[
\begin{array}{ccc}
(P, \theta) & \xrightarrow{(f, \chi)} & (Q, \phi) \\
\downarrow & & \downarrow \\
(P', \theta') & \xrightarrow{(f', \chi')} & (Q, \phi')
\end{array}
\]

commutes. We shall use the notation \((P, \theta) \leq (Q, \phi)\) both for a particular inclusion of \(\pm\) forms, and the subform of \((Q, \phi)\) it represents.

For example, a sublagrangian of a \(\pm\) form \((Q, \phi)\) is a subform \((L, \theta) \leq (Q, \phi)\) such that

\[(\theta') : Q \to \Theta^* \text{ is even.}\]
Isomorphisms of \( \pm \)-forms preserve subforms in the obvious way: an isomorphism
\[
(g, \nu): (Q, \phi) \to (Q', \phi')
\]
sends the subform \( (P, \theta) \subseteq (Q, \phi) \) to the subform
\[
(P, \theta) \subseteq (g, \nu)(Q, \phi).
\]
Subforms \( (P, \theta) \subseteq (Q, \phi) \), \( (P', \theta') \subseteq (Q, \phi) \)
are **orthogonal** if
\[
f^\ast(\phi \pm \phi')f = 0 \in \text{Hom}_A(P, P'^\ast)
\]
and maximally orthogonal if the sequences
\[
0 \to P \xrightarrow{f} Q \xrightarrow{f'^\ast(\phi \pm \phi')} P'^\ast \to 0
\]
\[
0 \to P' \xrightarrow{f'} Q \xrightarrow{f^\ast(\phi \pm \phi)} P^\ast \to 0
\]
are exact (that is, \( fP = (\phi'^\ast)^\ast \subseteq Q \), \( f'^\ast fP = (\phi \pm \phi') \subseteq Q \)).
(For non-singular \( \pm \)-forms \( (Q, \phi) \) the exactness of one of the sequences implies that of the other.)

For example, lagrangians \( (L, 0) \subseteq (Q, \phi) \) are the maximally self-orthogonal subforms of \( (Q, \phi) \).

The corresponding notions for \( \pm \)-formations, set out below, are rather more complicated.
It is convenient to consider only the \( \pm \)-formations of standard type
\[
(H_\pm(F), [F, 0], (G, \sigma) \subseteq H_0(F))
\]
which we shall abbreviate to
\[
(F, (\lambda, \theta)G).
\]
An isomorphism of \( \pm \)-formations
\[
(F, (\lambda, \theta)G) \to (F', (\lambda', \theta')G')
\]
is defined by a commutative square
\[
\begin{array}{ccc}
G & \xrightarrow{\beta} & G' \\
(\lambda) \downarrow & & \downarrow (\lambda')
\end{array}
\]
for some isomorphisms \( \alpha \in \text{Hom}_A(F, F') \), \( \beta \in \text{Hom}_A(G, G) \)
and some \( \psi \in \Pi_\mp(F^\ast) \) such that
\[
\beta^\ast \theta' \beta = \theta + \mu^\ast \psi \mu \in \Pi_\mp(G).
\]
A subformation of a $\pm$ formation $(F, ((\mathcal{I}), \delta) G)$ is an equivalence class of inclusions

$$(F', ((\mathcal{I}'), \delta') G') \subseteq_{\lambda, \mu} (F, ((\mathcal{I}), \delta) G)$$

under the relation

$$(F', ((\mathcal{I}'), \delta') G') \subseteq_{\lambda, \mu} (F, ((\mathcal{I}), \delta) G) \sim (F', ((\mathcal{I}'), \delta') G') \subseteq_{\lambda, \mu} (F, ((\mathcal{I}), \delta) G)$$

iff there exists an isomorphism $\omega : \text{Hom}_A(F', F')$ such that

$$(\omega, \delta, \mu') : (F', ((\mathcal{I}'), \delta') G) \rightarrow (F', ((\mathcal{I}'), \delta') G')$$

defines an isomorphism of $\pm$ formations such that $\frac{\omega}{\delta, \mu'}$ commutes.

An isomorphism of $\pm$ formations

$$(\omega, \delta, \mu) : (F, ((\mathcal{I}), \delta) G) \rightarrow (F', ((\mathcal{I}'), \delta') G')$$

sends a subformation

$$(F', ((\mathcal{I}'), \delta') G') \subseteq_{\lambda, \mu} (F, ((\mathcal{I}), \delta) G)$$

to the subformation

$$(F', ((\mathcal{I}'), \delta') G') \subseteq_{\lambda, \mu} (F, ((\mathcal{I}), \delta) G)$$

with

$$(\lambda, \mu', \lambda' \delta' \mu') : (F', ((\mathcal{I}'), \delta') G') \rightarrow (F', ((\mathcal{I}'), \delta') G')$$

defining an isomorphism of $\pm$ formations.

Composition is by $(\omega, \delta', \mu') : (F', ((\mathcal{I}'), \delta') G') \rightarrow (F', ((\mathcal{I}'), \delta') G')$ of $\delta' = \delta \circ \omega$. The graph $\pm$ formations will be denoted by $(P, ((\mathcal{I}), \delta) P)$, the graph $\pm$ formations $(P, ((\mathcal{I}), \delta) P)$.

Composition is by $(\omega, \delta', \mu') : (F, ((\mathcal{I}), \delta) G) \rightarrow (F', ((\mathcal{I}'), \delta') G')$ of $\delta' = \delta \circ \omega$. The graph $\pm$ formations will be denoted by $(P, ((\mathcal{I}), \delta) P)$, the graph $\pm$ formations $(P, ((\mathcal{I}), \delta) P)$.

An inclusion of $\pm$ formations

$$(F', ((\mathcal{I}'), \delta') G') \subseteq_{\lambda, \mu} (F, ((\mathcal{I}), \delta) G)$$

is defined by $\lambda : \text{Hom}_A(F, F')$ and an isomorphism $h : \text{Hom}_A(G, G')$ such that the squares

$$\begin{array}{ccc}
G & \xrightarrow{\gamma} & F \\
\downarrow h & & \downarrow h \\
G' & \xrightarrow{\gamma'} & F'
\end{array}$$

commute, and $h^* \gamma' = \theta \in \Pi_{\mathcal{I}}(G)$.
The annihilator of a formation \((F, ((\xi), \theta)G)\) is the formation
\[(F, ((\xi'), \theta)G)^\perp = (F^*, ((\xi'\mu), \theta)G)\]
(This is just an expression in the new notation of an isomorph of the formation given by \((H, (\xi'\mu), \theta), G, \theta))\) in the old notation).

Note that
\[(F, ((\xi), \theta)G)^\perp = (F^*, ((\xi'\mu), \theta)G)^\perp\]
and that if
\[(F, ((\xi), \theta)G) \subseteq_{x, h} (F', ((\xi'), \theta)G')\]
then
\[(F', ((\xi'), \theta)G')^\perp \subseteq_{x', k'} (F, ((\xi), \theta)G)^\perp\]

Subformations
\[(F', ((\xi'), \theta)G') \subseteq_{x, k} (F, ((\xi), \theta)G), \quad (F', ((\xi'), \theta)G') \subseteq_{x, k} (F, ((\xi), \theta)G)\]
are orthogonal if there exists \(\psi \in \text{Hom}_A(F^*, F')\) such that
\[(F', ((\xi'), \theta)G') \subseteq_{x', k', \psi} (F^*, ((\xi'\mu), \theta)G')^\perp\]
(in which case
\[(F^*, ((\xi'\mu), \theta)G') \subseteq_{x'\mu, \psi} (F', ((\xi'), \theta)G')^\perp)\]

and maximally orthogonal if, further,
\[
\psi \in \text{Hom}_A(F^*, F')\]
is an isomorphism.

In general, given an inclusion
\[(F', ((\xi), \theta)G') \subseteq_{x, h} (F, ((\xi), \theta)G)\]
we shall use the isomorphism \(\psi \in \text{Hom}_A(G, G')\) as an identification, writing
\[
(F', ((\xi'), \theta)G') \subseteq_{x} (F, ((\xi), \theta)G)\]

For example, for any formation \((F, ((\xi), \theta)G)\)
we have that
\[(G^*, ((\xi'\odot), \theta)G) \subseteq_{\mu, (F, ((\xi), \theta)G) \subseteq_{x, (G, ((\xi), \theta)G) \subseteq_{\psi, (G, ((\xi), \theta)G) \subseteq_{x, (G, \theta)})\]
§1 Cobordism of ± forms

Call ± forms \((\alpha,\phi),(\beta,\psi)\) cobordant if there exist a ± formation \((F, ((\alpha),\theta)G)\) and inclusions of ± forms
\[
(\begin{pmatrix} i^* \\ i_{kr} \end{pmatrix}, \iota) : (\alpha, (-)^*\psi) \rightarrow H_+(F) \quad (r=0,1)
\]
such that the sequence
\[
0 \rightarrow Q_0 \oplus G \xrightarrow{(i_{kr}, \iota)} F \oplus F^* \xrightarrow{(i_{kr}^*, i^*)} Q_0^* \oplus G^* \rightarrow 0
\]
is exact, a quintuple such as
\[
(\begin{pmatrix} F, ((\alpha),\theta)G \end{pmatrix}, (\alpha, \phi), (\beta, \psi), (\iota_{kr}, \iota), (\iota_0^{kr}, \iota))
\]
being a cobordism from \((\alpha,\phi)\) to \((\beta,\psi)\).

In other words, ± forms \((\alpha,\phi),(\beta,\psi)\) are cobordant iff \((\alpha,\phi),(\beta,-\psi)\) may be included as maximally orthogonal subforms of a stably trivial ± form
\[
(\alpha \wedge G, \phi) = 0 \quad F, ((\alpha),\theta)G
\]
by definition the ± form to which \((\alpha,\phi)\) \in \text{Hom}_+(F \otimes F^*, F \otimes F^*)
restricts on a direct complement to \(\text{im}((\iota_0^{kr}): G \rightarrow F \otimes F^*)\)
\[\text{ker } (\iota_{kr}^* : F^* \otimes F \rightarrow G^*).
\]

Theorem 1.1 Cobordism is an equivalence relation on ± forms.

Proof. A cobordism of ± forms
\[
((F, ((\alpha),\theta)G); (\alpha,\phi), (\beta,\psi), (\iota_{kr}^{0}, \iota), (\iota_0^{kr}, \iota))
\]
may be reversed, to give a cobordism
\[
((F, ((\beta),\theta)G); (\beta,\psi), (\alpha,\phi), (\iota_0^{kr}, \iota), (\iota_{kr}^{0}, \iota)),
\]
so that the relation is symmetric.

An isomorphism of ± forms
\[
(f, \iota) : (\alpha, \phi) \rightarrow (\beta, \psi)
\]
defines a cobordism,
\[
(\alpha, \phi); (\beta, \psi), (\iota, \psi), (\iota_0^{kr}, \psi), (\iota_{kr}^{0}, \psi)
\]
so that, in particular, the relation is reflexive.

Finally, suppose given adjoining cobordisms
\[
C = ((F, ((\alpha),\theta)G); (\alpha,\phi), (\beta,\psi), (\iota_{kr}^{0}, \iota), (\iota_0^{kr}, \iota))
\]
\[
C' = ((F', ((\beta),\theta)G'); (\beta,\psi), (\alpha,\phi), (\iota_0^{kr}, \iota), (\iota_{kr}^{0}, \iota))
\]

The join of \( c \) and \( c' \),
\[ c \circ c' = \left( F \otimes F', \left( \begin{pmatrix} \gamma & \nu \cr \nu & \theta \end{pmatrix} \right) \left( \begin{pmatrix} \gamma & \nu \cr \nu & \theta \end{pmatrix} \right) \right) \circ (Q, \Phi, G) \]
\[ = \left( \begin{pmatrix} \alpha & \beta \cr \beta & \delta \end{pmatrix} \right) ; \]
\[ \left( Q, \Phi, (Q, \Phi) \right), \left( \begin{pmatrix} \gamma \cr \nu \cr \nu \cr \theta \end{pmatrix} \right) \right) \]
defines a cobordism from \((Q, \Phi)\) to \((Q, \Phi)\), whence transitivity.

Given cobordisms of \( \pm \)-forms
\[ c = \left( (F, ((\xi, \theta) G) ; (Q, \Phi), (Q, \Phi) \right), \left( \begin{pmatrix} \xi \cr \theta \end{pmatrix} \right), \left( \begin{pmatrix} \xi \cr \theta \end{pmatrix} \right) \right) \]
\[ c' = \left( (F', ((\xi', \theta') \theta) G') ; (Q, \Phi), (Q, \Phi') \right), \left( \begin{pmatrix} \xi' \cr \theta' \end{pmatrix} \right), \left( \begin{pmatrix} \xi' \cr \theta' \end{pmatrix} \right) \right) \]
we can define their direct sum \( c \circ c' \) in the obvious way, so that cobordism respects the direct sum operation on \( \pm \)-forms.

**Corollary 1.2** The \( \pm \)-forms \((Q, \Phi)\) and \((Q', \Phi')\) are cobordant if
\[ (Q, \Phi) = (Q', \Phi') \in X_{2i}(A) / \text{im}(\Xi: X_{2i}(A) \rightarrow X_{2i}(A)) \]

**Proof:** A hamiltonian \( \pm \)-form is cobordant to 0 by
\[ (L, 0) ; H_{\pm}(L) ; 0 , (L, 0) ; 0 \]
As isomorphic \( \pm \)-forms are cobordant, and cobordism respects \( \otimes \), it follows from Theorem 1.1 that \( \pm \)-forms \((Q, \Phi)\) and \((Q', \Phi')\) related by an isomorphism
\[ (\alpha) : (Q, \Phi) \otimes H_{\pm}(L) \rightarrow (Q', \Phi') \otimes H_{\pm}(L) \]
are cobordant.

Conversely, suppose given a cobordism
\[ (F, ((\xi, \theta) \theta) G) ; (Q, \Phi), (Q, \Phi), (\xi, \theta) \]
Now,
\[ (\xi, \theta) \rightarrow (Q, \Phi) \otimes H_{\pm}(F) \]
is the inclusion of a sublagrangian with
\[ (Q, \Phi) \otimes H_{\pm}(F) \]
which is clear from the exact sequence
\[ \left( \begin{array}{cc} \Xi & 0 \\ \Phi & \Phi \end{array} \right) \rightarrow Q, \Phi \rightarrow Q \rightarrow Q, \Phi \rightarrow Q \rightarrow 0 \]
A direct application of Corollary 1.2 of I gives an isomorphism of \( \pm \)-forms
\[ (Q, \Phi) \otimes H_{\pm}(Q, \Phi) \rightarrow (Q, \Phi) \otimes H_{\pm}(F) \]
so that
\[ (Q, \Phi) = (Q, \Phi) \in X_{2i}(A) / \text{im}(\Xi: X_{2i}(A) \rightarrow X_{2i}(A)) \]
in particular, non-singular \( \pm \)-forms \((Q, \Phi)\) and \((Q', \Phi')\) are cobordant if
\[ (Q, \Phi) = (Q', \Phi') \in X_{2i}(A) \].
An isomorphism of cobordisms

\[ [\alpha, \beta, \psi] : (F, ((\gamma), \delta)G) \rightarrow \left( F', ((\gamma'), \delta')G' \right) \]

is defined by a stable isomorphism of \( \pm \)-formations

\[ [\alpha, \beta, \psi] : (F, ((\gamma), \delta)G) \rightarrow (F', ((\gamma'), \delta')G') \]

such that the induced isomorphism of \( \pm \)-form

\[ \alpha [\alpha, \beta, \psi] : (G^+ G, \phi) \rightarrow (G'^+ G', \phi') \]

sends the subforms \((G^r, (G^r, \phi)) \subseteq (G^+ G, \phi)\)

to \((G'^r, (G'^r, \phi')) \subseteq (G'^+ G', \phi')\) for \(r = 0, 1\).

The join construction

\[ (C, C') \mapsto C \times C' \]

defined in the proof of Theorem 1.1

may be characterized up to isomorphism by:

\[ \tilde{C} = ((F, ((\gamma), \delta)G) ; (Q_0, \phi_0), (\ldots, \phi_r), ((\gamma_0), \gamma_r), ((\gamma_1), \gamma_r)) \]

\[ \tilde{C}' = ((F', ((\gamma'), \delta')G) ; (Q_0, \phi_0), (\ldots, \phi_r), ((\gamma_0'), \gamma_r'), ((\gamma_1'), \gamma_r')) \]

\[ \tilde{C}'' = ((F'', ((\gamma''), \delta'')G) ; (Q_0, \phi_0), (\ldots, \phi_r), ((\gamma_0''), \gamma_r''), ((\gamma_1''), \gamma_r'')) \]

and an isomorphism of cobordisms

\[ \tilde{C} \times \tilde{C}' \rightarrow \tilde{C}'' \].
Proof: We are dealing with a commutative hexagon.

Choose any \( \phi_r \in \text{Hom}_A(Q_r,Q_r^*) \), \( x_r^0 \in \Pi_r(Q_r^*) \) \( (r = 0,1,2) \) to satisfy the identities

\[
\begin{align*}
&j_r^* x_r^0 \circ k_r = \phi_r = x_r^0 \circ \tau_r^* \\
& (j_{r+1}^* + \phi_r) \circ k_r = x_{r+1}^0 \tau_{r+1}^* \in \text{Hom}_A(Q_r,Q_{r+1}^*)
\end{align*}
\]

such that the following identities may be satisfied simultaneously:

\[
\begin{align*}
&j_r^* x_r^0 \circ k_r = 0 \\
&x_r^0 \circ \tau_r^* \in \text{Hom}_A(Q_r,Q_r^*)
\end{align*}
\]

in order to verify that

\[
\xi = \left( (F, (\gamma, \delta)_G), (\alpha, \beta), (\phi, \psi), ((\lambda, \mu), (\nu, \omega)), ((\phi, \psi), (\psi, \phi)) \right)
\]

defines a cobordism we have to show that the sequence

\[
\begin{array}{c}
0 \rightarrow Q_0 \oplus G \rightarrow F \oplus F^* \rightarrow Q_1^* \oplus G^* \rightarrow 0
\end{array}
\]

is exact.

This is done by diagram chasing round the commutative diagram.
so that
\[(x, y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (x, y) \in \text{im} \left( \begin{pmatrix} \lambda \circ \gamma \\ k_0 \circ \mu \end{pmatrix} : Q_0 \oplus G \to F \oplus F^* \right),\]
as required for the exactness of
\[Q_0 \oplus G \to F \oplus F^* \to G \oplus G^* \to Q_0 \oplus G \]
To verify that
\[\begin{pmatrix} \pm i \circ \gamma \\ \pm \mu \circ \gamma \end{pmatrix} : F \oplus F^* \to G \oplus G^* \]
is onto, start with any
\[(a, b) \in Q_0 \oplus G = \text{im} \left( \begin{pmatrix} i \circ \gamma \\ \mu \circ \gamma \end{pmatrix} : F \oplus F' \to G \oplus G^* \right),\]
let \((c, d) \in F \oplus F'\) be such that
\[(a, b) = \begin{pmatrix} i \circ c + \mu \circ d \\ \mu \circ c + i \circ d \end{pmatrix} \in F \oplus F', Q_0 \oplus G^* \]
Now \(d \in F' = \text{im} \left( \begin{pmatrix} i \circ \gamma \\ \mu \circ \gamma \end{pmatrix} : F \oplus F^* \to F' \right)\)
so that
\[d = x \epsilon - y \delta \]
for some \(\epsilon \in F^*, \delta \in F^*.\)
Then
\[\begin{pmatrix} i \circ \gamma \\ \mu \circ \gamma \end{pmatrix} \begin{pmatrix} x \epsilon + y \delta \\ -y \delta + x \epsilon \end{pmatrix} \in \text{im} \left( \begin{pmatrix} \pm i \circ \gamma \\ \pm \mu \circ \gamma \end{pmatrix} : F \oplus F^* \to G \oplus G^* \right)\]
as required.

Despite the notation we are in a situation with a high degree of symmetry (- this will be exploited more fully later on), as is clear from the hexagon drawn above.

For example, up to signs,
\[\begin{pmatrix} \lambda \circ \gamma \\ k_0 \circ \mu \end{pmatrix} : Q_0 \oplus G \to F \oplus F^* \]
is the dual of the morphism corresponding to
\[\begin{pmatrix} \pm i \circ \gamma \\ \pm \mu \circ \gamma \end{pmatrix} : F \oplus F^* \to G \oplus G^* \]
in the hexagon obtained by rotating the one above through 180°. It is therefore a split mono, and \(\mathcal{E}\) is a well-defined cobordism of \(\pm\) forms.

Similarly, we have that \(\mathcal{E}'\) and \(\mathcal{E}''\) are well-defined.

It now remains to exhibit an isomorphism
\[\mathcal{E} \times \mathcal{E}' \to \mathcal{E}''\]
where
\[\mathcal{E} \times \mathcal{E}' = \left( \left( F \oplus F', \left( \begin{pmatrix} \gamma \circ \lambda \circ \gamma \\ k_0 \circ \mu \circ \gamma \end{pmatrix} \Sigma \begin{pmatrix} \mu \circ \gamma \\ i \circ \gamma \end{pmatrix} \Sigma \begin{pmatrix} \gamma \circ \lambda \circ \gamma \\ k_0 \circ \mu \circ \gamma \end{pmatrix} \right) \mathcal{E} \right) \right),\]
by definition, where \(\mathcal{E} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \in \text{Hom}_A(G \oplus G, G \oplus G)\).
Choosing some left inverse \((\mu_i^*)_*: F^* \rightarrow F^* \oplus F^*\)
to \((\varepsilon^* - \varepsilon)^*: F^* \oplus F^* \rightarrow F^*\), let
\[
0 \rightarrow F^* \xrightarrow{\mu_i^*} F^* \oplus F^* \xrightarrow{(\mu_i^*)_{Q,G}} Q \oplus G \rightarrow 0
\]
be the short exact sequence defined by the corresponding right inverse \((\mu_i^*_i)_*: F^* \oplus F^* \rightarrow Q \oplus G\)
to \((\mu_i^*): Q \oplus G \rightarrow F^* \oplus F^*\).
Then it may be easily verified that
\[
(x, y, z) = \left( \begin{array}{c} \zeta_i \mu_i^* \\ \xi_i \mu_i^* \end{array} \right) \left( \begin{array}{ccc} 1 & 0 & -1 \\ \eta_i & 1 - \eta_i & \xi_i \\ \zeta_i & 0 & \xi_i \end{array} \right) \Sigma \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & \delta \end{array} \right)
\]
Conversely, we have:

**Theorem 1.4** Adjoining cobordisms

\[
\varepsilon' = ((F \oplus F', ((\varepsilon^*), (\theta^*)))_*: (Q, L_1, \lambda, \mu^*) \times (Q, \tilde{L}_1, \tilde{\lambda}, \tilde{\mu})\rightarrow (Q, \xi, \zeta), ((\varepsilon^*), (\theta^*)))
\]
may be replaced by isomorphic ones:

\[
\tilde{\varepsilon} = ((F \oplus F', ((\tilde{\varepsilon}^*), (\tilde{\theta}^*)))_*: (Q, L_1, \lambda, \mu^*) \times (Q, \tilde{L}_1, \tilde{\lambda}, \tilde{\mu})\rightarrow (Q, \xi, \zeta), ((\tilde{\varepsilon}^*), (\tilde{\theta}^*)))
\]
where

\[
(G', \theta') = (G \oplus Q, \Theta G', \left( \begin{array}{ccc} 0 & \delta & 0 \\ \delta & \xi & 0 \\ 0 & 0 & \xi \end{array} \right)),
\]
which together with

\[
\tilde{\varepsilon}' = \varepsilon' \times \varepsilon^* = ((F \oplus F', ((\varepsilon^*), (\theta^*)))_*: (Q, L_1, \lambda, \mu^*) \times (Q, \tilde{L}_1, \tilde{\lambda}, \tilde{\mu})\rightarrow (Q, \xi, \zeta), ((\varepsilon^*), (\theta^*)))
\]
fit into the scheme of Theorem 1.3,
according to the commutative hexagon

\[
(\varepsilon^*, \varepsilon, \varepsilon') : \tilde{\varepsilon} \times \varepsilon^* \rightarrow \tilde{\varepsilon}'.
\]
Proof: The commutative diagram

\[
(\lambda \oplus \lambda') \oplus (G \oplus \Delta', \oplus \Delta') \longrightarrow (\lambda \oplus \lambda') \oplus (G \oplus \Delta', \oplus \Delta')
\]

shows that

\[
(\alpha, \beta, \gamma) = (1, 1, (0, \frac{1}{\gamma}, 0) \oplus (0, \frac{1}{\gamma}, 0))
\]

defines an isomorphism of cobordisms

\[
[\alpha, \beta, \gamma] : \mathcal{C} \longrightarrow \mathcal{C}'
\]

Similarly for \( \mathcal{C}', \mathcal{C}'' \).
§2 Cobordism of $\pm$-forms

Call $\pm$-forms $(F_0, (\xi^i_0, e_0) G_0)$, $(F, (\xi^i, e) G)$ cobordant if there exists a $\mp$-form $(G, e)$, and an inclusion of $\pm$-forms

$$(\xi^i_0, e_0) : (G, e) \hookrightarrow H_+(F)$$

such that there are defined stable isomorphisms

$$[\xi^i_0, \xi^j_0, \psi] : (F_0, (\xi^i_0, e_0) G_0) \to (F, (\xi^j, e) G)$$

$$[\xi^i, \xi^j, \psi] : (F, (\xi^i, e) G) \to (F, (\xi^j, e) G),$$

a quintuple such as

$$((G, e) ; (F_0, (\xi^i_0, e_0) G_0), (F, (\xi^i, e) G), [G_0, \xi^i_0], [G, \xi^i], [\xi], [\xi^i_0], [\xi^i], [\psi])$$

being a cobordism from $(F_0, (\xi^i_0, e_0) G_0)$ to $(F, (\xi^i, e) G)$.

