THE HOMOTOPY GROUPS OF A TRIAD. II

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Introduction

The principal purpose of this paper is to state and prove a rather general theorem (Theorem I below) about triad homotopy groups. This theorem is a considerable generalization of the main theorem previously proved by the authors on this subject [1, Theorem I]\(^1\), but its proof is in many respects simpler than that of the previous theorem.

The notation and terminology of the present paper are the same as in [1]. For a proper understanding of this paper, the reader should be familiar with almost all of parts 1, 2, and 3, and the following sections of part 4 from [1]: Sections 4.1, 4.2, 4.3, 4.4, and 4.7.

The main theorem is stated in Section 1. In Section 2 several necessary lemmas are proved. The proof of Theorem I is given in Sections 3, 4, and 5. The remainder of the paper is devoted to several applications of Theorem I. Among these is Theorem III, which generalizes Theorem II of [1].

1. Statement of the main theorem

Let \((X; A, B)\) be a triad which satisfies the following conditions:

(a) \(A, B,\) and \(A \cap B\) are all arc-wise connected.

(b) \(X = (\text{Int } A) \cup (\text{Int } B)\).

(c) \((A, A \cap B)\) is \(m\)-connected, \((B, A \cap B)\) is \(n\)-connected, and \(m \geq n \geq 1\).

(Clearly, no generality is lost by assuming \(m \geq n\), since this condition may always be satisfied by a proper choice of notation).

(d) In case \(n = 1\) and \(m > n\), we assume either that \(\pi_1(B, A \cap B)\) is abelian, or that \((A, A \cap B)\) is simple in dimension \(m + 1\). In case \(m = n = 1\), we assume that \((B, A \cap B)\) is simple in dimension 2.

It is clear that condition (d) is satisfied if \(A \cap B\) is simply connected.

**Theorem I.** If the hypotheses (a)–(d) hold, then the triad \((X; A, B)\) is \((m + n)\)-connected.

One of the principal tools for the proof of this theorem is Lemma 4.7.1 of [1]. Before proceeding with the proof, we shall develop several auxiliary lemmas.

2. The supplement of a subcomplex

Let \(K\) be a simplicial complex and \(L\) a subcomplex. Denote by \((K', L')\) the first barycentric subdivision of \((K, L)\).

**Definition.** The supplement of \(L\) in \(K\), denoted by \(K + L\), is the subcomplex of \(K'\) spanned by all of the vertices of \(K' - L'\); i.e., a simplex of \(K'\) belongs to \(K + L\) if and only if none of its vertices is in \(L'\).

\(^1\) Numbers in square brackets refer to the bibliography at the end of the paper.
It follows that every simplex of $K'$ that is not a simplex of $L'$ or of $K \div L$, is the join \[8, p. 202\] of a simplex of $L'$ with a simplex of $K \div L$.

We will identify the spaces $|K|$ and $|K'|$, and the spaces $|L|$ and $|L'|$. Since every simplex of $K'$ is the join of a simplex of $L'$ and a simplex of $K \div L$, it follows that there exists a homeomorphic imbedding

$$h: |K| \rightarrow |L| \ast |K \div L|$$

where the symbol "\ast" denotes the join operation. By means of this homeomorphic imbedding we can introduce coordinates in $|K|$ as follows: a point $z \in |K|$ has coordinates $(x, t, y)$ where $x \in |L|$, $y \in |K \div L|$ and $0 \leq t \leq 1$. If $t = 0$, then $(x, t, y) \in |L|$, while if $t = 1$, then $(x, t, y) \in |K \div L|$. Let

$$N(L) = \{(x, t, y) \in |K| \mid 0 \leq t < \frac{1}{2}\},$$

$$N(K \div L) = \{(x, t, y) \in |K| \mid \frac{1}{2} < t \leq 1\}.$$

Then $N(L)$ and $N(K \div L)$ are disjoint open neighborhoods of $|L|$ and $|K \div L|$ in $|K|$. Define

$$\widetilde{N}(L) = \text{Cl} \ N(L) = \{(x, t, y) \in |K| \mid 0 \leq t \leq \frac{1}{2}\},$$

$$\widetilde{N}(K \div L) = \text{Cl} \ N(K \div L) = \{(x, t, y) \in |K| \mid \frac{1}{2} \leq t \leq 1\}.$$

Then $|K| = \widetilde{N}(L) \cup \widetilde{N}(K \div L)$. Furthermore, $|L|$ is a deformation retract\(^2\) of $\widetilde{N}(L)$ and $|K \div L|$ is a deformation retract of $\widetilde{N}(K \div L)$.

For later use we define a deformation

$$\phi_\tau: |K| \rightarrow |K|,$$

$$\phi_\tau(x, t, y) = \begin{cases} (x, (2 - \tau)t, y), & 0 \leq t \leq \frac{1}{2}, \\ (x, 1 - \tau + t\tau, y), & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Then $\phi_0 = \text{identity}$, $\phi_{\tau}(z) = z$ if $z \in |L|$ or $z \in |K \div L|$, $\phi_2(\widetilde{N}(L)) \subset L$, and $\phi_2(\widetilde{N}(K \div L)) = |K| \setminus |L|$. We will call $\phi_\tau$ the deformation of $|K|$ on itself toward $|L|$.

We will make use later of several additional properties of $K \div L$. These are contained in the following lemmas:

**Lemma 1.** Let $A$ be an arbitrary subcomplex of $K$. Then

$$A \div (A \cap L) = A \cap (K \div L).$$

**Proof.** Let $K', A', L'$ denote the barycentric subdivisions of $K$, $A$, $L$ respectively. Then it is readily seen from the definition of supplement, that a simplex of $K'$ belongs to $A \div (A \cap L)$ if and only if all its vertices are in $A'$ but not in $L'$. But this is precisely the condition that the simplex belong to $A \cap (K \div L)$.

\(^2\) Throughout this paper we use the term "deformation retract" in the strong sense, i.e., points of the retract remain fixed throughout the deformation.
Let \( M_1 \) denote \( K + L \), and \( M_2 = A + (A \cap L) = A \cap (K + L) \). Then \( \tilde{N}(M_1) \subset |K| \) and \( \tilde{N}(M_2) \subset |A| \).

**Lemma 2.** \( \tilde{N}(M_2) = |A| \cap \tilde{N}(M_2) \).

**Proof.** Let \( h : |K| \rightarrow |L| \ast |M_1| \) as before. Then it is readily seen that the map \( h \) of \( |A| \) gives the homeomorphic imbedding of \( |A| \) into \( |A \cap L| \ast |M_2| \). In other words, the coordinate system induced in \( |A| \) by that in \( |K| \), agrees with that defined in \( |A| \) in terms of \( A \cap L \) and \( M_2