In other words, $\pm$-forms $(F_0, (\xi^i_0, e_0) G_0), (F, (\xi^i, e) G)$ are cobordant iff $(F_0, (\xi^i_0, e_0) G_0), (F, (\xi^i, e) G)$ are stably isomorphic to maximally orthogonal sub-formations of the graph $\pm$-formation

$$(G, e) = (G, (\xi^i_0, e_0) G),$$

defined by some $\mp$-form $(G, e)$. 

\[\square\]
Theorem 2.1 Cobordism is an equivalence relation on $\pm$-formations.

Proof: A cobordism of $\pm$-formations

$$(G,\theta); (F,((\tilde{z}^*),\theta)G), (F,((\bar{z}^*),\theta)G), \alpha_*\beta_*\psi_*\eta_*$$

may be reversed to define a cobordism

$$(G,-\theta); (F,((\bar{z}^*),\theta)G), (F,((\tilde{z}^*),\theta)G), \alpha_*\beta_*\psi_*\eta_*$$

so that the relation is symmetric.

Every $\pm$-formation $(F,((\tilde{z}^*),\theta)G)$ defines a cobordism

$$(G,\theta); (F,((\tilde{z}^*),\theta)G), (F^*,((\bar{z}^*),\theta)G), \alpha_*\beta_*\psi_*\eta_*$$

so that it is sufficient to show transitivity to have reductivity. So let

$$C = (G,\theta); (F,((\bar{z}^*),\theta)G), (F,((\tilde{z}^*),\theta)G), \alpha_*\beta_*\psi_*\eta_*$$

$$C' = (G',\theta); (F,((\bar{z}^*),\theta)G), (F,((\tilde{z}^*),\theta)G), \alpha_*\beta_*\psi_*\eta_*$$

be adjoining cobordisms. We then have a stable isomorphism

$$[\alpha,\beta,\psi] = [\alpha',\beta',\psi'] [\alpha,\beta,\psi,\eta]^-1$$

$$: (F^*,((\tilde{z}^*),\theta)G) \rightarrow (F',((\bar{z}^*),\theta)G)$$

Let

$$(\alpha,\beta,\psi): (F^*,((\tilde{z}^*),\theta)G) \rightarrow (F,((\bar{z}^*),\theta)G)$$

be any representative, with

$$\alpha = (\bar{a}^*\bar{a}); F^* \rightarrow F, \beta = (\bar{b}^*\bar{b}); G \rightarrow G, \psi = (\bar{c}^*\bar{c}); \psi_0 \rightarrow \psi_0$$

so that we have a commutative diagram

$$\begin{array}{ccc}
G \otimes \psi & \xrightarrow{(b_1^*b_1)} & G \otimes \psi_0 \\
\downarrow & & \downarrow \\
(F^* \otimes \psi) \otimes (F \otimes \psi_0) & \xrightarrow{(a_1^*a_1)} & (F \otimes \psi_0) \otimes (F^* \otimes \psi_0)
\end{array}$$

with

$$(b_1^*b_1)(c_1^*c_1)(d_1^*d_1) = (a_1^*a_1) + (c_1^*c_1)(d_1^*d_1)(0_1^*0_1^*) (\in \mathcal{P}(G \otimes \psi_0))$$

Then

$$\begin{array}{ccc}
G \otimes \psi & \xrightarrow{(b_1^*b_1)} & G \otimes \psi_0 \\
\downarrow & & \downarrow \\
(F^* \otimes \psi) \otimes (F \otimes \psi_0) & \xrightarrow{(a_1^*a_1)} & (F \otimes \psi_0) \otimes (F^* \otimes \psi_0)
\end{array}$$

are inverse isomorphisms of $\pm$-forms over A.

Define

$$\xi = (\xi_0, \xi_1^*): FG \rightarrow FG$$

and let

$$\xi' = (\xi_0^*, \xi_1): FG^* \rightarrow FG^*$$
Then
\[
( (\xi, \eta), (\theta, 0) ) : (G \oplus F^*, 0) \to H_+(G \oplus F^*)
\]
and
\[
( (\xi, \eta^*), (\theta^*, 0^*) ) : (F \oplus G, 0) \to H_+(F \oplus G)
\]
are the inclusions of a sublagrangian \((G, 0) \subseteq H_+(F^*)\),
where \(F^* = F \oplus F^*\), because the diagram of \(\pm\)-forms and morphisms
\[
\begin{array}{ccc}
(G \oplus F^*, 0) & \xleftarrow{(\xi, \eta, \theta, 0)} & (G, 0) \\
\downarrow & & \downarrow \\
(F \oplus G, 0) & \xrightarrow{(\xi^*, \eta^*)} & H_+(F^*)
\end{array}
\]

commutes. The isomorphisms of \(\pm\)-formations
\[
(1, 1, (0, 0)) : (F, ((\xi, \eta, 0), (\xi^*, \eta^*))) \to (F \oplus F^*, (\xi, \eta, 0), (\xi^*, \eta^*))
\]
\[
((1, 1), (\xi^*, \eta^*)) : (F \oplus F^*, (\xi^*, \eta^*)) \to (F \oplus F^*, (\xi^*, \eta^*))
\]
define stable isomorphisms
\[
[\alpha_1, \beta_1, \eta_1] : (F, ((\xi, \eta, 0), 0)) \to (F^*, ((\xi^*, \eta^*), 0))
\]

where \(F^* = F \oplus F^*\), because the diagram of \(\pm\)-forms and morphisms
\[
\begin{array}{ccc}
(G \oplus F^*, 0) & \xleftarrow{(\xi, \eta, \theta, 0)} & (G, 0) \\
\downarrow & & \downarrow \\
(F \oplus G, 0) & \xrightarrow{(\xi^*, \eta^*)} & H_+(F^*)
\end{array}
\]

commutes. The isomorphisms of \(\pm\)-formations
\[
(1, 1, (0, 0)) : (F, ((\xi, \eta, 0), (\xi^*, \eta^*))) \to (F \oplus F^*, (\xi, \eta, 0), (\xi^*, \eta^*))
\]
\[
((1, 1), (\xi^*, \eta^*)) : (F \oplus F^*, (\xi^*, \eta^*)) \to (F \oplus F^*, (\xi^*, \eta^*))
\]
define stable isomorphisms
\[
[\alpha_1, \beta_1, \eta_1] : (F, ((\xi, \eta, 0), 0)) \to (F^*, ((\xi^*, \eta^*), 0))
\]

A different choice of representative of the stable isomorphism \([\alpha_1, \beta_1, \eta_1]\), as given by \((\alpha_1, \beta_1, \gamma_1) \oplus 1_{A_1, A_2}\), say, leads to precisely the same \([\alpha_1, \beta_1, \eta_1, \gamma_1] = [\alpha_1, \beta_1, \eta_1] \oplus 1_{A_1, A_2}\). Thus, defining stable isomorphisms
\[
[\alpha_2, \beta_2, \eta_2] = [\alpha_1, \beta_1, \eta_2] : (F, ((\xi, \eta, 0), (\xi^*, \eta^*)) \to (F^*, ((\xi^*, \eta^*), 0))
\]
we are justified in calling the cobordism
\[
C \times C^\prime = (G, 0) : (F, ((\xi, \eta, 0), (\xi^*, \eta^*))) \times (F^*, ((\xi^*, \eta^*), 0))
\]
the join of \(C\) and \(C^\prime\), the only arbitrariness in the definition of a representative \(G\) and being of the choice of \(\theta \in \text{Hom}_A(G_\xi, G_{\xi^*})\) to represent \(B \in \Gamma_G(G_\xi, G_{\xi^*})\).

In particular, we have the transitivity of the relation.

\[\square\]

**Corollary 2.2** The \(\pm\)-formations \((G, ((\xi, \eta), 0) G), (F, ((\xi^*, \eta^*), 0) G)\) are cobordant if
\[
(F, ((\xi, \eta), 0) G) = (F, ((\xi^*, \eta^*), 0) G) \in X_{\text{sing}}(A) / \sim_{\text{rel}(0) / \text{X_{\text{sing}}(A) / X_{\text{sing}}(A))}
\]

The converse holds for non-singular \(\pm\)-formations, so that \(\pm\)-formations \((G, ((\xi, \eta), 0) G), (F, ((\xi^*, \eta^*), 0) G)\) are cobordant iff
\[
(F, ((\xi, \eta), 0) G) = (F, ((\xi^*, \eta^*), 0) G) \in U_{\text{sing}}(A).
\]

**Proof:** A graph \(\pm\)-formation \(\delta(p, \theta)\) is cobordant to \(C\) if
\[
((p, \theta), \delta(p, \theta), 0, 1) \in \text{Hom}_A((\xi, \eta), (\xi^*, \eta^*))
\]

An evidently isomorphic \(\pm\)-formations are cobordant.
and cobordism respects the direct sum operation $\oplus$, it follows from Theorem 2.1 that $\pm$-formations which represent the same element of $X_{\mathbb{A}^n}(A) / \mathbb{A}^n \times X_{\mathbb{A}^n}(A) \rightarrow X_{\mathbb{A}^n}(A)$ are cobordant.

Although it does not seem possible to establish a stable isomorphism of the type

$$[\mathbb{A}^n, \mathbb{B}^n] : (f, (\xi, \theta)^* G) \oplus \mathbb{A}(\mathbb{A}, \mathbb{B}) \rightarrow (f^*, ((\xi)^*, \theta)^* G) \oplus \mathbb{A}(\mathbb{A}, \mathbb{B})$$

for general $\pm$-formations $(f, (\xi, \theta)^* G)$, such may be deduced for non-singular $\pm$-formations from the sum formula of Lemma 3.3 of I.

(Alternatively, note that $(f, (\xi, \theta)^* G) \oplus (f^*, ((\xi)^*, \theta)^* G) \cong (f, G) \oplus (f^*, G)$ by Theorem 2.3 of I.)

An isomorphism of cobordisms

$$(f, (\xi, \theta)) : (G, \theta) \rightarrow (G', \theta')$$

together with an isomorphism $f^* : \text{Hom}(f, F')$ such that the squares

$$\begin{array}{ccc}
G & \rightarrow & G' \\
\downarrow \gamma & & \downarrow \gamma' \\
F & \rightarrow & F'
\end{array} \quad \begin{array}{ccc}
G & \rightarrow & G' \\
\downarrow \mu & & \downarrow \mu' \\
F^* & \leftarrow & F'^*
\end{array}
$$

commute.

The join operation defined in the proof of Theorem 2.1 may be characterized up to isomorphism by the following:

**Theorem 2.3** Let $(G, \theta) \subseteq (G', \theta')$, $(G', \theta') \subseteq (G', \theta')$, be orthogonal subformations such that

$$
(\theta^* \oplus \theta^*) f : G \oplus G' \rightarrow G' \oplus G
$$

is a split mono.

Then $F = G'_{/f} G'$, together with

$$
\gamma : G \rightarrow G'_{/f} \; G' \; x \mapsto [fx]
$$

$$
\mu : G \rightarrow (G'_{/f} \; G')^* \; x \mapsto (\gamma x) \rightarrow (\gamma x, \theta_0)
$$

define a $\pm$-formation $(f, (\xi, \theta)^* G)$, as does $(f^*, ((\xi)^*, \theta)^* G)$, with

$$
\gamma' : G' \rightarrow (G'_{/f} \; G')^* \; x \mapsto (\gamma' x) \rightarrow (\gamma' x, \theta_0)
$$

$$
\mu' : G' \rightarrow G'_{/f} \; G' \; x \mapsto [fx'],
$$

where $F' = (G'_{/f} \; G')^*$. A particular choice of direct complements to $f, f'$ in $G'$ defines a $\pm$-formation $(f \oplus F', ((\xi)^*, \theta)^* G')$, and a stable isomorphism of $\pm$-formations

$$
[\mathbb{A}, \mathbb{B}, \mathbb{C}] : (f^*, ((\xi)^*, \theta)^* G') \rightarrow (f^*, ((\xi)^*, \theta)^* G')
$$
Then there are defined cobordisms of formations
\[ \mathcal{C} = (G, \theta, f, \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right), \theta G, [U], [U]) \]
\[ \mathcal{C}' = (G', \theta, f', \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right), \theta G', [U'], [U']) \]
\[ \mathcal{C}^* = (G^*, \theta^*, f^*, \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right), \theta^* G^*, [U], [U]) \]
\[ \mathcal{C}^{*'} = (G'^*, \theta^*, f'^*, \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right), \theta^* G', [U'], [U']) \]
and an isomorphism of cobordisms
\[ \mathcal{C} \times \mathcal{C}' \rightarrow \mathcal{C}^{*}. \]

A different choice of direct complements leads to isomorphic \( \mathcal{C}, \mathcal{C}', \mathcal{C}^* \).

Proof: A particular choice of direct complements in \( G^* \) to \( fG, f'G' \) corresponds to being given pairs
\[ F \oplus G' \xleftarrow{(h, f')} G^* \]
\[ F^* \oplus G \xleftarrow{(h', f)} G' \]
of inverse isomorphisms, with
\[ e: G^* \rightarrow F = S_{GF}^*, \quad e': G' \rightarrow F^* = S_{GF}' \]
the natural projections. Thus
\[ \gamma = ef : G \rightarrow F \quad \gamma' = h' (\theta^* \oplus \theta) f' : G' \rightarrow F' \]
\[ \mu = h (\theta \oplus \theta^*) f : G \rightarrow F^* \quad \mu' = e' f' : G' \rightarrow F'^* \]

The isomorphism of formations
\[ (\mu, \mu') : (G^*, \left( \begin{array}{cc} (-\theta^* \oplus \theta) f' & 0 \\ 0 & 0 \end{array} \right) \theta^* G^*) \rightarrow (G, \theta, \left( \begin{array}{cc} (f f') & 0 \\ 0 & 0 \end{array} \right) \theta G) \]

can now be expressed as
\[ \left( \begin{array}{cc} h (\theta^* \oplus \theta) f & 0 \\ 0 & 0 \end{array} \right) \theta^* G, \left( \begin{array}{cc} (f f') & 0 \\ 0 & 0 \end{array} \right) \theta G \]

\[ \xrightarrow{(F \oplus G^*), \left( \begin{array}{cc} h (\theta^* \oplus \theta) f & 0 \\ 0 & 0 \end{array} \right) \theta^* G^*} \left( \begin{array}{cc} (-\theta^* \oplus \theta) f' & 0 \\ 0 & 0 \end{array} \right) \theta^* G^* \rightarrow (F \oplus G, \left( \begin{array}{cc} (f f') & 0 \\ 0 & 0 \end{array} \right) \theta G) \]
We therefore have an isomorphism
\[
\left( \alpha', \beta', \gamma' \right) = \left( \left( \begin{array}{cc} k^* e^* & k^* \theta^* \theta' \\ f^* e^* & f^* \theta^* \theta' \end{array} \right), \left( \begin{array}{cc} -g' f & 1 \\ -g g' f & g \end{array} \right), \left( \begin{array}{cc} k' \theta' h & k' \theta' \theta' h' \\ k' \theta' h & k' \theta' \theta' h' \end{array} \right) \right)
\]

\[
: (F^*, \left( \begin{array}{c} k^* e^* \\ e \end{array} \right), \theta) G \otimes (G^*, G') \rightarrow (F', \left( \begin{array}{c} k^* e^* \\ e \end{array} \right), \theta') G' \otimes (G^*, G'),
\]
and so a stable isomorphism
\[
\left[ \alpha', \beta', \gamma' \right] : (F^*, \left( \begin{array}{c} \pm \rho \nu \gamma' \\ \pm \rho \nu \gamma' \end{array} \right), \theta) G \rightarrow (F', \left( \begin{array}{c} \gamma' \\ \mu \end{array} \right), \theta') G',
\]

incidentally verifying that
\[
\left( \begin{array}{c} \gamma' \\ \theta \end{array} \right) : (G, G) \rightarrow H_+(F), \left( \begin{array}{c} \gamma' \\ \theta \end{array} \right) : (G, G) \rightarrow H_+(F')
\]

are indeed inclusions of sublagrangians.

Applying the join construction to the cobordisms
\[
\bar{\alpha} = \left( \left( \begin{array}{c} \gamma' \\ \theta \end{array} \right) \right), \left( \begin{array}{c} (F, ((\gamma') \theta) G), (F^*, ((\gamma', \theta) G'), [1], [1]) \\ (G, \theta) \right)
\]
\[
\bar{\gamma} = \left( \left( \begin{array}{c} \gamma' \\ \theta \end{array} \right) \right), \left( \begin{array}{c} (G', \theta) \end{array} \right) \end{array} \right), (F^*, ((\gamma', \theta) G'), [1], [1])
\]

we obtain (choosing the obvious representatives)

\[
\tilde{\alpha} \times \tilde{\alpha}' = \left( \begin{array}{c} \begin{array}{cc} k^* e^* & k^* \theta^* \theta' \\ f^* e^* & f^* \theta^* \theta' \end{array} \\ \begin{array}{cc} -g' f & 1 \\ -g g' f & g \end{array} \begin{array}{cc} k' \theta' h & k' \theta' \theta' h' \\ k' \theta' h & k' \theta' \theta' h' \end{array} \right)
\]

where
\[
\left( \alpha', \beta', \gamma' \right) = \left( \begin{array}{cc} 1 & \theta \end{array} \\ \begin{array}{cc} k^* e^* & k^* \theta^* \theta' \end{array} \right)
\]

\[
: (F, ((\gamma') \theta) G) \otimes (F', F^*) \rightarrow (F \otimes F', \left( \begin{array}{cc} \gamma & e \end{array} \right), \left( \begin{array}{cc} \mu & 0 \\ 0 & 1 \end{array} \right), \left( \begin{array}{cc} \gamma' & e \end{array} \right), \left( \begin{array}{cc} k' \theta' h & k' \theta' \theta' h' \end{array} \right))
\]

\[
\left( \alpha_2^*, \beta_2^*, \gamma_2^* \right) = \left( \begin{array}{cc} 1 & \theta \end{array} \\ \begin{array}{cc} k^* e^* & k^* \theta^* \theta' \end{array} \right)
\]

\[
: (F^*, F) \otimes (F^*, ((\gamma', \theta) G') \rightarrow (F \otimes F^*, \left( \begin{array}{cc} \gamma & e \end{array} \right), \left( \begin{array}{cc} \mu & 0 \\ 0 & 1 \end{array} \right), \left( \begin{array}{cc} \gamma' & e \end{array} \right), \left( \begin{array}{cc} k' \theta' h & k' \theta' \theta' h' \end{array} \right))
\]

Now
\[
((f W), (\begin{array}{c} \chi & f \theta^* \theta' \end{array})) : (G, \theta) \rightarrow (G, \theta)
\]

defines an isomorphism of \( F \) forms, so that setting
\[
\tilde{\gamma} = \left( \begin{array}{cc} \gamma & e \end{array} \right), \left( \begin{array}{cc} \mu & 0 \\ 0 & 1 \end{array} \right) : G \rightarrow F \otimes F' = F^*
\]
\[
\tilde{\gamma} = \left( \begin{array}{cc} \gamma & e \end{array} \right), \left( \begin{array}{cc} \mu & 0 \\ 0 & 1 \end{array} \right) : G \rightarrow F \otimes F^* = F^*
\]

we have a cobordism of \( \pm \) forms.
\[ c'' = (G', \theta') ; (F', ((\tilde{\beta}'_{\gamma}), \theta')G') \]

isomorphic to \( c'\).

Change of direct complements to \( fG, fG' \) in \( G'' \) affects neither

\[ c = ((G, \theta) ; (F, ((\beta), \theta)G), (F^* ; ((\theta'), \theta)G), [C] , [\iota]) \]

\[ (G, \theta') ; (F', ((\tilde{\beta}'_{\gamma}), \theta')G'), (F'^* ; ((\theta')_{\gamma}), \theta)G), [\iota], [\iota]) \]

an isomorphism of \( c' \).

Change of the direct complement to \( fG' \) in \( G'' \) does not affect

\[ (\tilde{\beta}'_{\gamma}, \theta') : (G'', \theta) \rightarrow H_4(\mathbb{F}), \]

giving the same \( c'' \).

A different choice of direct complement to \( fG \) in \( G'' \)
replaces \( (\kappa', f') : \mathbb{F}^* \otimes G \rightarrow G'' \) by

\[ (\tilde{\kappa}'_{\gamma}, f') = (k' \cdot f)(\begin{pmatrix} 0 & 1 \\ \sigma & 1 \end{pmatrix}) = (k' + f \cdot \mu \cdot f') : \mathbb{F}^* \otimes G \rightarrow G'', \]

for some \( \sigma \in \text{Hom}_4(\mathbb{F}^*, \mathbb{G}) \), and \( (e')_{\gamma} : G'' \rightarrow \mathbb{F}^* \otimes \mathbb{G} \) by

\[ (e')_{\gamma} = \begin{pmatrix} 1 & 0 \\ -\sigma & 1 \end{pmatrix} \cdot (e')_{\gamma} = (e')_{\gamma} : G'' \rightarrow \mathbb{F}^* \otimes \mathbb{G}. \]

This gives

\[ \left( \begin{pmatrix} e \\ 0 \end{pmatrix} \right)_{\gamma} = \left( \begin{pmatrix} e \\ \beta^* \cdot \mu^* \cdot \gamma \end{pmatrix} \right) \]

\[ : G'' \rightarrow \mathbb{F}' \oplus \mathbb{F}^* \]

in place of \( \left( \tilde{\beta}'_{\gamma} \right) : G'' \rightarrow \mathbb{F}' \oplus \mathbb{F}^* \).

An isomorphism of bordisms

\[ \tilde{c}'' = ((G'', \theta'') ; (F', ((\tilde{\beta}'_{\gamma}), \theta')G'), (F'^* ; ((\theta')_{\gamma}), \theta)G'), [C] , [\iota]) \]

\[ \rightarrow \tilde{c}'' = ((G'', \theta'') ; (F'', ((\tilde{\beta}'_{\gamma}), \theta')G'), (F''^* ; ((\theta')_{\gamma}), \theta)G'), [C] , [\iota]) \]

is defined by the identity on \( (G'', \theta'') \) together with the automorphism

\[ \begin{pmatrix} 1 & 0 \\ -\sigma \cdot \mu^* & 1 \end{pmatrix} : \mathbb{F} \oplus \mathbb{F}' \rightarrow \mathbb{F} \oplus \mathbb{F}' \]

on \( \mathbb{F}' \).

\[ \square \]
Conversely, we have

**Theorem 2.4** Adjoining cobordisms

\[
\begin{align*}
C &= (C, \Theta) ; (G, \Theta) \ ; (F, (\ell^2, b^2) G) , (F, (\ell^2, b^2) G) , \ [a, b, c, d], \ [a, b, c, d] \] \\
C' &= (C', \Theta') ; (F, (\ell^2, b^2) G) , (F, (\ell^2, b^2) G) , \ [a, b, c, d], \ [a, b, c, d] \\
\end{align*}
\]

may be replaced by isomorphic ones

\[
\begin{align*}
\tilde{E} &= (C, \Theta) ; (F, \ell^2, \Theta) , (F^* , (\ell^2, b^2) G) , \ [a, b, c, d], \ [a, b, c, d] \\
\tilde{E}' &= (C', \Theta') ; (F^* , (\ell^2, b^2) G) , (F^* , (\ell^2, b^2) G) , \ [a, b, c, d], \ [a, b, c, d] \\
\end{align*}
\]

where

\[
\begin{align*}
[a, b, c, d] &= [a, b, c, d] \ [a, b, c, d] : (F^* , (\ell^2, b^2) G) \rightarrow (F^* , (\ell^2, b^2) G) \\
\end{align*}
\]

which together with an isomorphism \( \phi \) form \( C \times C' \)

\[
\begin{align*}
\tilde{E}^* &= (C', \Theta') ; (F, \ell^2, \Theta) , (F^* , (\ell^2, b^2) G) , \ [a, b, c, d], \ [a, b, c, d] \\
\tilde{E}'^* &= (C, \Theta) ; (F^* , (\ell^2, b^2) G) , (F^* , (\ell^2, b^2) G) , \ [a, b, c, d], \ [a, b, c, d] \\
\end{align*}
\]

fit into the scheme of Theorem 2.3, according to the subform inclusions

\[
\begin{align*}
((0), 0) : (G, \Theta) \rightarrow (G \oplus F^*, (0, \Theta)) \\
((0), 0) : (G', \Theta') \rightarrow (F \oplus G', (0, \Theta)) \\
\end{align*}
\]

in the isomorphs of \( (C, \Theta) \) occurring in the definition of \( C \times C' \).

**Proof**: For definiteness, set

\[
(G^*, \Theta^*) = (G \oplus F^*, (0, \Theta)) \\
\]

Then

\[
(f, \phi) = ((0), 0) : (G, \Theta) \rightarrow (G \oplus F^*, (0, \Theta)) \\
(f', \phi') = ((0), 0) : (G', \Theta') \rightarrow (G \oplus F^*, (0, \Theta)) \\
\]

are the inclusions of orthogonal subforms, for some \( \chi \in \Pi_2(G) \), because

\[
\begin{align*}
\tilde{f}^* (\phi \oplus \phi^*) \tilde{f} &= -\tilde{b}^* (\phi \oplus \phi^*) + \mu^* a \mu \\
\end{align*}
\]

employing the standard notation of the definition of \( C \times C' \).

The isomorphisms

\[
\begin{align*}
G \oplus F^* & \leftrightarrow F \oplus G' \\
\end{align*}
\]

can now be used to define inverse pairs of isomorphisms

\[
\begin{align*}
((\ell^2, b^2) G) & \leftrightarrow (G \oplus F^*) \\
F \oplus G' & \leftrightarrow G' \oplus F \\
\end{align*}
\]

such that

\[
\begin{align*}
\gamma &= e_f^* : G \rightarrow F \\
\gamma' &= e_f^* : G' \rightarrow F' \\
\mu &= e_f^* : G \rightarrow F^* \\
\mu' &= e_f^* : G' \rightarrow F^* \\
\end{align*}
\]

exactly as in the proof of Theorem 2.3.
Now
\[(a^* \circ b^*), (\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}), 0) : (F^*, ((4^t), -\theta)G) \oplus (G^*, G)\]

\[\rightarrow (G^*, \left((-\theta^t \circ \theta^t) f \circ 0\right), \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix})G \oplus G^*\]

defines an isomorphism of $\pm$ formations, so that
\[\begin{pmatrix} (\theta^t \circ \theta^t) f & 0 \\ f & f' \end{pmatrix} : G \oplus G \rightarrow G^* \oplus G^*\]
is a split mono.

The join of the cobordisms
\[\hat{C} = (G; \theta); (F, ((\chi), \theta)G), (F^*, ((\text{Id}), -\theta)G), [\alpha], [\ell] \]
\[\hat{C}' = (G^*; F^*, ((\text{Id}), \theta)G), (F^*, ((\text{Id}), -\theta)G), [\alpha], [\ell] \]
is given by Theorem 2.3 to be isomorphic to
\[\hat{C}'' = (G^*, \theta); (F^*, ((\text{Id}), \theta)G), (F^*, ((\text{Id}), -\theta)G), [\alpha], [\ell] \]
where $F^* = F \oplus F'$ and
\[\hat{\gamma}^* = \begin{pmatrix} \gamma^* & \alpha^* \\ \alpha & \gamma \end{pmatrix} : G \oplus F^* \rightarrow F \oplus F'\]
\[\hat{\mu}^* = \begin{pmatrix} \mu & 0 \\ 0 & 1 \end{pmatrix} : G \oplus F^* \rightarrow F \oplus F^*\]

Now $\hat{\gamma}^* = \xi \hat{\mu}^* \hat{\eta}$ (in the notation of Theorem 2.1), so that we do recover an isomorphism of the join $\hat{C} \oplus \hat{C}'$.

\textbf{Lemma 2.5} If $(F_i, ((\chi_i), \theta)G_i)$ is a trivial $\pm$ formation, the join of cobordisms
\[C = (G; \theta); (F_0, ((\chi_0), \theta)G_0), (F_1, ((\chi_1), \theta)G_1), [\alpha], [\ell] \]
\[C' = (G^*; F_0, ((\text{Id}), \theta)G_0), (F_1, ((\text{Id}), \theta)G_1), [\alpha], [\ell] \]
is isomorphic to their direct sum, $C \oplus C'$.

\textbf{Proof:} By Theorem 2.4 we have a situation as in Theorem 2.3, with orthogonal subform inclusions
\[\hat{\gamma} : G \rightarrow G \oplus G', \quad \hat{\alpha} : G \rightarrow G \oplus G'\]
such that $(F_i, ((\chi_i), \theta)G_i)$ is stably isomorphic to $(G \oplus G')^*, ((\text{Id}), \theta)G_i)$ (as defined in Theorem 2.3). Now $(F_i, ((\chi_i), \theta)G_i)$ is trivial, so that $\mu_i$, and hence
\[\gamma : G \rightarrow G \oplus G', \quad \alpha : 1 \rightarrow [f \alpha],\]
is an $A$-module isomorphism, allowing the identification $(G \oplus G') = \mathcal{G} \oplus \mathcal{G}$.

We can therefore apply Theorem 2.3 with the configuration
\[(h f')(\hat{\gamma} \hat{\alpha} \hat{\gamma}^*) = (\hat{\alpha} \hat{\gamma} \hat{\alpha}^*)(h f')(\hat{\gamma} \hat{\alpha} \hat{\gamma}^*)\]
\[G \oplus G' \Rightarrow \Rightarrow \Rightarrow \Rightarrow G \oplus G' \Rightarrow \Rightarrow \Rightarrow \Rightarrow G \oplus G'\]
\[(\hat{\alpha}^* \hat{\gamma}^*) = (\hat{\alpha}^* \hat{\gamma}^*)\]

\[\hat{\gamma}^* = (\hat{\gamma}^*)\]

\[\hat{\alpha}^* = (\hat{\alpha}^*)\]
This gives an isomorph of \( C \times C' \)
\[
(\xi, \xi') : (G, G') \rightarrow (K_0^+(G), K_0^+(G'))
\]
\[
\xi : \left( \begin{array}{c|c}
G & \xi \\
\hline
\xi & G'
\end{array} \right)
\]

\[
\xi : \left( \begin{array}{c|c}
\xi & G
\\
\hline
G & \xi'
\end{array} \right)
\]

which is in turn isomorphic to \( C \times C' \).

In connection with the later applications of Theorem 2.4, it should be pointed out that the isomorphism of \( \pm \)-formations
\[
(\alpha, \beta, \psi) = \left( \left( \begin{array}{c|c}
\alpha & 0 \\
\hline
0 & \alpha'
\end{array} \right), \left( \begin{array}{c|c}
\beta & 0 \\
\hline
0 & \beta'
\end{array} \right) \right)
\]

\[\psi : (F, (\pm \vartheta)) \rightarrow (F', ((\pm \vartheta'))\]

used in the definition of \( C \times C' \) is replaced, on passing to the isomorphs \( \xi, \xi' \) of \( C, C' \) by the isomorphism
\[
(\xi, \xi') = \left( \begin{array}{c|c}
\alpha & a \xi \\
\hline
\beta & b
\end{array} \right), \left( \begin{array}{c|c}
\xi' & a' \xi' \\
\hline
\xi' & b'
\end{array} \right), \left( \begin{array}{c|c}
\xi & a \xi' \\
\hline
\xi & b'
\end{array} \right)
\]

as given by Theorem 2.3. These isomorphisms give precisely the same join but, rather inconsequently, may not in general be representatives of the same stable isomorphism

\[\psi : (F, (\pm \vartheta)) \rightarrow (F', ((\pm \vartheta'))\]

\[\psi : (F, (\pm \vartheta)) \rightarrow (F', ((\pm \vartheta'))\]

Given a stable isomorphism of \( \pm \)-formations
\[\psi : (F, (\pm \vartheta)) \rightarrow (F', (\pm \vartheta'))\]

call the isomorphism of \( \pm \)-formations \( (\xi, \xi') \) (defined above) the normalization of \( [\alpha, \beta, \psi] \). Stable isomorphisms
\[\psi, \psi' : (F, (\pm \vartheta)) \rightarrow (F', (\pm \vartheta'))\]

are coherent if their normalizations coincide.

Suppose given stable isomorphisms
\[\psi, \psi' : (F, (\pm \vartheta)) \rightarrow (F', (\pm \vartheta'))\]

\[\psi, \psi' : (F, (\pm \vartheta)) \rightarrow (F', (\pm \vartheta'))\]
with representatives

\[(\alpha, \beta, \psi) = \left(\begin{array}{c}
\left(\begin{array}{cc}
\alpha_2 & \alpha_3 \\
0 & \beta_3
\end{array}\right), \left(\begin{array}{cc}
\beta_2 & \beta_3 \\
0 & \alpha_3
\end{array}\right), \left(\begin{array}{cc}
\alpha_1 & \alpha_2 \\
0 & \beta_2
\end{array}\right)\right) \right)

\]

: \( (F, ((\hat{\alpha}, \hat{\beta}), \hat{\psi}) \otimes (P, P^*) ) \rightarrow (F, ((\hat{\alpha}, \hat{\beta}), \hat{\psi}) \otimes (P, P^*) ) \)

\[(\theta, \hat{\psi}) = \left(\begin{array}{c}
\left(\begin{array}{cc}
\hat{\alpha} & \hat{\alpha}_4 \\
\hat{\alpha}_3 & \hat{\alpha}_2
\end{array}\right), \left(\begin{array}{cc}
\hat{\beta} & \hat{\beta}_4 \\
\hat{\beta}_3 & \hat{\beta}_2
\end{array}\right), \left(\begin{array}{cc}
\hat{\psi} & \hat{\psi}_4 \\
\hat{\psi}_3 & \hat{\psi}_2
\end{array}\right)\right) \right)

: \( (F, ((\hat{\alpha}, \hat{\beta}), \hat{\psi}) \otimes (P, P^*) ) \rightarrow (F, ((\hat{\alpha}, \hat{\beta}), \hat{\psi}) \otimes (P, P^*) ) \)

The composite stable isomorphism

\[ [\hat{\alpha}, \hat{\beta}, \hat{\psi}] [\alpha, \beta, \psi] : (F, ((\hat{\alpha}, \hat{\beta}), \hat{\psi}) \otimes (P, P^*) ) \rightarrow (F, ((\alpha, \beta), \psi) \otimes (P, P^*) ) \]

has representative isomorphism

\[\left(\begin{array}{c}
\left(\begin{array}{cc}
\alpha_1 & \alpha_2 \\
0 & 1
\end{array}\right), \left(\begin{array}{cc}
\beta_1 & \beta_2 \\
0 & 1
\end{array}\right), \left(\begin{array}{cc}
\alpha_3 & \alpha_4 \\
0 & 1
\end{array}\right)\right) \right) \]

: \( (F, ((\hat{\alpha}, \hat{\beta}), \psi) \otimes (P, P^*) ) \rightarrow (F, ((\alpha, \beta), \psi) \otimes (P, P^*) ) \)

with normalization

\[\left(\begin{array}{c}
\left(\begin{array}{cc}
\hat{\alpha}_1 & \hat{\alpha}_2 \\
\hat{\alpha}_3 & \hat{\alpha}_4
\end{array}\right), \left(\begin{array}{cc}
\hat{\beta}_1 & \hat{\beta}_2 \\
\hat{\beta}_3 & \hat{\beta}_4
\end{array}\right), \left(\begin{array}{cc}
\hat{\psi}_1 & \hat{\psi}_2 \\
\hat{\psi}_3 & \hat{\psi}_4
\end{array}\right)\right) \right) \]

: \( (F, ((\hat{\alpha}, \hat{\beta}), \psi) \otimes (P, P^*) ) \rightarrow (F, ((\alpha, \beta), \psi) \otimes (P, P^*) ) \)

The composite of the normalizations of

\[ [\alpha, \beta, \psi], [\hat{\alpha}, \hat{\beta}, \hat{\psi}] (stabilized in the obvious way), \]

\[\left(\begin{array}{c}
\left(\begin{array}{cc}
\alpha & \alpha_1 \\
0 & 1
\end{array}\right), \left(\begin{array}{cc}
\beta_1 & \beta_2 \\
0 & 1
\end{array}\right), \left(\begin{array}{cc}
\alpha_3 & \alpha_4 \\
0 & 1
\end{array}\right)\right) \right) \]

: \( (F, ((\alpha, \beta), \psi) \otimes (P, P^*) ) \rightarrow (F, ((\alpha, \beta), \psi) \otimes (P, P^*) ) \)

: \( \left(\begin{array}{c}
\left(\begin{array}{cc}
\alpha & \alpha_1 \\
0 & 1
\end{array}\right), \left(\begin{array}{cc}
\beta_1 & \beta_2 \\
0 & 1
\end{array}\right), \left(\begin{array}{cc}
\alpha_3 & \alpha_4 \\
0 & 1
\end{array}\right)\right) \right) \]

: \( (F, ((\alpha, \beta), \psi) \otimes (P, P^*) ) \rightarrow (F, ((\alpha, \beta), \psi) \otimes (P, P^*) ) \)
has precisely the same normalization.

Therefore the coherence classes of stable isomorphisms are the morphisms of a category with ± formations as objects.

§3 Lattices and n-ads

A lattice of ± forms, $\Sigma(Q_*, \phi_*)_{\iota \in I}$, is a finite collection of ± forms $(Q_\iota, \phi_\iota)$, together with inclusions $(f^\iota_j, \chi^\iota_j) : (Q_\iota, \phi_\iota) \rightarrow (Q_\j, \phi_\j)$ ($\iota, \j \in I$) such that the triangles

\[
\begin{array}{c}
(Q_\iota, \phi_\iota) \\
\downarrow (f^\iota_j, \chi^\iota_j) \\
(Q_\j, \phi_\j)
\end{array} \quad \begin{array}{c}
(f^\j_k, \chi^\j_k) \\
\downarrow (f^\iota_k, \chi^\iota_k) \\
(Q_\iota, \phi_\iota)
\end{array}
\]

commute, with

\[
(f^\iota_j, \chi^\iota_j) = (1,0) : (Q_\iota, \phi_\iota) \rightarrow (Q_\j, \phi_\j),
\]

and such that for disjoint $\iota, \j \in I$ the subforms

\[
(f^\iota_k, \chi^\iota_k) : (Q_\iota, \phi_\iota) \rightarrow (Q_{\iota k}, \phi_{\iota k})
\]

\[
(f^\j_k, \chi^\j_k) : (Q_\j, \phi_\j) \rightarrow (Q_{\j k}, \phi_{\j k})
\]

are orthogonal. By convention, $(Q_\iota, \phi_\iota) = 0$. 

An n-ad of ±forms, $\mathcal{E}(\mathcal{F}, \langle \mathcal{F}, \mathcal{G} \rangle), \mathcal{F}(\mathcal{A}, \mathcal{B}) \cup \mathcal{J} \mathcal{I} \mathcal{E} \mathcal{I}$, is a ±formation $\langle \mathcal{F}, \langle \mathcal{F}, \mathcal{G} \rangle \rangle$ together with a lattice of ±forms $\mathcal{E}(\mathcal{F}, \mathcal{A} \cup \mathcal{I} \mathcal{E} \mathcal{I})$, where $|\mathcal{J}| = n + 1$, and with an isomorphism of ±forms

$$(\mathcal{F}, \varphi): (\mathcal{A}, \mathcal{B}) \to \mathcal{E}(\mathcal{F}, \langle \mathcal{F}, \mathcal{G} \rangle),$$

and such that the subforms

$$(\mathcal{A}, \mathcal{B}, \mathcal{C}) \subseteq (\mathcal{A}, \mathcal{B}), (\mathcal{A}, \mathcal{B}, \mathcal{C}) \subseteq (\mathcal{A}, \mathcal{B})$$

are maximally orthogonal for all $\mathcal{J} \subseteq \mathcal{I}$. For example,

An isomorphism of n-ads of ±forms

$$[\omega, \psi]: \{ (\mathcal{A}, \mathcal{B}, \mathcal{C}) \} \to \{ (\mathcal{A}, \mathcal{B}) \}$$

is defined by a stable isomorphism of ±formations

$$(\mathcal{F}, \mathcal{A} \cup \mathcal{I} \mathcal{E} \mathcal{I}) \to (\mathcal{F}, \langle \mathcal{F}, \mathcal{G} \rangle)$$

together with isomorphisms of ±forms

$$h_{\mathcal{J}}: (\mathcal{A}, \mathcal{B}) \to (\mathcal{A}, \mathcal{B})$$

such that the squares

$$\begin{array}{ccc}
(\mathcal{A}, \mathcal{B}) & \to & (\mathcal{A}, \mathcal{B}) \\
\downarrow h_{\mathcal{J}} & & \downarrow h_{\mathcal{J}} \\
(\mathcal{A}, \mathcal{B}) & \to & (\mathcal{A}, \mathcal{B})
\end{array}$$

are commutative.

A lattice of ±formations $\mathcal{E}(\mathcal{F}, \langle \mathcal{F}, \mathcal{G} \rangle)$ is a finite collection of ±formations $\langle \mathcal{F}, \mathcal{G} \rangle$, together with inclusions

$$(\mathcal{F}, \mathcal{G}) \subseteq (\mathcal{F}, \mathcal{G})$$

such that

$$\lambda_{\mathcal{J}} \lambda_{\mathcal{K}} = \lambda_{\mathcal{J} \mathcal{K}} \in \text{Hom}_{\mathcal{A}}(\mathcal{F}, \mathcal{F})$$

and such that for disjoint $\mathcal{J}, \mathcal{K} \subseteq \mathcal{I}$ there are given orthogonality relations

$$(\mathcal{F}, \mathcal{G}) \subseteq (\mathcal{F}, \mathcal{G})$$

with

$$\lambda_{\mathcal{J}}^* = \lambda_{\mathcal{K}}^* \in \text{Hom}_{\mathcal{A}}(\mathcal{F}, \mathcal{F})$$

By convention,

$$\mathcal{F}^* = \mathcal{G}^* = \mathcal{F}^* = \mathcal{G}^*$$

with

$$\lambda_{\mathcal{J}} = \lambda_{\mathcal{K}} : \mathcal{F} \to \mathcal{G}$$

and

$$\lambda_{\mathcal{J}}^* = \lambda_{\mathcal{K}}^* : \mathcal{F} \to \mathcal{G}$$.
An n-ad of \(\pm\) formations \(\{(\mathbf{G},\mathbf{\theta}) ; \{f_{x} , ((\mathbf{G}_{x},\mathbf{\theta}_{x})\}_{x \in X}\} \) is a \(\mathfrak{F}_{\pm}\)m \((\mathbf{G},\mathbf{\theta})\) together with a lattice of \(\pm\) formations \(\{(\mathbf{F}_{x} , ((\mathbf{G}_{x},\mathbf{\theta}_{x})\}_{x \in X}\) such that \((\mathbf{F}_{x} , ((\mathbf{G}_{x},\mathbf{\theta}_{x})\}_{x \in X}\) are maximally orthogonal, that is
\[
\mu_{x}^{\pm} \in \text{Hom}_{H}(\mathbf{F}_{x}^{\pm}, \mathbf{F}_{x})
\]
is an isomorphism, for all \(J \subseteq I\). (In particular, it follows that
\[
(\mu_{x}^{\pm} , 1_{0}) : (\mathbf{F}_{x} , ((\mathbf{G}_{x},\mathbf{\theta}_{x})\}_{x \in X}\) \longrightarrow \mathfrak{J}(\mathbf{G},\mathbf{\theta})
defines an isomorphism of \(\pm\) formations.

For example, a 0-ad is (essentially) a \(\pm\) formation, while a 1-ad is a cobordism of such.

An isomorphism of n-ads of \(\pm\) formations
\[
\{(f_{x} , \mathbf{\theta}_{x}) ; \mathbf{\theta}_{x} \}_{x \in X}\}
: \{(\mathbf{G},\mathbf{\theta}) ; \{f_{x} , ((\mathbf{G}_{x},\mathbf{\theta}_{x})\}_{x \in X}\} \longrightarrow \mathfrak{J}(\mathbf{G}^{\pm},\mathbf{\theta})\}
\]
is an isomorphism of \(\pm\) forms
\[
(f_{x} , \mathbf{\theta}_{x}) : (\mathbf{G},\mathbf{\theta}) \longrightarrow (\mathbf{G}^{\pm},\mathbf{\theta}^{\pm})
\]
together with isomorphisms \(h_{x} \in \text{Hom}_{H}(\mathbf{F}_{x}^{\pm}, \mathbf{F}_{x}^{\pm})\) \((J \subseteq I)\) such that
\[
(h_{x} , f , \mathbf{G}) : (\mathbf{F}_{x} , ((\mathbf{G}_{x},\mathbf{\theta}_{x})\}_{x \in X}\) \longrightarrow (\mathbf{F}_{x}^{\pm}, ((\mathbf{G}_{x},\mathbf{\theta}_{x})\}_{x \in X}\)
defines an isomorphism of \(\pm\) formations for all \(J \subseteq I\) and the squares
\[
\begin{array}{ccc}
\mathbf{F}_{x}^{\pm} & \xrightarrow{h_{x}^{J}} & \mathbf{F}_{x}^{\pm} \\
\downarrow & & \downarrow \\
\mathbf{F}_{x} & \xrightarrow{h_{x}} & \mathbf{F}_{x}
\end{array}
\]
\[
\begin{array}{ccc}
\mathbf{F}_{x}^{\pm} & \xrightarrow{h_{x}^{J}} & \mathbf{F}_{x}^{\pm} \\
\downarrow & & \downarrow \\
\mathbf{F}_{x} & \xrightarrow{h_{x}} & \mathbf{F}_{x}
\end{array}
\]

commute, with
\[
h_{x} = f_{x}^{J^{-1}} : \mathbf{G}^{\pm} \longrightarrow \mathbf{G}^{\pm}
\]

We shall now define \(\text{facette operations}\), assigning to an n-ad \(y = \{x , \{x_{x}\}_{x \in X}\}\) of \(\pm\) forms satisfying some extra conditions one m-ad of \(\pm\) forms
\[
\mathcal{J}^{J} y = \{x_{x} , \{x_{x , x}\}_{x \in X , J}\}
\]
for each subset \(J\) of \(I\), where \(m = |I| - |J| - 1\).
Lemma 3.1 Given an $n$-ad of $\pm$-forms

\[ y = f(F, ((\xi, 0)), G) ; \{(Q, \phi)\}_I \]  

there is defined a morphism of $\pm$-forms

\[ ((\Omega^{\mu}x, \gamma_{\mu}), \phi_{\mu}) : (Q, 0) \longrightarrow H_{\pm}(F_{\mu}x) \]

for disjoint $J \subseteq I$ by

\[ \gamma_{J, K} : (Q, 0) \longrightarrow F_{J, K} = (Q_{\nu} \otimes \nu_{J, K}, Q) ; x \longmapsto (\xi) \longmapsto \langle \xi_{\nu} \otimes \nu_{J, K}, \gamma_{\nu} \rangle \]

\[ \mu_{J, K} : (Q, 0) \longrightarrow F_{J, K}^* ; x \longmapsto [f_{J, K} \otimes \nu_{J, K}] \]

which is the inclusion of a subalgebra of $\Omega^k_{J, K}$

\[ \left( \begin{array}{ccc}
\mathcal{F}^J_{J, K} & \mathcal{F}^J_{J, K} & 0 \\
0 & \mathcal{F}^J_{J, K} & \mathcal{F}^J_{J, K} \\
\mathcal{F}^J_{J, K} & 0 & \mathcal{F}^J_{J, K} \\
\end{array} \right) : Q_{J, K} \otimes \nu_{J, K} \otimes \nu_{J, K} \longrightarrow Q_{J, K} \otimes \nu_{J, K} \otimes \nu_{J, K} \]

is a split mono, where $L = I - (J \cup K)$.

If that is the case for some $J \subseteq I$, and all $K \subseteq I - J$, then there is defined an $m$-ad of $\pm$-forms

\[ \nu^J y = \{ (Q, \phi) ; \{(F, ((\Omega^{\mu}x, \gamma_{\mu}), \phi_{\mu}))_{J, K} \}_{K \subseteq I - J} \]  

with

\[ \lambda_{J, K} : F_{J, K} ; x \longmapsto [f_{J, K} \otimes \nu_{J, K}]^* x \quad (K \subseteq K' \subseteq I - J) \]

\[ \lambda_{J, L} : F_{J, L} \otimes \nu_{J, L} \longrightarrow \nu^J y ; \xi \longmapsto (\xi) \longmapsto \langle \xi_{\nu} \otimes \nu_{J, L}, \gamma_{\nu} \rangle \]

\[ (J, L \subseteq I \text{ disjoint}) \]

An isomorphism of $\pm$-form $n$-ads

\[ h : (\xi_1) \longrightarrow (\xi_2) \]

induces an isomorphism of $\pm$-formation $m$-ads

\[ \delta^J h : \delta^J (\xi) \longrightarrow \delta^J (\xi') \]

whenever $\delta^J (\xi)$ (and hence $\delta^J (\xi')$) is defined.

Proof: The construction of

\[ ((\Omega^{\mu}x, \gamma_{\mu}), \phi_{\mu}) : (Q, 0) \longrightarrow H_{\pm}(F_{\mu}x) \]

is exactly as in Theorem 2.3, and it was shown there that this defines an inclusion of

\[ \left( \begin{array}{ccc}
\mathcal{F}^J_{J, K} & \mathcal{F}^J_{J, K} & 0 \\
0 & \mathcal{F}^J_{J, K} & \mathcal{F}^J_{J, K} \\
\mathcal{F}^J_{J, K} & 0 & \mathcal{F}^J_{J, K} \\
\end{array} \right) : Q_{J, K} \otimes \nu_{J, K} \otimes \nu_{J, K} \longrightarrow Q_{J, K} \otimes \nu_{J, K} \otimes \nu_{J, K} \]

is a split mono.

Now, setting $L = I - (J \cup K)$, we have that

\[ \gamma_{J, L} : \frac{Q_{J, L}}{Q_{K, L}} \longrightarrow Q_{J, K}^* ; \xi \longmapsto (\xi) \longmapsto \langle \xi_{\nu} \otimes \nu_{J, L}, \gamma_{\nu} \rangle \]

is an $A$-module isomorphism (because, by definition,

\[ (Q_{J, K}, \phi_{J, K}) \subseteq (Q_{J, L}, \phi_{J, L}) \quad (Q_{L, K} \subseteq (Q_{L, K}, \phi_{L, K})) \]

are maximally orthogonal), and we have

a commutative triangle
If the condition is satisfied for some \( J \leq I \) and all \( K \leq I - J \), then
\[
\bar{y}_I = \{ (F, \phi_I) ; \xi (F, (\xi, \psi_j), \phi_I) Q_{I_1, k} \in \mathcal{K}_{I_0} - J \}
\]
defines a lattice of formations, provided only that
\[
\nu_{I_1, k}^J : F_{I_1, k} \rightarrow F_{I_1, k}
\]
is an \( A \)-module isomorphism for complementary \( J, K \leq I \).

But this is clear from the commutative diagram
\[
0 \rightarrow Q_L \xrightarrow{f_{I_1}^L} Q_{I_1, L} \xrightarrow{f_{I_1}^{L, J}} Q_{I_1, L, k} \rightarrow 0
\]
\[
\gamma_{L, k}^{I_1} \downarrow \quad \gamma_{L, k}^{I_1, J} \downarrow \quad \gamma_{I_1, k}^{I_1, J}
\]
\[
0 \rightarrow (Q_L^* f_{I_1}^L)^* \rightarrow (Q_{I_1, L}^* f_{I_1}^{L, J})^* \rightarrow (Q_{I_1, L, k}^* f_{I_1}^{L, J})^* \rightarrow 0
\]
in which the rows are exact and the other two vertical maps are isomorphisms.

An isomorphism of \( \mathcal{F} \)-form \( n \)-ads
\[
h = \{ [\alpha, \beta, \gamma], \xi (h_I, h_J) \} \}
\]

: \( y = \{ (F, (\xi, \phi_I), \phi_J) ; \xi (Q_I, \phi_I), Q_J \} \rightarrow \mathcal{K}_{I_0} - J \}
\]

\[
y = \{ (F, (\xi, \phi_J), \phi_I) ; \xi (Q_I, \phi_J) \} \}
\]

\( \alpha \) \( \mathcal{F} \)-form \( n \)-ads \( \mathcal{F}_J \) are defined, for each \( J \leq I \).
induces isomorphisms of ±-form n-ads

\[ \sigma^T h = \{ (h_\sigma, \nu_T), \sigma f_{j,k} \} \]

\[ \sigma^T y = \{ (\phi_T, \nu_T), \{ (\sigma_{j,k}, \phi_j)Q_j \}^{\exists_k \subseteq I-J} \} \]

\[ \sigma^T y' = \{ (\phi_T, \nu_T), \{ (\sigma_{j,k}, \phi_j)Q_j \}^{\exists_k \subseteq I-J} \} \]

with \( f_{j,k} \in \text{Hom}_A(F_{j,k}, F_{j,k}) \) the A-module isomorphism dual to

\[ Q_{juk}/f_{juk} \rightarrow Q_{juk}/f_{juk} \]

\[ [x] \mapsto [h_{juk}] \]

Lemma 3.2 Given an n-ad of ±-formations

\[ y = \{ (\phi_T, \nu_T), \{ (\sigma_{j,k}, \phi_j)G \}^{\exists_k \subseteq I-J} \} \]

such that the morphisms

\[ (\lambda_T^{juk} - \lambda_K^{juk}): F_T \rightarrow F_k \]

are onto for all \( K \subseteq I-J \), for some \( J \subseteq I \), there is defined an m-ad of ±-forms

\[ \sigma^T y = \{ (\phi_T, (\nu_T, \phi_j), \{ (\sigma_{j,k}, \phi_j)G \}^{\exists_k \subseteq I-J} \} \]

by

\[ \text{ker}(\lambda_T^{juk} - \lambda_K^{juk}): F_T \rightarrow F_k \]

\[ \text{im}(\lambda_T^{juk} - \lambda_K^{juk}): F_T \rightarrow F_k \]

with \( \sigma_{j,k} \in \text{Hom}_A(G_{j,k}, G_{j,k}) \) induced by

\[ \left( \begin{array}{c} 0 \\ \nu_T^{juk} \end{array} \right) \in \text{Hom}_A(F_T \oplus F_k, F_T \oplus F_k) \]

An isomorphism of ±-form n-ads

\[ h: y \rightarrow y' \]

induces an isomorphism of ±-form m-ads

\[ \sigma^T h: \sigma^T y \rightarrow \sigma^T y' \]

whenever \( \sigma^T y, \sigma^T y' \) are defined.

Proof: Choosing a direct complement to

\[ \text{im}(\lambda_T^{juk} - \lambda_K^{juk}): G \rightarrow F_T \oplus F_k \]

we have an exact sequence

\[ 0 \rightarrow Q_{juk} \oplus G \rightarrow F_T \oplus F_k \rightarrow F_k \rightarrow 0 \]

for each \( K \subseteq I-J \), and we set

\[ \phi_{j,k} = g^*_{j,k} \sigma_{j,k} \in \text{Hom}_A(G_{j,k}, G_{j,k}) \]

For \( K' \subseteq I-J \) let \( f_{j,k} \in \text{Hom}_A(Q_{j,k}, Q_{j,k}) \)

be the unique morphism making the diagram
For each $K \subseteq I - J$ we have a situation almost like that in Theorem 1.3, with a commutative hexagon:

The isomorphism of short exact sequences defines an isomorphism of $\pm$-forms:

$$(f, \delta) : (\mathbb{Q}_{j, I - J}, \mathbb{Q}_{j, I - J}) \rightarrow \mathcal{O}(F_J, (\mathbb{Q}_j, \delta_{G_J})).$$
Thus
\[ \Theta^* y = \{ (F, (\mu_\xi^*, \theta)G) : \Sigma (Q_{\mu_\xi^*, \phi_{\xi, \lambda}}) \} \]
is indeed an n-ad of \( \pm \)-forms.

An isomorphism of \( \pm \)-formation n-ads

\[ h = \{ (F, (\xi, \lambda)) : \Sigma (F_{\mu_\xi}^*(\xi, \lambda)G) \} \]

induces an isomorphism of \( \pm \)-form n-ads

\[ \Theta^* h = \{ (F, (\xi, \lambda)) : \Sigma (F_{\mu_\xi}^*(\xi, \lambda)G) \} \]

\[ \Theta^* y = \{ (F, (\mu_\xi^*, \theta)G) : \Sigma (Q_{\mu_\xi^*, \phi_{\xi, \lambda}}) \} \]

whenever \( \Theta^* y \), \( \Theta^* y' \) are defined, with \( \mu_{\xi, \lambda} \in \text{Hom}_A(Q_{\mu_\xi^*, \phi_{\xi, \lambda}}) \).

Further, given an isomorphism of n-ads of \( \pm \)-formation

\[ f = \{ \{ F_{\mu_\xi}^*(\xi, \lambda)G \} : y \to y' \}
\]

there is defined an isomorphism

\[- f = \{ \{ F_{\mu_\xi}^*(\xi, \lambda)G \} : y \to y' \}
\]

Given a \( \pm \)-formation \( x = \{ (F, (\xi, \lambda)G) \}
\]

let \( - x = \{ (Q, \Phi) : (F, (\mu_\xi^*, \theta)G) \} \), and given an isomorphism

\[ f = \{ (\xi, \lambda) : x \to x' \}
\]

More generally, given an n-ad of \( \pm \)-formations

\[ y = \{ x, \{ x_{\mu_\xi^*, \phi_{\xi, \lambda}} \} \}
\]

let

\[ - y = \{ - x, \{ - x_{\mu_\xi^*, \phi_{\xi, \lambda}} \} \}
\]
The join construction defined on coloroids in §§1,2 will be generalized to $n$-ads:

- an $n$-ad $y = \{x_i; x_i^J \in \mathbb{I}\}$
- an $n'$-ad $y' = \{x_i'; x_i'^J \in \mathbb{I}'\}$
- an isomorphism of $n'$-ads $f: \sigma y \to \sigma y'$

for some $J \subseteq I, J' \subseteq I'$ such that $I - J = I' - J' = I''$.

There will be defined an $n''$-ad $y \ast f y' = \{x_{i''}; x_{i''}^J \in \mathbb{I}'', x_{i''}^J \in \mathbb{I}''\}$

uniquely up to isomorphism.

Call two $n$-ad isomorphisms $f_1, f_2: \{z; z^J \in \mathbb{I}\} \to \{z'; z'^J \in \mathbb{I}'\}$ coherent if their definitions coincide in all respects but one: in dealing with $n$-ads of $\pm$-forms, the stable isomorphisms of $\pm$-formations $z \to z'$ need only be coherent (in the sense defined in §2). Coherence classes of isomorphisms are the morphisms of a category with $n$-ads as objects.

Theorem 3.3 Given $n$ (resp. $n'$) -ad isomorphisms

- $g: y \to \hat{y}$
- $g': y' \to \hat{y}'$

and $n''$-ad isomorphisms

- $f: \sigma y \to \sigma y'$
- $\hat{f}: \sigma \hat{y} \to \sigma \hat{y}'$

such that

$$
\begin{align*}
\sigma y & \xrightarrow{f} \sigma y' \\
\sigma g & \Downarrow \sigma g' \\
\sigma \hat{y} & \xrightarrow{\hat{f}} \sigma \hat{y}'
\end{align*}
$$

commutes up to coherence, there is defined an isomorphism of $n''$-ads

$$
g \ast g': y \ast f y' \to \hat{y} \ast \hat{f} \hat{y}'.
$$

This will be proved together with
Theorem 3.4 Let \( y = \sum x_i ; \sum x_i j \in I \) be an \( n \)-ad such that \( \vartheta_j y, \varphi_j y, \varphi \vartheta_j y \) are defined for some disjoint \( J, K \subseteq I \). Then \( \vartheta_j \varphi_j y, \varphi_j \vartheta_j y \) are defined, and there are defined isomorphisms

\[
\tau_{j,k} : \vartheta_j \varphi_j y \rightarrow \vartheta_j \varphi \vartheta_j y
\]
\[
\sigma_{j,k} : \varphi_j \vartheta_j y \rightarrow \varphi_j \varphi \vartheta_j y
\]

of \( m \)-ads, with \((\tau_{j,k})^{-1} = -\tau_{k,j} ; \varphi \vartheta_j y \rightarrow -\vartheta_j \varphi \vartheta_j y\).

We define first the join of \( n \)-ads of \( \pm \) forms. Let then

\[
y = \sum (f, (\chi_j, \delta_j) G) ; \sum (Q_j, \phi_j) j \in I\]
\[
y' = \sum (f', (\chi'_j, \delta'_j) G) ; \sum (Q'_j, \phi'_j) j \in I\]

be \( n \) (resp \( n' \))-ads of \( \pm \) forms, and let

\[
f = \sum (F, (\chi, \delta) G) ; \sum (Q_k, \phi_k) k \in I\]

be an isomorphism of \( n' \)-ads of \( \pm \) forms, for \( n \leq n' \).

Let

\[
y \# y' = \sum (F, ((\chi_j, \delta_j), (\phi_j, \chi'_j)) G) ; \sum (Q_k, \phi_k) k \in I\]

be the \( n \)-ad of \( \pm \) forms defined by

\[
(F, ((\chi_j, \delta_j), (\phi_j, \chi'_j)) G) = (F \# F', ((\chi_j, \delta_j), (\phi_j, \chi'_j)) G) = (Q_k, \phi_k) k \in I,
\]

where

\[
((\chi_j, \delta_j), (\phi_j, \chi'_j)) : (Q_j, \phi_j) \rightarrow H_+ (F)
\]

\[
((\chi_k, \delta_k), (\phi_k, \chi'_k)) : (Q_k, \phi_k) \rightarrow H_+ (F')
\]

are the inclusions defined by \((\chi_j, \delta_j) \), etc.

\[
(Q_k, \phi_k)
\]

\[
= (\ker ((e_k - g_k e_k^*) : Q_{j,k} \otimes Q_{j,k} \rightarrow F_{j,k}^*) / \ker ((e_k - g_k e_k^*) : Q_{j,k} \otimes Q_{j,k} \rightarrow F_{j,k}^*) )
\]

\[
[\psi_{j,k} \otimes \psi_{j,k}^*]
\]

where

\[
E_k : Q_{j,k} \rightarrow F_{j,k}^* ; \chi \mapsto [\chi]
\]

\[
e'_k : Q_{j,k} \rightarrow F_{j,k}^* ; \chi \mapsto [\chi]
\]

are the natural projections,

\[
(F_j^*, (\chi_k', \delta_k')) = \left( \left( \left( \chi_{j,k} \otimes \delta_{j,k} \right), \left( \phi_{j,k} \otimes \chi_{j,k}' \right) \right) \right) 
\]

for \( k \in I \).
Diagram chasing as in the proof of Theorem 1.3 verifies that $y \mapsto y'$ is indeed an $n^*$-ad of $\pm$-forms.

In proving Theorem 3.3 for $n$-ads of $\pm$-forms, it is sufficient to consider only those isomorphisms

$$ g: y \to \hat{y}, g': y' \to \hat{y}' $$

defined using actual (rather than stable) isomorphisms of $\pm$-formations: for if $(F, (((\mathcal{Q}),\Theta))G), (F', (((\mathcal{Q}'),\Theta))G')$ are stabilized by trivial $\pm$-formations, so is $(F', (((\mathcal{Q}'),\Theta))G')$ in the same way. Now for such $g,g'$ the definition of an isomorphism

$$ g \times g': y \times y' \to \hat{y} \times \hat{y}' $$

is obvious, proving Theorem 3.3 in this case.

We next prove Theorem 3.4 for $n$-ads of $\pm$-formations.

Applying Lemmas 3.2, 3.1 in turn, we have that $\varphi_k \varphi_y$ is defined for some $n$-ad of $\pm$-formations

$$ y = \varphi_k \varphi_y \in \varepsilon(F, (((\mathcal{Q}),\Theta))G)_I \sqcup \varepsilon(F, (((\mathcal{Q}'),\Theta))G')_I $$

and disjoint $I,K \sqcup I$ if $\varphi_y$ is defined and if

$$ \begin{pmatrix} f^k_{s,\text{kul}} & f^k_{s,\text{kul}} & 0 \\ 0 & f^k_{l,\text{um}} & f^k_{l,\text{um}} \\ f^k_{s,\text{muk}} & f^k_{s,\text{muk}} & 0 \end{pmatrix} \cdot \varepsilon(I \sqcup K; G) \to \varepsilon(Q; G) $$

is a split mono for all disjoint $L \sqcup I - (J \cup K)$, which corresponds to having

$$ \begin{pmatrix} \lambda^*_{J,JK} & -\lambda^*_{J,JK} & 0 & 0 \\ \lambda^*_{J,JK} & 0 & -\lambda^*_{J,JK} & 0 \\ \lambda^*_{J,JK} & 0 & -\lambda^*_{J,JK} & 0 \\ 0 & -\lambda^*_{L,M} & 0 & -\lambda^*_{L,M} \end{pmatrix} $$

$$ \begin{pmatrix} 0 & 0 & -\lambda^*_{L,M} & 0 \\ 0 & -\lambda^*_{L,M} & 0 & -\lambda^*_{L,M} \\ 0 & -\lambda^*_{L,M} & 0 & -\lambda^*_{L,M} \\ 0 & -\lambda^*_{L,M} & 0 & -\lambda^*_{L,M} \end{pmatrix} $$

$$ \begin{pmatrix} F^*_{s} \otimes F^*_{k} \otimes F^*_{l} \otimes F^*_{m} \\ \text{im} \left( \begin{pmatrix} \mu_{J} \\ \mu_{K} \\ \mu_{L} \\ \mu_{M} \end{pmatrix} : G \to F^*_{s} \otimes F^*_{k} \otimes F^*_{l} \otimes F^*_{m} \right) \end{pmatrix} $$

$$ \to F^*_{s} \otimes F^*_{k} \otimes F^*_{l} \otimes F^*_{m} $$

is a mono.
The symmetry of this condition makes apparent that \( \Theta^k \Theta^j y \) is defined whenever \( \Theta^j y, \Theta^k \Theta^j y \) are defined, if that is the case then

\[
\Theta^k \Theta^j y = \{ q_{j,k}, \Phi_{j,k} \}; \{ e_{j_k,l}^{*}, (\lambda_{j_k,l}^{*}, \chi_{j_k,l}^{*}) q_{j_k,l} \}_{k \in \Lambda_j} \}
\]

is an \( m \)-ad of \( \pm \) formations which we can express more directly in terms of \( y \), as follows:

we have a commutative diagram (for each \( k \in \Lambda_j \))

\[
\begin{array}{ccc}
O & \longrightarrow & Q_{j,k} \oplus G \\
\downarrow & & \downarrow \left( \begin{array}{c} g_j^* \\
H_j^* \end{array} \right) \\
F_j^* \oplus F_L^* & \longrightarrow & F_{j,k,l}^*
\end{array}
\]

\[
\begin{array}{ccc}
F_{j,k,l}^* & \longrightarrow & F_{j,k,l}^* \\
\downarrow & & \downarrow \left( \begin{array}{c} \alpha_{j,k,l}^* \\
\chi_{j,k,l}^* \end{array} \right) \\
0 & \longrightarrow & O
\end{array}
\]

\[
\begin{array}{ccc}
O & \longrightarrow & Q_{j,k,l} \oplus G \\
\downarrow & & \downarrow \left( \begin{array}{c} g_j^* \\
H_j^* \end{array} \right) \\
F_{j,k,l}^* \oplus F_{k,l}^* & \longrightarrow & F_{j,k,l}^* \\
\downarrow & & \downarrow \left( \begin{array}{c} \alpha_{j,k,l}^* \\
\chi_{j,k,l}^* \end{array} \right) \\
0 & \longrightarrow & O
\end{array}
\]

with the top two rows exact, whence the exactness of the bottom row (by diagram chasing).

We can therefore identify

\[
F_{j,k,l}^* = \text{ker}(\lambda_{j,k,l}^{*}, \chi_{j,k,l}^{*}) : F_{j,k,l}^* \oplus F_{k,l}^* \longrightarrow F_{j,k,l}^*
\]

\[
\text{im}(\lambda_{j,k,l}^{*}, \chi_{j,k,l}^{*}) : F_j^* \longrightarrow F_{j,k,l}^* \oplus F_{k,l}^*
\]

\[
\begin{array}{ccc}
Q_{j,k} & \longrightarrow & Q_{j,k} \\
\downarrow & & \downarrow \left( \begin{array}{c} g_j^* \\
H_j^* \end{array} \right) \\
F_j^* \oplus F_k^* & \longrightarrow & F_{j,k,l}^* \\
\downarrow & & \downarrow \left( \begin{array}{c} \alpha_{j,k,l}^* \\
\chi_{j,k,l}^* \end{array} \right) \\
0 & \longrightarrow & 0 \\
\end{array}
\]

It is clear that we can identify

\[
Q_{j,k} = Q_{j,k}^* F_{j,k,l} = F_{j,k,l}^* y_{j,k,l} = -y_{j,k,l} \quad H_{j,k,l} = H_{j,k,l}^*
\]

and as

\[
\Phi_{j,k} = g_j^* \sigma_k \Phi_j^*
\]

\[
\tau_{j,k} = \Phi_{j,k} \in \text{Hom}_g(Q_{j,k}, Q_{j,k}^*)
\]

there is defined an isomorphism of \( \pm \) forms

\[
(1, \Phi_{j,k}) : (Q_{j,k}, - \Phi_{j,k}) \longrightarrow (Q_{j,k}, \Phi_{j,k})
\]

and hence an isomorphism

\[
\tau_{j,k}^{-1} : - \Phi_{j,k} \longrightarrow \Phi_{j,k}
\]

of \( n \)-ads of \( \pm \) formations, with

\[
(\tau_{j,k})^{-1} = - \tau_{j,k} : \Phi_{j,k} \longrightarrow \Phi_{j,k}
\]
The stable isomorphism of ±-forms

\([\alpha, \beta, \psi]: \mathbb{F} \oplus \mathbb{F}_k, \left( \begin{array}{ccc} \psi \cdot \mathbb{F}_k \cdot \psi & 0 \\ 0 & \mathbb{F}_k \cdot \psi \end{array} \right), \left( \begin{array}{ccc} \theta \cdot \mathbb{F}_k \cdot \theta & 0 \\ 0 & \mathbb{F}_k \cdot \theta \end{array} \right) \right) \rightarrow (\mathbb{F}_{\theta_k}, \left( \begin{array}{ccc} \psi_{\theta_k} \cdot \mathbb{F}_{\theta_k} \cdot \psi_{\theta_k} & 0 \\ 0 & \mathbb{F}_{\theta_k} \cdot \psi_{\theta_k} \end{array} \right) \right) \mathbb{G} \mathbb{G}_{\theta_k, \mathbb{G}})

defined in the proof of Theorem 1.3

now gives an isomorphism

\(\sigma_{\theta_k} : \partial^\theta y \times \tau_{\theta_k} \partial^\theta y \rightarrow \partial^\theta y\)

of \(n\)-ads of ±-forms, an exact sequence

\[0 \rightarrow Q_{\theta_k, \mathbb{F}_k} \oplus Q_{\theta_k} \rightarrow Q_{\theta_k} \oplus Q_{\theta_k} \rightarrow \mathbb{F}_{\theta_k} \rightarrow 0\]

being readily established.

This proves Theorem 3.4 for \(n\)-ads of ±-forms.

We next define the join of \(n\)-ads.

Let then

\[y = \xi(G, \theta), \xi(F_\theta, ((\frac{1}{\mu}), \theta)G) \in \mathcal{I}\]

\[y' = \xi(G', \theta), \xi(F_\theta, ((\frac{1}{\mu}), \theta)G) \in \mathcal{I}\]

be \(n\) (resp. \(n'\)) - ads of ±-forms, and let

\[f = \xi[\alpha, \beta, \psi] : \xi(F_\theta, \chi) \in \mathcal{I}\]

\[-\partial^\theta y = \xi(F_\theta, ((\frac{1}{\mu}), \theta)G), \xi(Q_{\theta_k} \oplus Q_{\theta_k}) \in \mathcal{I}\]

\[\rightarrow \partial^\theta y = \xi(F_\theta, ((\frac{1}{\mu}), \theta)G), \xi(Q_{\theta_k} \oplus Q_{\theta_k}) \in \mathcal{I}\]

be an isomorphism.

Using the standard notation of the proof of Theorem 2.1 to describe \([\alpha, \beta, \psi]\), we have a commutative diagram:

\[\begin{array}{ccc}
Q_{\theta_k} \oplus \mathbb{G} \oplus \mathbb{P}^* & \overset{(f_k, 0)}{\longrightarrow} & Q_{\theta_k} \oplus \mathbb{G} \oplus \mathbb{P}^* \\
\downarrow & & \downarrow \\
(\mathbb{F}_\theta \oplus \mathbb{P}) \oplus (\mathbb{F}_\theta \oplus \mathbb{P}^*) & \longrightarrow & (\mathbb{F}_\theta \oplus \mathbb{P}) \oplus (\mathbb{F}_\theta \oplus \mathbb{P}^*) \\
\end{array}\]
(As in the proof of Theorem 1.3) choose a left inverse
\[
\binom{l_k}{m_k}: F_{juk}^* \rightarrow F_j^* \oplus F_k^*
\]
to
\[
\lambda_{juk} (\lambda_{juk}^*) : F_j^* \oplus F_k^* \rightarrow F_{juk}^*
\]
for each $K \in I^*$, and let
\[
\binom{n_k}{p_k} \binom{q_k}{r_k} : F_j^* \oplus F_k^* \rightarrow Q_{j,k} \oplus G
\]
be the corresponding right inverse to
\[
\binom{g_{j}^k}{\mu_j} \binom{g_{k}^j}{\mu_k} : Q_{j,k} \oplus G \rightarrow F_j^* \oplus F_k^*,
\]
so that
\[
\binom{g_{j}^k}{\mu_j} \binom{g_{k}^j}{\mu_k} \binom{l_k}{m_k}
\]
\[
\big( (g_{j}^k, \mu_j) (g_{k}^j, \mu_k) \big) = Q_{j,k} \oplus G \oplus F_{juk}^* \mathrel{\leftrightarrow} F_j^* \oplus F_k^*
\]
are inverse isomorphisms. (By convention, $l_*=1$, $m_*=0$, $q_* = 0$, $r_* = 1$.)

It will not be true, in general, that the diagram
Similarly, choose left inverses
\[(V'_k) : F_{j_uk}^* \rightarrow F_{j_uk}^* \oplus F_k^* \]
to
\[(X_{j_uk}^* - X_{j_uk}^*): F_{j_uk}^* \oplus F_k^* \rightarrow F_{j_uk}^* ,\]
let
\[(n'_{j_uk}, p'_{j_uk}) : F_{j_uk}^* \oplus F_k^* \rightarrow Q_{j_uk} \oplus G' \]
be the corresponding right inverses to
\[(g'_{j_uk}, h'_{j_uk}) : Q_{j_uk} \oplus G \rightarrow F_{j_uk}^* \oplus F_k^* ,\]
and let
\[(t'_{j_uk}, \omega'_{j_uk}) : F_{j_uk}^* \rightarrow Q_{j_uk} \oplus G' \]
be such that
\[
\begin{pmatrix}
1 & 0 \\
0 & \chi_{j_uk}^*
\end{pmatrix}
\begin{pmatrix}
V'_{j_uk} \\
p'_{j_uk} \chi_{j_uk}^*
\end{pmatrix} =
\begin{pmatrix}
1 & 0 \\
0 & \chi_{j_uk}^*
\end{pmatrix}
\begin{pmatrix}
V'_{j_uk} \\
p'_{j_uk} \chi_{j_uk}^*
\end{pmatrix} \in \text{Hom}_A (F_{j_uk}^*, F_{j_uk}^* \oplus F_k^*).\]

Setting
\[
\begin{align*}
(g'' \theta') &= (G \oplus F_{j_uk}^*, (\theta, 0)) \quad F_k' = F_k \oplus F_{j_uk}^* \quad (K \subseteq I^*) \\
\gamma_k' &= \begin{pmatrix}
\gamma_k' & \underline{\alpha} \\
0 & \underline{\alpha}^* \gamma_k'
\end{pmatrix} : G \oplus F_{j_uk}^* \rightarrow F_k \oplus F_{j_uk}^* \\
\mu' &= \begin{pmatrix}
\mu_k & 0 \\
0 & \chi_{j_uk}^* \mu_k
\end{pmatrix} : G \oplus F_{j_uk}^* \rightarrow F_k \oplus F_{j_uk}^* \\
&\quad \text{where } f_k' = f_k : Q_{j_uk} \rightarrow Q_{j_uk}
\end{align*}
\]

\[
\begin{align*}
\psi_k' &= \begin{pmatrix}
\psi_k' & \underline{\alpha} \\
0 & \underline{\alpha}^* \psi_k'
\end{pmatrix} : G \oplus F_{j_uk}^* \rightarrow F_k \oplus F_{j_uk}^* \quad (K \subseteq I^*) \\
\nu_k' &= \begin{pmatrix}
\nu_k' & 0 \\
0 & \chi_{j_uk}^* \nu_k'
\end{pmatrix} : G \oplus F_{j_uk}^* \rightarrow F_k \oplus F_{j_uk}^* \quad (K \subseteq I^*)
\end{align*}
\]

\[
\psi_k' = \begin{pmatrix}
\psi_k' & 0 \\
0 & \chi_{j_uk}^* \psi_k'
\end{pmatrix} : G \oplus F_{j_uk}^* \rightarrow F_k \oplus F_{j_uk}^* \quad (K \subseteq I^*)
\]

\[
\begin{align*}
\text{it is claimed that} \quad Y^* Y' &= \xi(G'', \theta''), \xi(F_k', ((\omega'_{j_uk} \omega_{j_uk}^*)), (\gamma_k', \gamma_k')) \in \text{Hom}_A (F_{j_uk}^*, F_{j_uk}^* \oplus F_k^*).
\end{align*}
\]
To see that each
\[
((\gamma_k^*), (\theta^*_k)) : (G^*, O) \rightarrow H_\pm(F_k^*) \quad (k \in \mathcal{I}^*)
\]
is the inclusion of a sublagrangian, note that
\[
\begin{pmatrix}
1 & 0 & 0 \\
-f_k & g_k^* & r_k^* \\
(k^*_k, j^*_k, m_k^*) & (r_k^* \xi_k^* - r_k^* \xi_k^*) & (r_k^* \xi_k^* + r_k^* \xi_k^*)
\end{pmatrix},
\begin{pmatrix}
1 & b_k^* & 0 \\
0 & f_k & r_k \\
0 & g_k & r_k^* \\
0 & j_k^* & m_k^*
\end{pmatrix},
\begin{pmatrix}
1 & b_k^* & 0 \\
0 & f_k & r_k \\
0 & g_k & r_k^* \\
0 & j_k^* & m_k^*
\end{pmatrix},
\begin{pmatrix}
1 & b_k^* & 0 \\
0 & f_k & r_k \\
0 & g_k & r_k^* \\
0 & j_k^* & m_k^*
\end{pmatrix}
\]

defines a stable isomorphism of deformations (with that on the right being of the kind used to define the join of cobordisms of deformations in $\mathcal{I}^*$).

The inverse isomorphisms
\[
\delta = \begin{pmatrix}
\mu^* & \alpha^* \\
-b & b\alpha^*
\end{pmatrix}
\]

considered in the proof of Theorem 2, may be generalized, to define inverse isomorphisms

\[
\delta_k = \begin{pmatrix}
\lambda^*_{j,k} & \lambda^*_{j,k} \alpha^*l_k \\
-(\mu^*_k b_{j,k} + g_{j,k}^* r_k^* p_k) & \mu^*_k b_{j,k} + g_{j,k}^* r_k^* p_k + m_k^*
\end{pmatrix}
\]

for all $k \in \mathcal{I}^*$, with $\delta_x = \delta$, $\delta_x = \delta'$.
The factorization
\[ F_k^* \oplus F_{juk}^* \xrightarrow{\nu_k^*} F_L \oplus F_{jul}^* \]
now shows that \( \nu_k^* \in \text{Hom}_A(F_k^*, F_L^*) \)
is an isomorphism whenever \( k, l \) are complementary in \( I^- \).

Therefore
\[ y'' = \Xi(G''; \theta'') \Xi(F_k^*, (\chi_k^p, \theta'') G'') \Xi(F_k^*) \]
does define an \( n'' \)-ad of \( \pm \)-formations.

So far, we have made use of
\((l_k^p, n_k^p)\) only in the definition of \( \delta_k^p \),
and not at all \( y'' \). It is clear that we could equally well have defined an \( n'' \)-ad
\[ y''' = \Xi(G''; \theta'') \Xi(F_k^*, (\chi_k^p, \theta'') G'') \Xi(F_k^*) \]
with
\[(G'', \theta'') = (F_j^* \oplus G', (\xi_0 \theta'', \xi)) F_k'' = F_{juk}^* \oplus F_k^* \text{ etc.} \]
using \((l_k^p, n_k^p)\), \((q_k^p, r_k^p)\) instead of \((m_k^p, q_k^p)\), \((n_k^p, p_k^p)\).

Now
\[ y'' \xrightarrow{\Xi(G''; \theta'') \Xi(F_k^*)} y'''' \]
are inverse isomorphisms of \( n'' \)-ads of \( \pm \)-formations, where
\[
\xi = \left( \begin{array}{cc} \mu_k^p \mu_k^p \theta & \mu_k^p \mu_k^p \theta \\ 0 & 0 \end{array} \right) \in \Gamma^+(F_k^* \oplus G')
\]
\[
\xi' = \left( \begin{array}{cc} \mu_k^p \mu_k^p \theta & 0 \\ 0 & \mu_k^p \mu_k^p \theta \end{array} \right) \in \Gamma^+(F_k^* \oplus G').
\]
Hence different choices of \((m_k^p)\) etc. lead to isomorphic expressions for \( y \times y''\), and we are justified in calling it the join.

It is readily verified that this join operation is natural in the sense of Theorem 33.

We are now in a position to complete the proof of Theorem 3.4.
Applying Lemmas 3.1, 3.2 in turn we have
that $\Theta^x y$ is defined for some $n$-ad of $\pm$-forms
$y = \{(F, ((\gamma^m_i)^{\alpha})_i, \Theta, G) ; (L, Q^x_\gamma, \phi)_{\gamma L, i}\}$
and disjoint $J, K \subseteq I$ if $\Theta y$ is defined and if

$$\lambda_J^x = \lambda_J^x : F_{\overline{J}, J}^* \oplus F_{\overline{L}, L}^* \rightarrow F_{\overline{J}, L}^*$$

is onto for all $L \subseteq I - (J \cup K)$, which corresponds
to having

$$(F_{\overline{J}, J}^* \oplus F_{\overline{L}, L}^* : Q_{\overline{J}, J} \oplus Q_{\overline{L}, L} \rightarrow Q_{\overline{J}, L}^*)$$

be pairs of inverse isomorphisms, with $e_J e_K$ the natural projections (as before)
and $f_J = F_{\overline{J}, J}^*$, $f_K = F_{\overline{K}, K}^*$ we have that

$$\gamma_{J, K} = h_J (F_{\overline{J}, J}^* \pm \phi_{\overline{J}, J}) f_J : Q_J \rightarrow F_{\overline{J}, J}$$

$$\mu_{J, K} = e_K F_J : Q_J \rightarrow F_{\overline{J}, J}^*$$

We are now in the situation of Theorem 2.3
and the stable isomorphism of $\pm$-forms

$$\left[\left(\begin{array}{c} f_J^L \\ \lambda_J^L \\
\end{array}\right) : (F_{\overline{J}, J}^*, (\gamma_J^m)_{J, J} \rightarrow (\gamma_{\overline{J}, J})_{J, J}) \right]$$
defined there, by

\[(\alpha, \beta, \psi) =
\left((h^*_e^* h^*_g^*), \left(\begin{array}{cc}
-g_k f^* & 1 \\
-f_k g^* & f_k^* g^*
\end{array}\right) (h^*_k h^*_g h^*_k^* h^*_g^* f_k^* g^*)\right)
\]

\[\left(\begin{array}{cc}
h^*_k h^*_g h^*_k^* h^*_g^* f_k^* g^* & 0 \\
0 & f^*_k g^* h^*_k^* h^*_g^* f_k^* g^*
\end{array}\right)
\]

\[\left((\gamma_{j,k}^*, \phi_J) Q_J \Theta (\gamma_{k,j}^*, \phi_k^*) Q_k \right) \rightarrow \left((\gamma_{k,j}^*, \phi_k^*) Q_k \Theta (\gamma_{j,k}^*, \phi_J) Q_J \right)
\]

which is such that

\[\Theta[\alpha, \beta, \psi] : \Theta((\gamma_{j,k}^*, \phi_J) Q_J) \rightarrow \Theta((\gamma_{k,j}^*, \phi_k^*) Q_k)
\]

restricts to isomorphisms

\[(t_{j,k} : Q_{j,k,l} \rightarrow Q_{k,j,l}, \phi_{j,k,l}) \rightarrow (Q_{k,j,l}, \phi_{k,j,l})
\]

for each \(L \in I - (juk)\).

We have therefore defined an isomorphism

\[\tau_{j,k} : \Theta \rightarrow \Theta \rightarrow \Theta \rightarrow \Theta
\]

of \(m\)-ads of \(\mathcal{L}\) forms. The identity

\[(\tau_{j,k})^{\dagger} = -\tau_{k,j} : \Theta \rightarrow \Theta \rightarrow \Theta \rightarrow \Theta \]

is clear.
for some pair of inverse isomorphisms
\[
(Q_{j,k}, \Phi_{j,k}) \otimes F_{j,k}^* \cong (\text{Hom}_A(F_{j,k}^*, Q_{j,k}), \alpha_{j,k})
\]
which defines an isomorphism of \(\pm\) forms
\[
(\alpha, \phi) \mapsto (Q_{j,k}, \Phi_{j,k})
\]
Further, dualizing
\[
0 \rightarrow Q_{j,k}, \Phi_{j,k} \rightarrow Q_{j,k} \rightarrow Q_{j,k}, \Phi_{j,k} \rightarrow 0
\]
we have a short exact sequence
\[
0 \rightarrow F_{j,k}, \Phi_{j,k} \rightarrow F_{j,k} \rightarrow Q_{j,k}, \Phi_{j,k} \rightarrow 0
\]
and hence isomorphisms
\[
F_{j,k} \cong F_{j,k}^* \cong F_{j,k}^* \otimes F_{j,k}
\]
An appropriate choice of such isomorphisms, together with the isomorphism of \(\pm\) forms defined above, gives an isomorphism
\[
\sigma_{j,k} : \delta^y, \bar{y} \mapsto \delta^y, \bar{y}
\]
of \(\pm\) forms of \(m\)-ads of \(\pm\) forms.
This completes the proof of Theorem 3.3.4.
The join operation commutes with the facette operations, in the following sense:

Theorem 3.5 Let
\[ y = \Xi(F, (\phi, \theta, G), \Xi(Q, \phi))_{J \in J} \]
\[ y' = \Xi(F', (\phi', \theta', G'), \Xi(Q', \phi'))_{J' \in J'} \]
be \( n \) (resp \( n' \)) ads, related by an isomorphism
\[ f : \Xi y \to \Xi y' \]
of \( n' \)-adss, for some \( J \in J, J' \in J' \) such that \( J = I \cdot J' = I'' \).

If \( K \subseteq I'' \) is such that \( \Xi y, \Xi_y K y \)
and \( \Xi y', \Xi_{y'} K y' \) are defined, then \( \Xi_y (y x f y) \)
is defined, and there is defined an isomorphism of \( m \)-ads
\[ \epsilon^k : \Xi_y (y x f y) \to \Xi_{y'} K y \]
for any isomorphism \( f_k : \Xi y \to \Xi_{y'} y \)
in the coherence class of the composite
\[ \Xi_{y'} y \to \Xi_{y'} y \]
\[ \Xi_{y'} y \to \Xi_{y'} y \]

Proof: i) Let
\[ y = \Xi(F, (\phi, \theta, G), \Xi(Q, \phi))_{J \in I} \]
\[ y' = \Xi(F', (\phi', \theta', G'), \Xi(Q', \phi'))_{J' \in I'} \]
be \( n \) (resp \( n' \)) ads of \( + \)-forms, and let
\[ f = \Xi(F, \chi), \Xi_k \exists \phi \]
\[ : \Xi y = \Xi(Q, \phi) \]
\[ \Xi(F_k, ((\phi_k, \theta), Q))_{J \in I} \]
\[ \Xi y' = \Xi(Q', \phi') \]
\[ \Xi(F_k', ((\phi', \theta'), Q'))_{J' \in I'} \]
be an isomorphism of \( n \)-ads of \( + \)-forms.

Let
\[ F_{k, k} \to Q \]
\[ F_{k, k} * \to Q \]
\[ F_{k, k} \to Q \]
\[ F_{k, k} * \to Q \]
\[ F_{k, k} \to Q \]
\[ F_{k, k} * \to Q \]
be pairs of inverse isomorphisms, with \( f_k = f_{k, k} \) etc., as before.
The composite
\[ f_L : \mathcal{Q}^k y \xrightarrow{\alpha} \mathcal{Q} y \xrightarrow{\beta} \mathcal{Q}^k y' \xrightarrow{\gamma} \mathcal{Q}^k y' \]

is defined by
\[ \Sigma \{ \alpha, \beta, \gamma \} ; \Sigma (\mathcal{Q} y, \mathcal{Q}^k y') \]
\[ : \Sigma (F_{\mathcal{K}}, ((\mathcal{Q}^k y'), \phi_k) \mathcal{Q}^k) ; \Sigma (\mathcal{Q}^k y, \mathcal{Q} y') \]
\[ \xrightarrow{\Sigma (F_{\mathcal{K}}, ((\mathcal{Q}^k y), \phi_k) \mathcal{Q}^k)} \]

where \((\alpha, \beta, \gamma)\) is the composite
\[ (F_{\mathcal{K}}, ((\mathcal{Q}^k y), \phi_k) \mathcal{Q}^k) \oplus (\mathcal{Q}^k \oplus \mathcal{Q}^k, \mathcal{Q} y \oplus \mathcal{Q} y') \]
\[ \left( \begin{array}{ccc}
\left( h^* e_k^* h^* e_k^* & 0 & 0 \\
0 & 0 & 1
\end{array} \right), \left( \begin{array}{ccc}
h^* e_k^* h^* e_k^* & 0 & 0 \\
0 & 0 & 1
\end{array} \right), \left( \begin{array}{ccc}
h^* e_k^* h^* e_k^* & 0 & 0 \\
0 & 0 & 1
\end{array} \right) \right) \]
\[ \left( \begin{array}{ccc}
(d_k & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array} \right), \left( \begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{array} \right), \left( \begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 1
\end{array} \right) \right) \]
\[ \left( \begin{array}{ccc}
(f_{\mathcal{K}} & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array} \right) \]
\[ \left( \begin{array}{ccc}
(h^* e_k^* & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array} \right), \left( \begin{array}{ccc}
(h^* e_k^* & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array} \right), \left( \begin{array}{ccc}
(h^* e_k^* & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array} \right) \right) \]
\[ \left( \begin{array}{ccc}
(d_k & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array} \right), \left( \begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{array} \right), \left( \begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 1
\end{array} \right) \right) \]

and \((\mathcal{Q} y, \mathcal{Q}^k y')\) the composite
\[ (F_{\mathcal{K}}, ((\mathcal{Q}^k y), \phi_k) \mathcal{Q}^k) \oplus (\mathcal{Q}^k \oplus \mathcal{Q}^k, \mathcal{Q} y \oplus \mathcal{Q} y') \]
\[ \xrightarrow{\Sigma (F_{\mathcal{K}}, ((\mathcal{Q}^k y), \phi_k) \mathcal{Q}^k)} \]

where \((t_L, \chi), (t'_L, \chi')\) are as defined in the proof of Theorem 3.4, and \(u_L \in \text{Hom}(\mathcal{Q}^k y, \mathcal{Q}^k y')\) is the unique isomorphism making the diagram commute:
\[ \xrightarrow{(g_{L}^k, \mu_{L}^k)} \]
\[ \xrightarrow{(\mathcal{Q}^k y, \mathcal{Q} y')} \]
\[ \xrightarrow{(F_{\mathcal{K}}^*, \mathcal{Q}^k y \oplus \mathcal{Q}^k y')} \]
\[ \xrightarrow{(\alpha^* \mathcal{Q}^k y \oplus \mathcal{Q}^k y', \mathcal{Q} y \oplus \mathcal{Q} y')} \]
\[ \xrightarrow{\Sigma (F_{\mathcal{K}}^*, ((\mathcal{Q}^k y), \phi_k) \mathcal{Q}^k)} \]

\[ \xrightarrow{\Sigma (F_{\mathcal{K}}^*, ((\mathcal{Q}^k y), \phi_k) \mathcal{Q}^k)} \]

\[ \xrightarrow{(\mathcal{Q}^k y, \mathcal{Q} y')} \]
\[ \xrightarrow{(F_{\mathcal{K}}^*, \mathcal{Q}^k y \oplus \mathcal{Q}^k y')} \]
\[ \xrightarrow{(\alpha^* \mathcal{Q}^k y \oplus \mathcal{Q}^k y', \mathcal{Q} y \oplus \mathcal{Q} y')} \]
\[ \xrightarrow{(F_{\mathcal{K}}^*, \mathcal{Q}^k y \oplus \mathcal{Q}^k y')} \]
\[ \xrightarrow{\Sigma (F_{\mathcal{K}}^*, ((\mathcal{Q}^k y), \phi_k) \mathcal{Q}^k)} \]

\[ \xrightarrow{(\mathcal{Q}^k y, \mathcal{Q} y')} \]
\[ \xrightarrow{(F_{\mathcal{K}}^*, \mathcal{Q}^k y \oplus \mathcal{Q}^k y')} \]
\[ \xrightarrow{\Sigma (F_{\mathcal{K}}^*, ((\mathcal{Q}^k y), \phi_k) \mathcal{Q}^k)} \]

Commute.
Constructing the $n^\ast$-add of $\mathcal{F}$-formations

$\partial^k y \times F_{*} \partial^k y = \sum (Q_{*} \otimes s(F_{*}, ((V_{*}), (\phi_{*})))_{s, s})$

we have that

$(Q_{*}, \phi_{*}) = (Q_{*} \oplus F_{*} \otimes (\phi_{*} \otimes s(J)))$

where

$s = K_{*} e_{*}^k d_{*} h_{*} a_{*}^k e_{*} h_{*}^k - K_{*} e_{*} h_{*} a_{*}^k e_{*} h_{*}^k$

$F_{*} = F_{*} \oplus F_{*} \otimes J$

$\gamma_{*} = \begin{pmatrix} 0 & 1 \\ K_{*} e_{*}^k d_{*} h_{*} a_{*}^k e_{*} h_{*}^k & 1 \end{pmatrix}$

$\delta_{*} = \begin{pmatrix} 1 & 0 \\ K_{*} e_{*}^k d_{*} h_{*} a_{*}^k e_{*} h_{*}^k & 1 \end{pmatrix}$

$\delta_{*} = \begin{pmatrix} 1 & 0 \\ K_{*} e_{*}^k d_{*} h_{*} a_{*}^k e_{*} h_{*}^k & 1 \end{pmatrix}$

$F_{*} \otimes F_{*} \otimes J \rightarrow F_{*} \otimes F_{*} \otimes J$

for some choice of inverse isomorphism

$\begin{pmatrix} (g_{*} \otimes J) & (h_{*}) \\ (g_{*} \otimes J) & (h_{*}) \end{pmatrix}$

and the usual definition of

$(t_{*}) \in \text{Hom}_{(F_{*} \otimes F_{*} \otimes J)}(Q_{*}, Q_{*} \otimes J)$

with

$(1, 0) (L_{*}) \neq (L_{*}) (m_{*})$
Now
\[ y \times_f y = \Sigma (F^r ((\chi^r), (\Phi^r)) G^r, \Sigma (G^r \Delta^r (\chi^r), (\Phi^r)) G^r) \]
with
\[ (G^r \Delta^r (\chi^r), (\Phi^r)) \]
\[ = \left( \frac{\ker((e^f_d - d^f_k) : Q_{juk} \otimes Q_{juk} \rightarrow E^f_{juk})}{\im((f^j_{j^f}, f^k_{j^f}) : Q^j \rightarrow Q_{juk} \otimes Q^j_{juk})}, [\Phi^j_{j^f} \otimes \Phi^j_{j^f}] \right) \]

The isomorphism of ± forms considered in the proof of Theorem 3.4 (on page 3.32) then generalizes to an isomorphism
\[ \left( \begin{array}{c} (e^f_k h^f_k e^f_k') \\ (0) \end{array} \right), \left( \begin{array}{c} (\chi^j_{j^f} f^f_k h^f_k d^f_k e^f_k') \\ (0) \end{array} \right) \]
\[ : (Q, \Phi) \longrightarrow (Q^r, \Phi^r) \]

The inverse of this, together with isomorphisms
\[ F_{k,l} = (Q_{juk} \otimes Q_{juk})^* = F_{k,l} \otimes (Q_{juk} \otimes Q_{juk}) \rightarrow F_{k,l} \otimes F_{j,l} = F_{k,l} \]
given by the isomorphisms \( F_{k,l} \otimes F_{j,l} \rightarrow F_{k,l} \otimes F_{j,l} \) also considered there, now defines an isomorphism
\[ \sigma^r : \Sigma (y \times_f y') \rightarrow \Sigma (y \times_f y') \]
\[ \sigma^r \Delta^r \]  

of the ± forms, verifying that \( \sigma^r \Delta^r \) is the 3.47

ii) Let
\[ y = \Sigma (G, \Theta) \Sigma (F^r, ((\chi^r), (\Phi^r)) G^r) \]
\[ y' = \Sigma (G, \Theta) \Sigma (F^r, ((\chi^r), (\Phi^r)) G^r) \]
be \( n \) resp. \( n' \) - ads of ± formations, and let
\[ f = \Sigma (G, \Theta) \Sigma (F^r, ((\chi^r), (\Phi^r)) G^r) \]
\[ : \sigma^r y = \Sigma (F^r, ((\chi^r), (\Phi^r)) G^r) \]
\[ \rightarrow \sigma^r y' = \Sigma (F^r, ((\chi^r), (\Phi^r)) G^r) \]
be an isomorphism of \( n' \)- ads of ± forms.

Let
\[ y \times_f y' = \Sigma (G, \Theta) \Sigma (F^r, ((\chi^r), (\Phi^r)) G^r) \]
and note that there is a factorization
\[ F^r_{j^f} = F^r_{j^f} \otimes F^r_{j^f} \]
\[ F^r_{j^f} \otimes F^r_{j^f} \rightarrow F^r_{j^f} \otimes F^r_{j^f} = F^r_{j^f} \]
\[ (1 \otimes \chi^r_{j^f}) \otimes (0 \otimes \chi^r_{j^f}) \rightarrow (0 \otimes \chi^r_{j^f}) \otimes (1 \otimes \chi^r_{j^f}) \]
It follows that
\[(\lambda^*_{KL} - \lambda^*_{L}) : F^* \rightarrow F^* \rightarrow F^*_{KL}\]
is onto for all \(L \leq I^* - K\) whenever \(Y \leq I^*\)
is such that \(\delta^*_{L}y, \delta^*_{K}y\) are defined, implying that \(\delta^*_{K}(y \circ y)\) is defined as well.
If that is the case, we can identify
\[Q_{KL}^* = \frac{\ker((\lambda^*_{KL} - \lambda^*_{L}) : F^* \rightarrow F^*_{KL})}{\text{im}((\lambda^*_{KL} - \lambda^*_{L}) : G \rightarrow F^*_{KL})}\]
\[= \frac{\ker((\lambda^*_{KL} - \lambda^*_{L}) : F^* \rightarrow F^*_{KL})}{\text{im}((\lambda^*_{KL} - \lambda^*_{L}) : G \rightarrow F^*_{KL})}\]
\[\oplus \frac{\ker((\lambda^*_{KL} - \lambda^*_{L}) : F^* \rightarrow F^*_{KL})}{\text{im}((\lambda^*_{KL} - \lambda^*_{L}) : G \rightarrow F^*_{KL})}\]
\[= Q_{KL} \oplus F^*_{KL},\]

using the description of \(F^*_{KL}\), given in the proof of Theorem 3.4.

The composite
\[f_1 : \partial^*_{L}y \rightarrow \partial^*_{K}y \rightarrow \partial^*_{L}y \rightarrow \partial^*_{K}y\]
is defined by
\[\mathcal{E}(\mathcal{F}_L, \lambda^*) \mathcal{E}(\mathcal{F}_L, \lambda^*)\]
\[\mathcal{E}((\mathcal{F}_L, \lambda^*)((\mathcal{F}_L, \lambda^*) \mathcal{F}_L, \lambda^*))\]
\[\rightarrow \mathcal{E}(\mathcal{F}_L, \lambda^*) \mathcal{E}((\mathcal{F}_L, \lambda^*) \mathcal{F}_L, \lambda^*)\]
\[\mathcal{E}(\mathcal{F}_L, \lambda^*) \mathcal{E}((\mathcal{F}_L, \lambda^*) \mathcal{F}_L, \lambda^*)\]
identifying \((\mathcal{F}_L, \lambda^*) \mathcal{F}_L, \lambda^*) = (\mathcal{F}_L, \lambda^*) \mathcal{F}_L, \lambda^*)\), \(F^*_{KL} = F^*_{KL}\) etc.

Now
\[\delta^*_{L}y \otimes \delta^*_{K}y\]
\[= \mathcal{E}(\mathcal{F}_L \mathcal{F}_L, \lambda^*) \mathcal{E}(\mathcal{F}_L \mathcal{F}_L, \lambda^*)\]
\[\mathcal{E}(\mathcal{F}_L \mathcal{F}_L, \lambda^*) \mathcal{E}(\mathcal{F}_L \mathcal{F}_L, \lambda^*)\]
\[\mathcal{E}(\mathcal{F}_L \mathcal{F}_L, \lambda^*) \mathcal{E}(\mathcal{F}_L \mathcal{F}_L, \lambda^*)\]
is an \(n\)'-ad of \(\pm\)forms.
The stable isomorphism of \( \pm \)formations

\[(F_\pm, (C_{\pm k}^\mu), \theta) \in \Gamma \]

\[- \mapsto \left( F_\pm \mathcal{G}_\pm, \left( \begin{array}{cc} \left( \begin{array}{cc} C_{\pm k}^\mu & 0 \\ \theta & 0 \end{array} \right) \\ 0 & \theta \end{array} \right) \right) \]

\( \Gamma \in \mathcal{Q}_{\pm k, \pm l} \in \mathcal{G}_\pm \)

defined in the proof of Theorem 3.3

(on page III 3.30, to be precise), and the isomorphisms of \( \pm \)forms (defined for \( L \in I^* - K \))

\[
\left[ \begin{array}{cc} f_{kl} & h_{kl} \\ k_{kl} & 0 \\ 0 & k_{kl} \end{array} \right] \rightarrow \left( \begin{array}{cc} f_{kl} & k_{kl} \\ k_{kl} & 0 \end{array} \right)
\]

\( \Theta_{k,l} \mathcal{T}_{kl}^\phi \) \( \phi \) defined as on page III 3.44

(with \( k \) replaced by \( k_l \), etc.)

\[
\ker \left( \mathcal{T}_{k,l}^\phi \right) : Q_{k_{j,l}} \mathcal{T}_{k_{j,l}} \rightarrow \mathcal{F}_+^* \\
\text{im} \left( \mathcal{T}_{k,l}^\phi \right) : Q_{k_{j,l}} \rightarrow Q_{k_{j,l}} \mathcal{T}_{k_{j,l}}^\phi
\]

\[
\left[ \begin{array}{cc} (Q_{k_{j,l}} \mathcal{T}_{k_{j,l}}^\phi) & 0 \\ 0 & (Q_{k_{j,l}} \mathcal{T}_{k_{j,l}}^\phi) \end{array} \right]
\]

together define an isomorphism

\( \Theta^k : \Theta^k (y \times_f y') \rightarrow \Theta^k y \times_f \Theta^k y' \)

of \( n^\circ \)-ads of \( \pm \)forms.

The isomorphisms given in Theorems 3.3, 3.4, 3.5 enjoy the following obvious naturality properties:

**Corollary 3.6**

1) Let \( g : y \rightarrow y' \) be an isomorphism of \( n \)-ads, and let \( J, K \) be such that \( r_{-J}, r_{-K} \) are defined for \( y \) (and hence \( y' \)).

Then the squares

\[
\begin{array}{ccc}
\Theta^J y & \xrightarrow{r_{-J}} & \Theta^J y \\
\downarrow & & \downarrow \\
\Theta^K y' & \xrightarrow{r_{-K}} & \Theta^K y'
\end{array}
\]

\[
\begin{array}{ccc}
\Theta^J y & \xrightarrow{r_{-J}} & \Theta^J y \\
\downarrow & & \downarrow \\
\Theta^K y' & \xrightarrow{r_{-K}} & \Theta^K y'
\end{array}
\]

commute.

2) Let \( g : y \rightarrow y' \), \( g' : y \rightarrow y' \)

be isomorphisms of \( n \)-ads such that

\( g \times g' : y \times_f y' \rightarrow y' \times_f y' \)

is defined (for any \( f, \Theta^J y \rightarrow \Theta^J y', f' : \Theta^K y' \rightarrow \Theta^K y' \)).
and let \( K \) be such that \( \rho^K \) is defined for \( y^* y' \) (and hence for \( y^*_f y' \)). Then the square

\[
\begin{array}{ccc}
\vartheta^K(y^*_f y') & \xrightarrow{\rho^K} & \vartheta^K y^*_f \vartheta^K y' \\
\vartheta^K(y^*_f y') & \xrightarrow{\vartheta^K y^*_f \vartheta^K y'} & \vartheta^K y^*_f \vartheta^K y'
\end{array}
\]

commutes.

The join operations are commutative and associative in the following sense:

**Theorem 3.7**

i) Let \( y, y' \) be \( n \) (resp. \( n' \)) -ads and let \( f: -\vartheta y \rightarrow -\vartheta f y' \) be an isomorphism of \( n' \)-ads. Then there is defined a natural isomorphism

\[
k: y^*_f y' \rightarrow y'^*_f \rho^{-1} y
\]

of \( n'-\)ads.

ii) Let \( x, y, z \) be \( m \) (resp. \( n, p \)) -ads, and suppose given isomorphisms

\[
f: -\vartheta x \rightarrow -\vartheta y, \ g: -\vartheta y \rightarrow -\vartheta z, \ h: -\vartheta x \rightarrow -\vartheta z
\]

such that the composites

\[
e: -\vartheta x \vartheta f \rightarrow -\vartheta x \vartheta y \rightarrow -\vartheta x \vartheta y \rightarrow -\vartheta x \vartheta y \rightarrow -\vartheta x \vartheta z
\]

\[
\tilde{e}: -\vartheta x \vartheta f \rightarrow -\vartheta x \vartheta y \rightarrow -\vartheta x \vartheta y \rightarrow -\vartheta x \vartheta z
\]

are coherent (and defined). Then there is defined a natural isomorphism

\[
\alpha: (y^*_f y')^*_{(h \ast g)_*} \rightarrow \alpha_{(g \ast h)_*} (y^*_f y')
\]

of \( q \)-ads, where \( (h \ast g)_* \) is the composite

\[
-\vartheta x^* f \vartheta g \vartheta h \vartheta x \vartheta f \vartheta y \vartheta f \vartheta g \vartheta h \vartheta x \vartheta f \vartheta g \vartheta h \vartheta x \vartheta f \vartheta g \vartheta h \vartheta x
\]

and \( (f \ast h)_* \) is the composite

\[
-\vartheta x \vartheta f \vartheta g \vartheta h \vartheta x \vartheta f \vartheta g \vartheta h \vartheta x \vartheta f \vartheta g \vartheta h \vartheta x
\]

Proof: i) If \( y, y' \) are \( n \)-ads of \( \pm \)forms, we can take for \( \kappa \) the \( \{ \text{identity isomorphism} \}

\[
\} \text{isomorphism of forms on } y
\]

\[
\} \text{on form } y
\]
ii) Exercise for the reader.

\[ X_n(A) \text{ of isomorphism classes of } \{ \pm \text{forms} \} \]

where \( \pm = (-1)^n \) if \( n \in \{ 2i \}_{i \in \mathbb{Z}^+} \). An n-ad

\[ y = \{ x ; x \in \mathbb{R} \} \]

will be said to be of finite dimension if \( y \in X_m(A) \),

so that we are dealing with an n-ad of \( \{ \pm \text{forms} \} \)

if \( n + f - 1 = \{ 2i \}_{i \in \mathbb{Z}^+} \), with \( \pm = (-1)^f \), and \( f(\text{mod} 4) \).

An n-ad \( y = \{ x ; x \in \mathbb{R} \} \) is connected

if \( \exists y \) is defined for all \( j \in I \), and highly-connected

if \( \exists \exists y \exists z \ldots \exists y \) is defined for all disjoint \( j_1, j_2, \ldots, j_r \in I \).

By Theorem 3.4 \( y \) is highly-connected if \( \exists y \) is defined for all \( j_1, j_2, \ldots, j_r \in I \). By Theorem 3.5

the join of highly-connected n-ads is again highly connected.
An oriented $n$-ad

$$y = \varepsilon x; \{x_3; x_3 x_1\}$$

is one for which $I = \varepsilon_0, \ldots, n_3$.

An orientation of an $n$-ad

$$y = \varepsilon x; \{x_3 x_1; x_3 x_1\}$$

is the oriented $n$-ad

$$y_# = \varepsilon x; \{x_3 x_1; x_3 x_1\}$$

defined by relabelling, according to some bijection

$$\omega : \varepsilon_0, \ldots, n_3 \rightarrow I$$

In applications, it will always be the case that $I \subseteq \{0, 1, 2, \ldots, k\}$, with

$$\omega = \omega_\pi : \varepsilon_0, \ldots, n_3 \rightarrow I$$

the unique order-preserving bijection.

Define face operations

$$\partial_j : \{\text{oriented connected } n\text{-ads of fibre dimension } f\} \rightarrow \{\text{oriented } (n-1)\text{-ads of fibre dimension } f\}$$

for $0 \leq j \leq n$, $n \geq 1$ by

$$\partial_j y = (\varepsilon^j (\partial x^j x))_#$$

Theorem 3.4 gives isomorphisms

$$\partial_{jk} = (\varepsilon^j \varepsilon^k \varepsilon^k) \partial_{k} \partial_j y \rightarrow \partial_j \partial_k y$$

for $0 \leq j < k \leq n$, whenever $\partial_j \partial_k y, \partial_k \partial_j y$ are defined.

For $f \equiv 0 \pmod{4}$, let $\mathcal{L}_f(A)$ be the $\Delta$-set defined by:

An $n$-simplex, $s \in \mathcal{L}_f(A)^{(m)}$, is a collection

$$s = \varepsilon (Y_r, h_{jk} : Y_{r} \rightarrow \mathcal{L}_f(A))$$

of oriented highly connected $(IJ-1)$-ads $Y_r$, of fibre dimension $f$, one for each non-empty subset $J$ of $\varepsilon_0, \ldots, n_3$, and $(IJ-2)$-ad isomorphisms $h_{jk}$ defined for $0 \leq k < IJ-1$ with $IJ \geq 2$, such that:

i) the $(IJ-2)$-ad isomorphisms

$$\partial_k \cdot Y_r \rightarrow \partial_j Y_r$$

defined by requiring the diagrams

$$\begin{array}{ccc}
\partial_k \cdot Y_r & \rightarrow & \partial_j Y_r \\
\partial_{jk} \cdot Y_r & \rightarrow & \partial_{j} Y_r \\
h_{r, \varepsilon_0, \varepsilon_j} & \rightarrow & \partial_j Y_r
\end{array}$$

for $0 \leq j < k \leq n$, whenever $\partial_k \partial_j \partial_{jk} y, \partial_j \partial_k \partial_{jk} y$ are defined.

ii) the $0$-ads $Y_{sij} = \varepsilon x_{ij} \in \mathcal{L}_n(A)$ have $x_{ij}$ non-singular, that is

$$x_{ij} \in \ker(\partial: X_f(A) \rightarrow X_{f-1}(A))$$
The base $n$-simplex, $o \in \Sigma_f(A)^{(n)}$, is the unique such collection with all the $A$-modules $0$.

Denoting $\omega_{i_0, \ldots, i_n-i}$ by $\partial_j$, so that

$$\partial_j: \{i_0, \ldots, i_n-1 \rightarrow i_0, \ldots, i_n-1\}; i \mapsto \{i \mapsto i < j \},$$

define face maps

$$\partial_j: \Sigma_f(A)^{(m)} \rightarrow \Sigma_f(A)^{(m-1)};$$

$$\{ (y_{ij}, h_{i,j}, k) \}; \text{ where } 0 \leq k \leq i 
\rightarrow \{ (y_{ij}, h_{i,j}, k) \}; \text{ where } 0 \leq k \leq i$$

for $0 \leq j \leq n, n \geq 1$. Then

$$\partial_j \partial_k = \partial_{k-1} \partial_j: \Sigma_f(A)^{(n)} \rightarrow \Sigma_f(A)^{(n-2)}$$

(by construction).

We shall establish that

i) the Kan extension condition holds for $\Sigma_f(A)$

ii) $\pi_n(\Sigma_f(A)) = U_{n+f}(A) \quad (n \geq 0)$

iii) there are defined homotopy equivalences

$$\Omega \Sigma_f(A) \rightarrow \Sigma_f(A) \quad (f \text{ mod } 4),$$

in a natural fashion.

The Kan extension condition is that every $n$-tuple of $(n-1)$-simplexes

$$s = \{ \{ s_j \in \Sigma_f(A)^{(n)} \}, \{ 0 \leq j \leq n, j \neq i \} \} \quad (0 \leq i \leq n)$$

with

$$\partial_j s_k = \partial_{k-1} s_j \in \Sigma_f(A)^{(m)} \quad (if \, m \geq 2)$$

is such that

$$s_j = \partial_j s \in \Sigma_f(A)^{(n)} \quad (0 \leq j \leq n, j \neq i)$$

for some $s \in \Sigma_f(A)^{(m)}$.

In order to verify it for $\Sigma_f(A)$ we have to show that for every collection

$$\{ (y_{ij}, h_{i,j}, k) \}; \text{ where } 0 \leq k \leq i + 1 
\rightarrow \{ (y_{ij}, h_{i,j}, k) \}; \text{ where } 0 \leq k \leq i + 1$$

of oriented highly connected $(111-1)$-ads $Y_{ij}$, and $(111-2)$-ad isomorphisms $h_{i,j,k}$, such that

$$s_j = \{ (y_{ij}, h_{i,j}, k) \}; \text{ where } 0 \leq k \leq 1 \quad (0 \leq j \leq n, j \neq i)$$

there is defined an oriented highly connected $n$-ad $Y_{i_0, \ldots, i_n}$ of fibre dimension $f$, with

$(n)$-ad isomorphisms

$$h_{i_0, \ldots, i_n}^*: Y_{i_0, \ldots, i_n} \rightarrow Y_{i_0, \ldots, i_n}$$

$(0 \leq i \leq n, j \neq i)$.
such that
\[ s = \{ (y_{ij}, h_{ij}) \}_{j \leq i} \in \mathfrak{S}_2(A) \],
with
\[ h_{y_{0}, y_{1}, \ldots, y_{n-1}, y_{n}} \rightarrow \mathfrak{S}_2(A) \]
the identity.

For the cases \( n=1,2 \) this programme was carried out in \( \mathfrak{S}_1(A) \), as will be detailed below. The general case proceeds by analogy.

A \( 0 \)-simplex of \( \mathfrak{S}_{2i}(A) \) is a non-singular \( \pm \)form \((a,i)\) together with a stable isomorphism \((E,(y_i)_{i \in 2i}) \rightarrow \mathfrak{S}_2(A)\)
of trivial \( \pm \)formations. A \( 1 \)-simplex of \( \mathfrak{S}_{2i}(A) \) is a cobordism of non-singular \( \pm \)formations, together with such stable isomorphisms.

A \( 0 \)-simplex of \( \mathfrak{S}_{2i+1}(A) \) is a non-singular \( \pm \)formation, and a \( 1 \)-simplex is a cobordism of such.

In verifying that cobordism of \( \mathfrak{S}_{2i} \) \( \pm \)formations is an equivalence relation, in Theorem \( \mathfrak{S}_{2i} \), we constructed cobordisms
\[ \{ (a,0), (a,1), (a,2), (a,0), (a,1), (a,2) \} \text{ for every } (a,0) \in \mathfrak{S}_{2i}(A) \]
for every \( (a,0) \in \mathfrak{S}_{2i}(A) \). In particular, for non-singular \( \mathfrak{S}_{2i} \) \( \pm \)formations, we have the extension condition for \( n=1 \), for each \( \mathfrak{S}_{2i}(A) \).
Moreover, the \( 0 \)-simplexes of \( \mathfrak{S}_{2i}(A) \) defined by the same non-singular \( \pm \)form \((a,i)\) are all seen to lie in the same path-component of \( \mathfrak{S}_{2i}(A) \). Corollary \( \mathfrak{S}_{2i} \).

Now show that
\[ \pi_i(\mathfrak{S}_f(A)) = \mathfrak{S}_f(A) \]
for all \( f \) (mod 4).

The above construction of cobordisms generalizes to

**Lemma 4.1** A lattice of \( \mathfrak{S}_{2i} \) \( \pm \)formations \( \mathfrak{S}_{2i}(A) \) can be embedded in an \( n \)-ad of \( \mathfrak{S}_{2i} \) \( \pm \)formations
\[ \mathfrak{S}_{2i}(A) \in \{ \mathfrak{S}_{2i+1}(A), \mathfrak{S}_{2i-1}(A), \mathfrak{S}_{2i-3}(A), \ldots \} \]

**Proof:**

1. Let \( \mathfrak{S}_{2i}(A) \) be a lattice of \( \pm \)forms.
2. For \( J \subseteq I \) set \( J_0 = J \cup \omega 2^I \) and define a \( \pm \)form
\[ (Q_{J_0}, \phi_{J_0}) = (H_{\pm}(\mathfrak{S}_{2i}(A)), \mathfrak{S}_{2i}(A)) \]

In particular, we have
\[ (Q_{J_0}, \phi_{J_0}) \in \mathfrak{S}_{2i}(A) \]
and
\[ (Q_{J_0}, \phi_{J_0}) = (Q_{J_0}, \phi_{J_0}) \]

where
\[ (Q_{J_0}, \phi_{J_0}) = H_{\pm}(\mathfrak{S}_{2i}(A)) \]
For $J \subseteq I$ let
\[(f^*_J, \gamma^*_J) = (f^*_I, \gamma_I) : (Q_\mathbb{I}, \phi_\mathbb{I}) \to (Q_{\mathbb{I}_J}, \phi_{\mathbb{I}_J})\]
and
\[(f_{J,\mathbb{I}_J}, \chi_{J,\mathbb{I}_J}) = (f_{I,\mathbb{I}_J}, \chi_I) : (Q_\mathbb{I}, \phi_\mathbb{I}) \to (Q_{\mathbb{I}_J}, \phi_{\mathbb{I}_J})\].

For $J \subseteq J' \subseteq I$ let
\[(f_{J,\mathbb{I}_J}, \varnothing) : (Q_{\mathbb{I}_J}, \phi_{\mathbb{I}_J}) \to (Q_{\mathbb{I}_{J'}}, \phi_{\mathbb{I}_{J'}})\]
be the subform inclusion defined by the $A$-module inclusion $f_{J,\mathbb{I}_J} \in \text{Hom}_A(Q_{\mathbb{I}_J}, Q_{\mathbb{I}_{J'}})$, and let
\[(f_{J,\mathbb{I}_J}, \varnothing) : (Q_\mathbb{I}, \phi_\mathbb{I}) \to (Q_{\mathbb{I}_{J'}}, \phi_{\mathbb{I}_{J'}})\].

Now, for each $J \subseteq I$,
\[(f_{I,\mathbb{I}_J}, \varnothing) : (Q_\mathbb{I}, \phi_\mathbb{I}) \to (Q_{\mathbb{I}_{J}}, \phi_{\mathbb{I}_{J}}), (f_{I,\mathbb{I}_J}, \varnothing) : (Q_{\mathbb{I}_{J}}, \phi_{\mathbb{I}_{J}}) \to (Q_{\mathbb{I}_{J'}}, \phi_{\mathbb{I}_{J'}})\]
are the inclusions of maximally orthogonal subforms (by construction). Therefore
\[\tilde{y} = \{ (Q_I, \varnothing) \ ; \ \{ (Q_J, \phi_J) \}_{J \subseteq I} \}\]
is an $n$-adj of ±-forms containing $y$.

\[\text{ii) let } y = \xi(F_J, ((\gamma_I), (\Theta)G) \mathbb{I}_{J \subseteq I} \text{ be a lattice of ±-formations.} \]

For $J \subseteq I$ set $J_0 = J \cup \mathbb{I}_{J \subseteq I}$, and define a ±-formation
\[(F_{J_0}, ((\gamma_{J_0}), (\Theta)G) = (F^*_I, ((\gamma_{J_0}), (\Theta)G)\]
and $A$-module morphisms
\[\lambda_{J_0}^{J_0} = \chi^{-J}_I \ast \in \text{Hom}_A(F^*_I, F^*_J)\]
\[\lambda_J^{J_0} = \psi_J^{J_0} \ast \in \text{Hom}_A(F^*_I, F_J) \quad (J \subseteq J' \subseteq I)\]
\[\gamma_{J_0}^{J_0} = \gamma_J^{J_0} \ast \in \text{Hom}_A(F^*_I, F_J) \quad (J \subseteq J' \subseteq I)\]
\[\gamma_{J_0}^{J_0} = \gamma_J^{J_0} \ast \in \text{Hom}_A(F^*_I, F_J) \quad (J \subseteq J' \subseteq I)\]

In particular, $\gamma_{I_{J_0}}^{I_{J_0}} = 1 \in \text{Hom}_A(F^*_I, F^*_I)$ is an isomorphism for all $J \subseteq I$.

Therefore
\[\tilde{y} = \{ (Q_I, \varnothing) \ ; \ \{ (Q_J, \phi_J) \}_{J \subseteq I} \}\]
is an $n$-adj of ±-formations containing $y$. 

A lattice of $\pm$-forms $\{ (\alpha, \beta) \}_{J \subseteq I}$ is connected if
\[
\begin{pmatrix}
(\phi^{+}_{J} \pm \phi^{-}_{K}) f_{J} & 0 \\
(\phi^{-}_{J} \pm \phi^{+}_{K}) f_{J}
\end{pmatrix} : Q_{J} \oplus Q_{K} \rightarrow Q_{J \cup K} \oplus Q_{K}
\]
is a split mono for disjoint $J, K \subseteq I$.

A lattice of $\pm$-formations $\{ (F_{J}, (\mu_{J}), \theta_{J}) \}_{J \subseteq I}$ is connected if
\[
\begin{pmatrix}
\lambda^{+}_{J} & -\lambda^{-}_{J} \\
\lambda^{-}_{K} & \lambda^{+}_{K}
\end{pmatrix} : F_{J} \oplus F_{K} \rightarrow F_{J \cup K}
\]
is onto for disjoint $J, K \subseteq I$.

Lemma 4.1 now specializes to:

Lemma 4.2 A connected lattice of $\xi \pm$-formations can be embedded in a connected $n$-ad of $\xi \pm$-formations.

Proof: Let $y = \{ (\alpha, \beta) \}_{J \subseteq I}$ be a connected lattice of $\pm$-formations, and let $\tilde{y} = \{ (\alpha, \beta) \}_{J \subseteq I}$ be the $n$-ad defined in the proof of Lemma 4.1.

By Lemma 3.2, the morphism
\[
\begin{pmatrix}
(\phi^{+}_{J} \pm \phi^{-}_{K}) f_{J} & 0 \\
(\phi^{-}_{J} \pm \phi^{+}_{K}) f_{J}
\end{pmatrix} : Q_{J} \oplus Q_{K} \rightarrow Q_{J \cup K} \oplus Q_{K}
\]
\[
\begin{pmatrix}
\lambda^{+}_{J} & -\lambda^{-}_{J} \\
\lambda^{-}_{K} & \lambda^{+}_{K}
\end{pmatrix} : F_{J} \oplus F_{K} \rightarrow F_{J \cup K}
\]
is a split mono, for some disjoint $J, K \subseteq I$.

If the same is true of the corresponding morphism with $J$ or $K$ replaced by $L = I \setminus (J \cup K)$.

Therefore it is sufficient to consider the cases with $J, K \subseteq I$, where this is true by hypothesis.

Thus $\forall \tilde{y}$ is defined for all $J \subseteq I$.

$\square$
Given $i \in \Xi_0,1,23$ and a collection of $0$-ads and $1$-ads and isomorphisms

$$s = \{ (y_j, h_{jk}, y_{jk} - 2y_{jk-1}) \rightarrow \theta_{y_{jk}} \}_{0 \leq k < 15}^{j \leq 80,1,23} \cup \{ z_{jk}, -z_{jk} \}_{j \in \Xi_0,1,23} \cup \{ x_{jk} \}_{j \in \Xi_0,1,23 - 313}$$

such that for both $j \in \Xi_0,1,23 - 313$

$$s_j = \{ (y_{j0}, h_{j0}, a_{j0}) \}_{j \in \Xi_0,1,23, 0 \leq k < 15} \in \mathcal{L}_y(A)^2$$

write

$$(-1)^j y_{j0,1,23} - x_j^3 \equiv \Xi_j, \Xi_{jk}, x_{jk} - 2x_{jk-1} - x_{jk-2}$$

and let

$$g_{jk} : -z_{jk} \rightarrow z_{jk} \quad (0 \leq j < k \leq 2, j + i)$$

be the (stable) isomorphism defined by the composite

$$h_{j0,1,23} = \theta_{y_{jk}} \rightarrow \theta_{x_{jk}} \rightarrow \theta_{y_{jk}}^3.$$  

The $1$-ads $(-1)^j y_{j0,1,23}$ and $(-1)^k y_{j0,1,23}$ can be then regarded as cobordisms (of $\Xi$-forms if $f = \{ \Xi_i \}

$$c = (-1)^j y_{j0,1,23} = \Xi_j, z_{jk}, -z_{jk}, \{16,1\}_{710}$$

$$(-1)^k y_{j0,1,23} = \Xi_j, z_{jk}, -z_{jk}, \{16,1\}_{710}$$

and $g_{jk} : -z_{jk} \rightarrow z_{jk}$ defines an isomorphism of cobordisms

$$g_{jk} : C = \Xi_k, -z_{jk}, z_{jk}, 1 \rightarrow (-1)^j y_{j0,1,23}.$$  

Now $c, c'$ are adjoining cobordisms of $\Xi$-forms and Theorem $\Xi 2.4$ shows that such a pair determines a connected lattice $w = \Xi w_0 ? j \leq \Xi_j, k_3$ of $\Xi$-forms as is considered in Theorem $\Xi 2.3$ with $1$-ad (alias cobordism) isomorphisms

$$f : c \rightarrow \theta_{j0} \quad f' : c' \rightarrow \theta_{j0} \quad w$$

where $w$ is the connected $2$-ad given by Lemma 4.2. Moreover, the $0$-ad isomorphisms

$$g_{jk}, g_{jk} : \theta_{j0} y_{j0,1,23} \rightarrow \theta_{j0} y_{j0,1,23}$$

defined by commutative diagrams.
In order to establish it for \( n \geq 3 \), we shall need the following generalization of Theorem 4.2.4:

**Theorem 4.3** Let \( I \) be a set of \( n \) elements, \( n \geq 3 \), and let \( \infty \in I \). Suppose given connected \((n-1)\)-ads \( Y_i = \{x_i, z_{x_i, j} \in I - \{i\}, (i \in I, I_\omega = I \cup \{\infty\}) \) and \((n-2)\)-ad isomorphisms

\[
g_{ik} : -\partial^k y_j \rightarrow \partial^i y_k \quad (j, k \in I, i \neq k)
\]

(writing \( \partial^k y_j \) for \( \partial^{(k, \omega)} y_j \)) such that

\[
g_{ik} = -g_{ik}^{-1} : -\partial^i y_k \rightarrow -\partial^k y_j
\]

with the \((n-3)\)-ad isomorphisms

\[
e_{ijk} : -\partial^i \partial^k y_j \rightarrow \partial^i \partial^j y_k \rightarrow -\partial^i \partial^k y_k \rightarrow \partial^i y_k
\]

\[
e_{ijk} : -\partial^i \partial^k y_j \rightarrow \partial^i \partial^j y_k \rightarrow -\partial^i \partial^k y_k \rightarrow \partial^i y_k
\]

defined and coherent for distinct \( j, k, l \in I \) (writing \( z_{ik} \) for \( z_{j, i, k, l} \)).

In short, setting

\[
y_{50, 1, 23} = \tilde{w} \quad y_{5 j, k l} = \partial_j \tilde{w}
\]

\[
h_{50, 1, 23, i} = 1 : y_{5 i, k l} \rightarrow \partial_i \tilde{w}
\]

\[
h_{50, 1, 23, j} = (\partial^j f^i)(\partial^j g_{ik})^{-1} : y_{5 i, k l} \rightarrow \partial_j \tilde{w}
\]

\[
h_{50, 1, 23, k} = (\partial^k f^i) : y_{5 i, k l} \rightarrow \partial_k \tilde{w}
\]

we have defined a \( 2 \)-simplex,

\[
\Delta = \{ (Y_i, h_{5, i}) \}_{5 \leq 50, 1, 23, 0 \leq i < 13} \in L_5(A)^{(n)}
\]

containing the horn \( s \).

This verifies the Kan condition for \( n = 2 \).
Then there is defined a connected lattice
\[ w = \oplus_{j \in J} w_j \text{ with (n-1)-ad isomorphisms} \]
\[ f_j : y_j \rightarrow \mathcal{G}_j w \ (j \in J) \]
where \( \hat{w} \) is the connected n-ad given by
Lemma 4.2, such that the (n-2)-ad isomorphisms
\[ g_{jk} : - \mathcal{G}_j y_j \rightarrow \mathcal{G}_k y_k \]
\[ \tilde{g}_{jk} : - \mathcal{G}_j y_j \rightarrow \mathcal{G}_k y_k \rightarrow \mathcal{G}_k y_k \rightarrow \mathcal{G}_j y_j \]
are coherent, for distinct \( j, k \in I \).

**Proof:** For any ordering of \( J \) define n-ads
\[ y_J = \oplus_{j \in J} \oplus_{k \in I, j < k} y_j \]
for all \( J \subset I \), and isomorphisms of p-ads
\[ g_{j,k} : - \mathcal{G}_j y_j \rightarrow \mathcal{G}_k y_k \ (J, k \subset I \text{ disjoint}, j \leq k \forall j, k) \]
such that the composites
\[ c_{jk} : - \mathcal{G}_j y_j \rightarrow \mathcal{G}_k y_k \rightarrow \mathcal{G}_k y_k \rightarrow \mathcal{G}_j y_j \]
\[ \tilde{c}_{jk} : - \mathcal{G}_j y_j \rightarrow \mathcal{G}_k y_k \rightarrow \mathcal{G}_k y_k \rightarrow \mathcal{G}_j y_j \]
are defined and coherent for disjoint \( J, k \subset I \)
with \( j < k < l \) for all \( j \in J, k \in K, i \in L \).

Set
\[ y_{\{j\}} = Y, \quad g_{j,k} = g_{j,k} \ (j, k \in I, j < k) \]
Assume inductively that \( y_J, g_{j,k} \) have already been defined for some \( J \subset I \), and all \( k \in I, k \leq j, j \in J \),
with \( c_{j,k,l} \), \( \tilde{c}_{j,k,l} \) defined and coherent for distinct \( k, l \in I - J \). Now set
\[ y_{\{j\}} = y_J \times g_{j,k} y_k \ (k \in J \text{ max } j < k) \]
\[ g_{j,k,l} : - \mathcal{G}_j y_j \rightarrow - \mathcal{G}_j y_j \times g_{k,l} y_k \rightarrow \mathcal{G}_j y_j \times g_{k,l} y_k \]
where we are successively applying Theorems 3.5, 3.4, 3.3 to define \( g_{j,k,l} \), and \( (g_{j,k,l})_l \)
stands for the composite
\[ - \mathcal{G}_j y_j \rightarrow \mathcal{G}_j y_j \rightarrow \mathcal{G}_j y_j \rightarrow \mathcal{G}_j y_j \]
(exactly as in the statement of Theorem 3.5)
and the diagram
\[ \delta^k \delta^l y_J^{(-\delta^k_\lambda)} \xrightarrow{\delta^2_\delta} - \delta^l y_k \]

\[ -\delta^k \delta^l y_J \xrightarrow{\delta^2_\delta} - \delta^l y_k \]

\[ \delta^2_\delta \delta^l y_J \xrightarrow{\delta^2_\delta} - \delta^l y_k \]

commutes up to coherence (as required for \( g_{J+1} \) to be defined) because the composites

\[ e_{J,k,l} : -\delta^k y_J \xrightarrow{\delta^2_\delta} - \delta^l y_k \]

\[ \delta^2_\delta \delta^l y_J \xrightarrow{\delta^2_\delta} - \delta^l y_k \]

\[ e_{I,k,l} : -\delta^k y_J \xrightarrow{\delta^2_\delta} - \delta^l y_k \]

\[ \delta^2_\delta \delta^l y_J \xrightarrow{\delta^2_\delta} - \delta^l y_k \]

are coherent. The isomorphisms defined by the composites

\[ e_{J,\delta^k,\delta^l,m} : -\delta^k y_J \xrightarrow{\delta^2_\delta} - \delta^l y_k \]

\[ \delta^2_\delta \delta^l y_J \xrightarrow{\delta^2_\delta} - \delta^l y_k \]

\[ e_{J,\delta^k,\delta^l,m} : -\delta^k y_J \xrightarrow{\delta^2_\delta} - \delta^l y_k \]

\[ \delta^2_\delta \delta^l y_J \xrightarrow{\delta^2_\delta} - \delta^l y_k \]

are coherent, being both coherent with

\[ -\delta^m \delta^j y_J \xrightarrow{\delta^2_\delta} - \delta^m y_J \]

\[ \delta^2_\delta \delta^j y_J \xrightarrow{\delta^2_\delta} - \delta^m y_J \]

writing \( \delta^m (\delta^j) \) for the isomorphism given by a double application of Theorem 3.5, and using the naturality properties given in Corollary 3.6 to ensure that \( e_{J,\delta^k,\delta^l,m} \) is defined, where \( k, l, m \in I - J \) are such that

\[ \max \{ j \in J : j < k < l \} < m \]

This completes the induction step, so we have defined \( y_J, g_{J,k}, e_{J,k,l} \). There is then a similar inductive construction for \( g_{J,k} \) given by

\[ g_{J,k,\delta^j} : -\delta^k \delta^j y_J \xrightarrow{\delta^2_\delta} - \delta^k y_J \]

\[ \delta^2_\delta \delta^j y_J \xrightarrow{\delta^2_\delta} - \delta^k y_J \]

with \( e_{J,k,l}, e_{J,k,l} \) defined and coherent.
Theorem 3.7 now shows that a different ordering of $I$ leads to essentially the same $y_j, g_{j,k}, e_{j,k,}\ell$ whenever there is overlap in the range of definitions. These are now defined for all disjoint $J,K,L\subseteq I$.

For disjoint $J,K\subseteq I$ we have defined cobordisms

$$
C = \Sigma x_j, x_{j,\ell}, -x_{j,k}, 1, 1^3,
$$

$$
C' = \Sigma x_k, -x_{j,k}, x_{k,\ell}, 3^3,
$$

$$
C'' = \Sigma x_{j,k}, x_{j,\ell}, x_{k,\ell}, ?, ?, 3
$$

such that $C \times C' = C''$ (by construction).

Theorem $\S 1.4$ now shows how to consider $x_j, x_k \subseteq x_{j,k}$ as orthogonal $\Sigma$ subforms $\Sigma$ subformations satisfying the condition for $\{x_{j}, x_{j,k}, x_{k}, x_{j,k}\}$ to be a connected lattice of $\Sigma$ subformations, with $x_{j,\ell}$ a trivial subformation where $x_{j,\ell} = 0$.

been also used to denote a (stable) isomorphism. Therefore $w = \Sigma x_j x_j \subseteq I$ is a connected lattice.

Isomorphisms $f_i : y_j \rightarrow \partial_i^i w$ (i.e., $I$) now follow on noting that

$$
\partial_i^i w = \Sigma x_j, \Sigma x_j, x_j \subseteq I \subseteq x_j (i.e., I)
$$

from the construction of Lemma 4.1.

The coherence of

$$
g_{i,k} : -\partial_i y_j \rightarrow \partial_i y_k $$ (i.k \in I, j+k)

$$
g_{j,k} : -\partial_j y_j \rightarrow -\partial_j \partial_j w \rightarrow \partial^1 \partial^1 w \rightarrow \partial^1_1 y, \quad \partial^1_1 w
$$

follows from the identity

$$
y_{j,k} = y_j \ast g_{j,k} y_k = \partial^1_{j,k} w
$$

$\square$
let \( i \in I = \{0, 1, \ldots, n\} \), \( n > 3 \).

Suppose given a collection
\[
S = \{ (y_{ij}, h^i_{jk}) : \mathbb{R}^m \xrightarrow{\partial} y_{ij} \} \big| 0 \leq k < |j| < |i|
\]
of oriented highly connected \( m \)-ads \( y_{ij} \) of fibre dimension \( f \), and isomorphisms \( h^i_{jk} \), such that
\[
S = \{ (y_{ij}, h^i_{jk}) \} \big| \mathbb{R}^{m-1} \xrightarrow{\partial} y_{ij} \in \mathfrak{L}_{\partial}(A)^{(n-1)}
\]
(j \( \in I - i3 \))

Define isomorphisms of \( (n-2) \)-ads
\[
g_{jk} : \mathbb{R}^{m-1} \xrightarrow{\partial} y_{ij} \rightarrow \partial_j y_{ij} \quad (j, k \in I - i3)
\]
by requiring the commutativity of
\[
\begin{array}{c}
h_{ij, k-1} \\
y_{ij, l} \\
h_{ik, j} \end{array} \xrightarrow{\partial_{k-1} y_{ij, l}} \xrightarrow{\partial_j y_{ij, l}} \xrightarrow{\partial_j h_{ik, j}}
\]

The diagrams
commute up to coherence, showing that 
\[ e_{j,k,l}, e_{j,k,l} : \partial_{l} \partial_{k} \partial_{j} y_{i_{1}i_{2}i_{3}} \rightarrow \partial_{j} \partial_{k} y_{i_{1}i_{2}i_{3}} \]
are well-defined coherent isomorphisms. Therefore we can apply Theorem 4.3 to obtain a connected lattice \( W = \Sigma W_{3} x I \), and isomorphisms
\[ f_{j} : y_{i_{1}i_{2}i_{3}} \rightarrow \partial_{j} W \quad (j \in I - i_{3}) \]
(setting \( a = i_{3} \)), such that the composite
\[ f_{j} \circ \partial_{k} \circ y_{i_{1}i_{2}i_{3}} \rightarrow \partial_{j} \partial_{k} W \]
is coherent with \( g_{j,k} \) (for \( j,k \in I - i_{3} \) with \( j \neq k \)).

Setting
\[ y_{i_{1}i_{2}i_{3}} = \tilde{W} \quad y_{i_{1}i_{2}i_{3}} = \partial_{i_{1}} \tilde{W} \]
\[ h_{i_{1}i_{2}i_{3}, i_{1}} : y_{i_{1}i_{2}i_{3}} - \partial_{i_{1}} \tilde{W} \]
\[ h_{i_{1}i_{2}i_{3}, i_{3}} = \tilde{f}_{j} : y_{i_{1}i_{2}i_{3}} - \partial_{i_{3}} \tilde{W} \quad (j \in I - i_{3}) \]
we have defined an \( n \)-simplex
\[ S = \Sigma (y_{i_{1}i_{2}i_{3}}, h_{i_{1}i_{2}i_{3}, i_{1}}) \in \mathfrak{L}_{f}(A) \]
containing the hom \( s \).

This completes the verification of the Kan condition.

We can now consider the homotopy theory of \( \mathfrak{L}_{f}(A) \).

- The homotopy groups are defined as usual, by
\[ \pi_{n}(\mathfrak{L}_{f}(A)) = \{ s \in \mathfrak{L}_{f}(A)^{m+1} \mid \partial_{i}s = 0 \in \mathfrak{L}_{f}(A)^{m} \}, 0 < i < n \]
where
\[ s = s' \iff \exists t \in \mathfrak{L}_{f}(A)^{m+1} \text{ s.t. } \partial_{i}t = \begin{cases} s & \text{if } i = 0 \\ s' & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases} \]
with the group law given by
\[ \left[ s \right] \times \left[ s' \right] = \left[ s' \right] \in \pi_{n}(\mathfrak{L}_{f}(A)) \]

Recalling the definition of the groups \( U_{x}(A) \) from §3 of I, we have:
Theorem 4.4 \( \pi_n(\mathbb{L}_f(A)) = \mathbb{U}_{n+f}(A) \) \((n \geq 0)\).

Proof: The case \( n = 0 \) was considered on III 4.7, so assume \( n > 1 \), and let \( I = \{0, 1, \ldots, n\} \).

An \( n \)-simplex
\[
S = \{ (y, h_{i,k} : y = \xi_0 y_i \mapsto \partial_k y) | \exists I, \exists k < |I|, \exists y \in \mathbb{L}_f(A)^{(n)} \}
\]
such that \( \partial_k s = 0 \in \mathbb{L}_f(A)^{(n-1)} \) for all \( k \in I \), is determined by an oriented highly connected \( n \)-ad of fibre dimension \( F \)
\[
Y = y \times \mathbb{S} \times \mathbb{I} \times \mathbb{I}
\]
and \( (n-1) \)-ad isomorphisms
\[
h_k \equiv h_{i,k} : 0 \to \partial_k y \quad (k \in I).
\]

Each
\[
\partial^j y = \{ x_j : \xi_0 x_j \in \mathbb{L}_f(A)^{(n)} \}
\]
is isomorphic to the zero \((n-1)\)-ad of \( \sum \) forms, on
\[
\text{if } n + f = \left\{ \begin{array}{ll}
2i, & i > 1, \\
2i + 1, & i = 0
\end{array} \right. \]
Thus each \( x_j \) is a trivial \( \sum \) form, with each \( x_{j,k} \), the zero \( \sum \) form. It now follows from Theorem 3.4 that each \( x_j \) is a trivial \( \sum \) form, the zero \( \sum \) form.

In particular, this is true of \( x_i \), an isomorph of \( \partial_i x \), so that \( x \) is a non-singular \( \sum \) form.

\[
\bigcup_{k \in I} \ker(\partial_k : \mathbb{L}_f(A)^{(n)} \to \mathbb{L}_f(A)^{(n-1)})
\]
is determined not only by the non-singular \( \sum \) form \( x \), but also by the \((n-1)\)-ad isomorphisms \( h_k : 0 \to \partial_k y \quad (k \in I) \). We shall now show that in fact this extra structure does not enter the homotopy groups.

Note first that given any trivial \( \sum \) form \((F, (\xi, \delta)G)\) there is defined an isomorphism
\[
(\mu^*, 1, 0) : (F, (\xi, \delta)G) \to (\mathbb{S}^*, (\xi^*, \delta^*)G)
\]
Given a stable isomorphism of trivial formations

\[ [\alpha, \beta, \psi] : (F, (\gamma^0), \theta)G \rightarrow (F', (\gamma^0'), \theta)G' \]

we can characterize the coherence class of the composite

\[ (G^*, ((\gamma^0)^*, \theta)G) \rightarrow (\gamma^0, (\mu^*), \theta)G \]

\[ \rightarrow (F', ((\gamma^0'), \theta)G') \rightarrow (G^*, ((\gamma^0)^*, \theta)G) \]

as follows. Let

\[ (\alpha, \beta, \psi) = ((a, a^*), (b, b^*), (s, s^*)) \]

be any representative of \([\alpha, \beta, \psi]\) (i.e. use the notation of the proof of Theorem 2.1). Then the normalization of the composite above is given by

\[
\left( \begin{array}{cccc}
\mu^*a & \mu^*a & b^* \\
1 & -b^* & 0 \\
0 & 0 & 0 \\
\end{array} \right), \left( \begin{array}{cccc}
b \end{array} \right), \left( \begin{array}{cccc}
\mu^*s & \mu^*s & \mu^*s & \mu^*s \\
0 & 0 & 0 & 0 \\
\end{array} \right)
\]

This is just the isomorphism (and, indeed, the normalization of) given by Theorem 2.3, when applied to the orthogonal subforms

\[(G, \theta) \leq (G \otimes G, (\theta \otimes \theta)), \quad (G', \theta') \leq (G' \otimes G', (\theta' \otimes \theta'))\]

with the choice of direct complements given by the configuration

\[ (a, f) = \left( \begin{array}{c}
a \\
f(a) \\
\end{array} \right), \quad (b, f) = \left( \begin{array}{c}
b \\
f(b) \\
\end{array} \right) \]

\[ G \otimes G \rightarrow G \otimes G', \quad G' \otimes G \rightarrow G' \otimes G' \]

\[ (\gamma) = \left( \begin{array}{c}
\frac{1}{2} \\
\gamma(a) \\
\end{array} \right), \quad (\gamma') = \left( \begin{array}{c}
\frac{1}{2} \\
\gamma'(b) \\
\end{array} \right) \]

\[ (\beta) = \left( \begin{array}{c}
b \\
\beta(a) \\
\end{array} \right), \quad (\beta') = \left( \begin{array}{c}
b' \\
\beta'(b) \\
\end{array} \right) \]
Returning to the problem in hand, suppose given two simplexes
\[ s, s' \in \bigcap_{k \in I} \ker(\partial_k : L_f(A)^m \to L_f(A)^{m-n}) \]
with \( s \) (resp. \( s' \)) defined by the \( n \)-ad
\[ y = \xi(G, \theta) ; \xi(F_t, ((x^i_t), \theta)G) \mid_{j \in I} \]
(resp. \( y' = \xi(G', \theta') ; \xi(F_{t'} , ((x^i_{t'}), \theta')G') \mid_{j \in I} \)).

and \( (n-1) \)-ad isomorphisms \( h_k : 0 \to \partial_k y \) (resp. \( h_k' : 0 \to \partial_k y' \)) \( (k \in I) \), with the non-singular \( \pm \) forms \( (G, \theta), (G', \theta') \) isomorphic.

For any isomorphism \( (f, \chi) : (G, \theta) \to (G', \theta') \),
\[ \text{im}(\xi(((\frac{i}{f}), \chi) : (G, \theta) \to (\mathbb{G} \otimes G', (\theta^i \theta')))) \]
defines a lagrangian of \( (G, \mathbb{G} \otimes G', \theta') \), which is therefore a trivial \( \pm \) form, isomorphic to \( H^\pm(G) \). We can therefore define an

\[(n+1)-\text{ad of } \pm \text{forms}
\[ z = \xi(G, \theta) ; \xi(Q_j, \phi_j) \mid_{j \in I} \]
by
\[(Q_j, \phi_j) = \begin{cases} 
0 & 0 \notin J, 1 \in J \\
(G, \theta) & 0 \in J, 1 \notin J \\
(G', \theta') & 0 \in J, 1 \notin J \\
\mathbb{G} \otimes G', (\theta^i \theta') & 0 \notin J
\end{cases} \]
(J \in I)_\infty
with
\[(f_{j^*}, \chi^*_{j^*}) = \begin{cases} 
0 & c_{i \notin J} \\
((\frac{i}{f}), \chi) : (G, \theta) \to (\mathbb{G} \otimes G', (\theta^i \theta')) & c_{i \in J} \notin J \\
((\frac{i}{f}), \chi) : (G', \theta') \to (\mathbb{G} \otimes G', (\theta^i \theta')) & c_{i \in J} \in J \\
1 & 0 \notin J
\end{cases} \]
(J \in I)_\infty

Defining a collection of \( \pm \text{-ads and isomorphisms}
\[ t = \xi(y_j, h_{j^*} : y_j \otimes \omega_0 \mid_{j \in I}) \mid_{j \in I} \]
for \( 0 \leq k, n \leq 1 \).
by

\[ y_J = \begin{cases} 
  z & J = I_\infty \\
  y' & J = I_\infty - \{0\} \\
  y & J = I_\infty - \{1\} \\
  0 & \text{otherwise}
\end{cases} \]

\[ h_{J,k} = \begin{cases} 
  \varepsilon(1,0), \varepsilon(\mu_k^* \lambda_k I) : y' \to \partial_0 z & J = I_\infty, k = 0 \\
  \varepsilon(1,0), \varepsilon(\mu_k^* \lambda_k I) : y \to \partial_1 z & J = I_\infty, k = 1 \\
  h_k : 0 \to \partial_k y' & J = I_\infty - \{0\}, k \in I \\
  h_k : 0 \to \partial_k y & J = I_\infty - \{1\}, k \in I \\
  0 & \text{otherwise}
\end{cases} \]

Note that the diagrams commute up to coherence: certainly, this

is clear when $J \neq I_\infty$. For $J = I_\infty$, it follows from the remarks made above, which ensure that for each $Jk$ there is an appropriate choice of direct complements in the definition of $T_{j,k}$ (as in the proof of Theorem 3.4, and hence Theorem 2.3) to make the corresponding diagram commute up to coherence.

Now $t \in \bigcap_{k \in I_{\infty} \setminus \{0\}} \ker(\partial_k : L_p(A)^{\langle n+1 \rangle} \to L_p(A)^{\langle n \rangle})$

defines (an $(n+1)$-simplex which is) a homotopy between $s$ and $s'$, so that

\[ [s] = [s'] \in \Pi_n(L_p(A)) \]

and it is only the (isomorphism class) of the $\pm$-form $(\gamma, \delta)$ which matters.
Corollary 1.2 now shows that
\[ \pi_n(L_f(A)) = U_{2i+1}(A) \]
as sets: the group laws coincide by Lemma 1.5 and Theorem 3.4.

If \( n + f = 2i + 1 \), so \( n \in \ker(\partial_k : \mathcal{L}_f(A)^m \to \mathcal{L}_s(A)^m) \)
is determined solely by the non-singular formation \( x \). Corollary 2.2 shows that
\[ \pi_n(L_f(A)) = U_{2i+1}(A) \]
as sets: the group laws coincide by Lemma 2.5 and Theorem 3.4.

Given a Kan \( \Delta \)-set \( \mathcal{X} \) we can define its "loop space" \( \Omega \mathcal{X} \) to be the Kan \( \Delta \)-set given by
\[ (\Omega \mathcal{X})^m = \{ s \in \mathcal{X}^{n+1} \mid \partial_{n+1} s = 0 \in \mathcal{X}^m \} \quad (n \geq 0) \]
\[ \partial_i : (\Omega \mathcal{X})^m \to (\Omega \mathcal{X})^{m-1}, \quad s \mapsto \partial_i s \quad (0 \leq i \leq n, n \geq 1) \]

Then
\[ \pi_n(\Omega \mathcal{X}) = \pi_{n+1}(\mathcal{X}) \]
as usual.

**Corollary 4.5** There are defined homotopy equivalences
\[ L_{f+1}(A) \to \Omega L_f(A) \quad (f \mod 4) \]
in a natural fashion.

**Proof:** Given an oriented \( n \)-ad of fibre dimension \( f \)
\[ y = \exists x; \exists x_3 \exists i \in I \quad (I = \{0,1,\ldots,n\}) \]
we shall define an oriented \((n+1)\)-ad of fibre dimension \( f+1 \)
\[ y_0 = \exists x; \exists x_3 \exists i \in I_0 \quad (I_0 = \{0,1,\ldots,n+2\}) \]
by "introducing an extra vertex", such that
\[ \forall y_0 = (y_{i,x}, y_{i+2,x}) \ (x \in \text{even}) \]
there is defined an isomorphism of \textit{n-ads}

\[ \omega_{\infty} : 0 \longrightarrow \vartheta_{\infty} y_{\infty} \]

The operation will be natural in that an isomorphism of oriented \textit{n-ads}

\[ h : y \longrightarrow y' \]

will induce an isomorphism

\[ h_{\infty} : y_{\infty} \longrightarrow y'_{\infty} \]

of oriented \textit{(n+1)-ads}.

If

\[ y = \mathcal{E}(F, ((\gamma_{\infty}), (\theta)G)) ; \mathcal{E}(Q_{\infty}, \Phi_{\infty}) \] \[ \forall \mathcal{E}I \] is an \textit{n-ad} of \textit{\pm forms}, let

\[ y_{\infty} = \mathcal{E}(F, ((\gamma_{\infty}), (\theta)G)) ; \mathcal{E}(Q_{\infty}, \Phi_{\infty}) \] \[ \forall \mathcal{E}I \] extending the definitions by setting

\[ (Q_{\infty}, \Phi_{\infty}) = (Q, \Phi) \quad (\mathcal{E}I, J_{\infty} = \mathcal{E}I \cup \mathcal{E}(n+1)) \]

\[ (f_{\infty}^{\mathcal{E}}, \chi_{\infty}^{\mathcal{E}}) = (f_{\mathcal{E}}^{\mathcal{E}I}, \chi_{\mathcal{E}I}^{\mathcal{E}}) \quad (\mathcal{E}I, J_{\infty} = \mathcal{E}I \cup \mathcal{E}(n+1)) \]

The definitions of \( \omega_{\infty}, h_{\infty} \) are obvious.

If

\[ y = \mathcal{E}(G, (\theta)G) ; \mathcal{E}(F, ((\gamma_{\infty}), (\theta)G)) \]

is an \textit{n-ad} of \textit{\pm formations}, let

\[ y_{\infty} = \mathcal{E}(G, (\theta)G) ; \mathcal{E}(F, ((\gamma_{\infty}), (\theta)G)) \]

extending the definitions by setting

\[ (F_{\infty}, (\gamma_{\infty}), (\theta)G) = (F, ((\gamma_{\infty}), (\theta)G)) \quad (\mathcal{E}I) \]

\[ \lambda_{\mathcal{E}I}^{\mathcal{E}I} = \lambda_{\infty}^{\mathcal{E}I} = \lambda_{\mathcal{E}I}^{\mathcal{E}I} \in \text{Hom}_{A}(F_{\mathcal{E}I}, F_{\mathcal{E}I}) \quad (\mathcal{E}I \subset \mathcal{E}I) \]

\[ \lambda_{\mathcal{E}I}^{\mathcal{E}I} = \lambda_{\mathcal{E}I}^{\mathcal{E}I} \in \text{Hom}_{A}(F_{\mathcal{E}I}, F_{\mathcal{E}I}) \quad (\mathcal{E}I \subset \mathcal{E}I) \]

Then

\[ \vartheta_{\infty} y_{\infty} = \mathcal{E}(F, ((\gamma_{\infty}), (\theta)G)) ; \quad \mathcal{E}I \]

\[ = \mathcal{E}(G^{*}, ((\gamma_{\infty}), (\theta)G)) ; \quad \mathcal{E}I \]

and

\[ \omega = \mathcal{E}[(1, 1, (\theta)G) \longrightarrow (G^{*}, ((\gamma_{\infty}), (\theta)G))] ; \quad \mathcal{E}I \]

\[ : \sigma \longrightarrow \vartheta_{\infty} y_{\infty} \]

defines an isomorphism of \textit{n-ads}.

The definition of \( h_{\infty} \) is obvious.
Define a $\Delta$-map

$$\mathbb{L}_{f+1}(A) \longrightarrow \Omega \mathbb{L}_f(A)$$

for each $f \mod 4$ by setting

$$\mathbb{L}_{f+1}(A)(n) \longrightarrow (\Omega \mathbb{L}_f(A))(n),$$

$$\xi(\tilde{y}_f,h_{j,k}: \tilde{y}_f - \tilde{y}_{j+1}(k)) \longrightarrow \partial_k \tilde{y}_f \big|_{j \in I, \ 0 \leq k < 1\mathbb{I}} \big)

\quad \longrightarrow \{(\tilde{y}_f, \tilde{h}_{j,k}: \tilde{y}_f - \tilde{y}_{j+1}(k)) \longrightarrow \partial_k \tilde{y}_f \big|_{j \in I, \ 0 \leq k < 1\mathbb{I}} \big) \}

$$

with

$$\tilde{y}_f = \begin{cases} (y_f)_\infty & \text{if } \tilde{f} = J_\infty, J \in I \text{ non-empty} \\ 0 & \text{otherwise} \end{cases}$$

$$\tilde{h}_{j,k} = \begin{cases} (h_{j,k}: y_f - y_{j+1}(k))_\infty & \text{if } \tilde{f} = J_\infty, J \in I \big| \mathbb{I}, k = \mathbb{I}, 0 \leq k < 1\mathbb{I} \\ 0 & \text{otherwise} \end{cases}$$

Applying Theorem 4.4 it should be clear that these $\Delta$-maps induce the identity isomorphisms

$$U_{n+f+1}(A) \longrightarrow U_{n+f+1}(A)$$

in the homotopy groups

$$\pi_n(\mathbb{L}_{f+1}(A)) \longrightarrow \pi_n(\Omega \mathbb{L}_f(A)).$$

The Kan $\Delta$-set analogue of Whitehead’s theorem (appearing as Theorem 6.6 in Rezk & Sandersson “$\Delta$-sets I: Homotopy theory” Quart. J. Math. Oxford (2), 22(1971), 321-332) now shows that we are dealing with homotopy equivalences

$$\mathbb{L}_{f+1}(A) \longrightarrow \Omega \mathbb{L}_f(A).$$

Naturality (w.r.t. $A$) obvious. □

Iterating four times we have natural homotopy equivalences

$$\Omega^4 \mathbb{L}_f(A) \cong \mathbb{L}_f(A) \big| \mathbb{I}, 0 \mod 4 \big)$$

giving a purely algebraic interpretation to the

...
Given a $\Delta$-map of Kan $\Delta$-sets
$$h: \mathcal{K} \to \mathcal{L}$$
we can define its "path space" to be the Kan $\Delta$-set $P_h$ with
$$P_h^{(n)} = \{(x,y) \in \mathcal{K}^{(n)} \times \mathcal{L}^{(n+1)} | f_x = \partial_n y \in \mathcal{L}^{(n)}\}$$
$(n \geq 0)$
$$\partial_i: P_h^{(n)} \to P_h^{(n-1)}; \ (x,y) \mapsto (\partial_i x, \partial_i y)$$
$(0 \leq i \leq n)$.

Defining $\Delta$-maps
$$\alpha: P_h \to \mathcal{K}, \ \beta: \Omega \mathcal{L} \to P_h$$
by
$$\alpha: P_h^{(n)} \to \mathcal{K}^{(n)}; \ (x,y) \mapsto x$$
$$\beta: \Omega \mathcal{L}^{(n)} \to P_h^{(n)}; \ z \mapsto (0, z)$$
$(z \in \mathcal{L}^{(n)}), \ \partial_n z = \alpha z$.

We have a homotopy exact sequence
$$\ldots \to \pi_n(P_h) \xrightarrow{\alpha} \pi_n(\mathcal{K}) \xrightarrow{h} \pi_n(\mathcal{L}) \xrightarrow{\beta} \pi_n(P_h) \xrightarrow{\alpha} \ldots$$
as usual.

**Corollary 4.6** Let
$$h: A \to B$$
be a $1$-preserving morphism of rings with involution. There is defined a Kan $\Delta$-set $L_f(h)$ for each $f(\text{mod} 4)$ with $\Omega L_f(h) \cong L_{f+2}(h)$, and $\Delta$-maps
$$\alpha: L_f(h) \to L_f(A), \ \beta: \Omega L_f(B) \to L_f(h)$$
inducing homotopy exact sequences
$$\ldots \to U_{n+1}(A) \xrightarrow{h} U_{n+1}(B) \xrightarrow{\beta} U_{n+1}(h) \xrightarrow{\alpha} U_{n+1}(A) \to \ldots$$
where $\pi_n(L_f(h)) = U_{n+1}(h)$ (definition).

**Proof:** The morphism
$$h: A \to B$$
induces a $\Delta$-map
$$h: L_f(A) \to L_f(B) \ (f(\text{mod} 4))$$
by the definitions of $\Sigma 6$ in the obvious way.

Let $L_f(h)$ be the path space of this $\Delta$-map, and apply Theorem 4.4 & Corollary 4.5.
(Alternatively, we could have defined the relative U-groups by generalizing the formulation of Theorem 3.1 of I. Given a morphism of ground rings

\[ h : A \rightarrow B \]

let \( X_n(h) \) be the abelian monoid of isomorphism classes of pairs \((x, c)\) with \( x \in \text{form} \) over \( A \)

\[ n = \begin{cases} 2i & \text{if } i \geq 0 \\ 2i + 1 & \text{if } i < 0 \end{cases} \]

and \( c = \xi y \circ z \cdot h x, g x, g y \).

A cobordism of \( \xi \text{-formations} \) over \( B \) from \( h x \)

(also a \( 1\)-ad \( c \) with \( c, c = 10x \)), an isomorphism of such pairs

\[ (e, f) : (x, c) \rightarrow (x', c') \]

being defined by isomorphisms \( e : x \rightarrow x', f : c \rightarrow c' \)

such that the square

\[
\begin{array}{ccc}
  h x & \xrightarrow{g} & c \\
  \downarrow h y & & \downarrow f \\
  h x' & \xrightarrow{g} & c'
\end{array}
\]

commutes up to cobordism.

The monoid law is the direct sum

\[(x, c) \oplus (x', c') = (x \oplus x', c \oplus c')\]

with \((0, 0)\) as zero. Boundary maps

\[ \partial : X_n(h) \rightarrow X_{n-1}(h) ; \]

\[ (x, c \circ l y \circ z \cdot h x, g x, g y) \rightarrow (\partial x, \partial y, h x, g x, 0, 0) \]

such that \( \partial^2 = 0 \) may be defined, with

\[ \tau : \partial \partial c \rightarrow \partial \partial c \]

as given by Theorem 3.4.

We can then define an abelian group

\[ U_n(h) = \ker(\partial : X_n(h) \rightarrow X_{n-1}(h)) / \im(\partial : X_{n+1}(h) \rightarrow X_n(h)) \]

The exact sequences

\[ \ldots \rightarrow U_n(A) \xrightarrow{h} U_n(B) \xrightarrow{\beta} U_n(h) \xrightarrow{\alpha} U_{n-1}(h) \xrightarrow{h} \]

are induced by chain maps

\[ \alpha : X_n(h) \rightarrow SX_n(A) \quad \text{with} \quad \alpha : X_n(h) \rightarrow SX_n(A) \equiv X_{n+1}(A), \quad \alpha : (x, c) \rightarrow \alpha \]

\[ \beta : X_n(A) \rightarrow X_n(h) \quad \beta : X_n(A) \rightarrow X_n(h) \quad \beta : (x, \xi, \delta, c, s, e) \rightarrow \]

Of course, the two definitions of \( U_n(h) \) coincide.
By considering stably fg-free (not based) $A$-modules instead of fg-projective ones, we can define functors

$L^V_f$ (resp. $L^W_f$) : rings with involution $\rightarrow$ Kan $\Delta$-sets

for $V$- (resp. $W$-) theory, generalizing the definition of $L^U_f = L_f$ for $U$-theory, with

$\pi_n(L^V_f(A)) = V_{n+f}(A)$ (resp. $\pi_n(L^W_f(A)) = W_{n+f}(A)$).

Theorem 4.3 of I. can be regarded as a calculation of

$\pi_n(L^V_f(A) \rightarrow L^V_f(A)) \simeq \Sigma_n^{U+f}(A)$

where $L^V_f(A) \rightarrow L^V_f(A)$ is the $\Delta$-map defined by the inclusion (stably fg-free $A$-modules) $\subseteq$ (fg-projective $A$-modules).

Theorem 5.7 of I. calculates

$\pi_n(L^W_f(A) \rightarrow L^V_f(A)) \simeq \Delta_n^{U+f}(A)$

where $L^W_f(A) \rightarrow L^V_f(A)$ is the $\Delta$-map defined by the forgetful map (based $A$-modules) $\rightarrow$ (fg-free $A$-modules).

Theorem 1.4 of II calculates

$\pi_n(L^V_f(A) \xrightarrow{\Sigma} L^V_f(A_0)) \simeq U_{n+f}(A)$

$\pi_n(L^W_f(A) \xrightarrow{\Sigma} L^W_f(A_0)) \simeq V_{n+f}(A)$.
An algebraic formulation of surgery

by A. A. Ranicki

Summary

The subject of my dissertation is some pure algebra to which I was led by a study of the invariants of manifolds, which are higher-dimensional analogues of curves and curved surfaces. The surgery referred to in the title is the name of a recently developed technique of cutting up manifolds. In order to make the abstractness of the work more palatable, it should be noted that it was mathematical physics which first motivated mathematicians towards this kind of geometry.

The differential calculus was invented by Newton (ca. 1666) as a mathematical tool with which to handle equations of motion, particularly those of the planets. The formulation of the equations was then generalized and perfected by Lagrange (1787) and Hamilton (1835). It turned out that the set of equations governing the motion of a dynamical system with more than two bodies does not admit a complete solution: while every set of initial conditions does determine a solution, it is not in general possible to work out that solution, except approximately, by numerical methods.

Poincaré (1880) developed new ways of obtaining information about the orbits which do not require a complete solution—trying to answer general questions such as

- Will the planets ever collide?
- Rather than particular ones such as
- Where will the planets be tomorrow at noon?

Instead of attempting a quantitative solution.
for a particular set of initial data, he described some of the qualitative features of all the solutions for all possible initial data. Essential use was made of the concept of an n-dimensional manifold introduced earlier by Riemann (1854).

Supposing that the initial data are given by n numbers (for at least such n), it is possible to consider the corresponding solution as a point of a geometrical object (the manifold in question) in which there is a notion of distance in the neighbourhood of each point, a small change in the initial data leading to only a small shift along the manifold. For example, a 1-dimensional manifold is just a curve, and a 2-dimensional manifold is a curved surface. Poincaré went on (in about 1895) to study manifolds for their own sake, thus founding the branch of pure mathematics now known as topology.

There are far too many manifolds for it to be feasible to describe them all. As indicated above, this would be the same as asking for the solution of all differential equations—an impossible task. The best one can hope for is a classification according to some scheme under which manifolds sharing certain properties are grouped together. For the purposes of topology two manifolds are held to be the same if they are homeomorphic, that is, if one can be distorted into the other by a stretching-compression, but without tearing, and in such a way as to keep distinct points apart.

"Rubber-sheet geometry" is the traditional description of the subject. For example, any two closed curves (without intersections) are homeomorphic.

\[
\begin{align*}
\text{as is clear to anyone who has played with a rubber band.}
\end{align*}
\]
A more mathematical description of when two manifolds are homeomorphic runs as follows:

A map \( f \) from a manifold \( M \) to another such \( N \), is a transformation which assigns to each point \( x \) of \( M \) its image \( f(x) \), a point of \( N \), a small change in \( x \) leading to only a small change in \( f(x) \). It is denoted by \( f: M \rightarrow N \), and may be pictured as

Given maps \( f: M \rightarrow N \), \( g: N \rightarrow P \), there is defined a map

\[ gf: M \rightarrow P \]

by sending each point \( x \) of \( M \) to the image \( g(f(x)) \) in \( P \) of its image \( f(x) \) in \( N \).

Two manifolds \( M, N \) are homeomorphic precisely when they are related by maps

\[ f: M \rightarrow N, \quad g: N \rightarrow M \]

such that

\[ gf = 1_M: M \rightarrow M, \quad f g = 1_N: N \rightarrow N \]

where

\[ 1_M: M \rightarrow M \]

is the map from \( M \) to \( M \) taking each point back to itself, and similarly for \( 1_N: N \rightarrow N \).

Even when one identifies homeomorphic manifolds, there are still a lot of them. It is possible to classify (up to homeomorphism) only 1- and 2-dimensional manifolds. The requirements have to be reduced yet further.

Call two maps

\[ f: M \rightarrow N, \quad f': M \rightarrow N \]

homotopic if it is possible to slide along \( N \) from the image \( f(x) \) to the image \( f'(x) \) of each point.
$x$ of $M$ in such a way that the path chosen varies only a little for small changes of $x$:

Homotopic maps are denoted by $f \simeq f' : M \rightarrow N$.

Two manifolds are homotopic when they are related by maps (called homotopy equivalences)

$$f : M \rightarrow N, \quad g : N \rightarrow M$$

such that

$$gf \simeq 1_M : M \rightarrow M, \quad fg \simeq 1_N : N \rightarrow N.$$  

Less mathematically, two manifolds are homotopic when one can be distorted into the other by a deformation which may be more violent than that allowed for a homeomorphism: it is no longer required that distinct points be kept apart. For example, the M"obius band

(obtained by introducing a twist into an ordinary band) is homotopic to a circle

by squeezing it down to one.

Homeomorphic manifolds are homotopic: the last example above shows the converse to be false. This is because the M"obius band is a two-dimensional manifold, whereas the circle is one-dimensional: it is a non-trivial theorem of topology, first proved by Brouwer (1911) that an $m$-dimensional manifold cannot be homeomorphic to an $n$-dimensional manifold, if $m$ is different from $n$. 

It is the aim of *algebraic topology* to classify manifolds up to homotopy, by first reducing the problem to algebra, and then doing it.

The first such reduction is due to Poincaré: given a manifold $M$ there is defined its *fundamental group*, $\pi(M)$, an algebraic entity which measures how complicated $M$ is. It is defined by considering how many different non-homotopic maps

$$(\text{Circle}) \rightarrow M$$

there are. It is a tractable enough quantity to be computable. Homotopic manifolds have the same fundamental groups, so that manifolds with different fundamental groups cannot be homotopic. For example, the surface of a doughnut

is not homotopic to the surface of a sphere

For that reason, unfortunately, it is possible to have manifolds with the same fundamental group which are not homotopic.

Carrying on from there, algebraic topologists have produced a host of ever finer such algebraic invariants with which to tackle the homotopy classification of manifolds. Up to 1960 it seemed that the difficulty of such a classification of $n$-dimensional
manifolds increased with \( n \). In that year came a remarkable discovery of Smale: it is possible to break up manifolds of dimension six and above into handles, which are manifolds of a particularly simple kind. This made the study of high-dimensional manifolds very much easier: the cases \( n = 3, 4, 5 \) remain difficult. From now on, all manifolds will be assumed to have dimension six and over, and to be "compact", which means not too large, and "framed", which means not too twisted.

**Surgery obstruction theory** is the investigation of the homotopy properties of manifolds by means of their handle decompositions. It does this by supplying an answer, in every possible case, to the question:

given a map of manifolds of the same dimension \( f: M \rightarrow N \) is it possible to make \( f \) into a homotopy equivalence by performing a sequence of surgeries on \( M \)? or (more brutally) can we kill the homotopy of \( f: M \rightarrow N \) by surgery?

A surgery on \( M \) is the addition or removal of a handle.

A research paper of Kervaire and Milnor (1963) reduced the problem to one in the algebraic theory of quadratic forms, and solved it, in the special case when \( \pi(N) \) is trivial. Wall (1965) went on to consider the general case, for any \( \pi(N) \). He showed that to answer the question posed it is necessary as well as sufficient to consider the surgery obstruction group \( L^n(\pi) \), which can be defined purely algebraically (or by quadratic forms) with:

\[
\pi = \pi(N), \quad n = \text{dimension of } N
\]
Wall gave an account of his surgery obstruction theory and some of its important applications in his book "Surgery on compact manifolds"

However, to quote the author,
"The algebra in this work, particularly in §6, is complicated, and even so it is not altogether satisfactory: most obviously, in §8."

Further on:
"... an independent algebraic treatment gives a payoff in the topology too: topologically motivated results like (12.6), proved in an algebraic setting, can apply more generally, and lead to new results with a different topological application."
Theorem 12.6 is a description of the groups $\pi_1(\mathbb{S}^k)$ in the special case when $N$ is made up of circles; the surface of a doughnut is such a manifold:

In a recent paper Novikov (1970) proposed an algebraic theory of the $L$-groups in terms of the formalism of hamiltonian and lagrangian physics, and "algebraic K-theory" (an earlier algebraic offshoot of topology). Unfortunately, Novikov's theory did not quite capture the $L$-groups: on the other hand, it did give an approximate algebraic proof of a special case of Wall's Theorem 12.6 (including the surface of a doughnut, a type of space which is very important in the topology of manifolds).
Parts I and II of my dissertation arose out of an attempt to understand Novikov's paper. Part I, "Foundations of L-theory", deals with an algebraic theory which does capture the L-groups. Part II, "Algebraic L-theory", goes on to improve Novikov's methods to give an absolutely precise algebraic proof of the case of Wall's Theorem 12.6 he considers.

Part III, "Geometric L-theory", is entirely original. It contains another, more sophisticated, characterization of the L-groups, using the geometrical background for motivation, but purely algebraic methods in execution. Hitherto, such a characterization had only been possible geometrically, in which form it was first considered in the Princeton PhD thesis.

"A geometric formulation of surgery" of Quinn (1969). There is a standard technique, "geometrical realization", leading back from my algebra to Quinn's geometry.

The work of Part III resolves the difficulties of §§6, 8 of Wall's book, to do with the definition of $L_n(n)$ for $n$ odd. One of the main results of Wall's book is that the group $L_n(n)$ is the same as $L_{n+4}(n)$. There is a simple algebraic interpretation of this periodicity in my theory.
It is to be hoped, however, that
my L-theory does more than give new
proofs of old facts, and that there
is indeed a payoff in the topology.

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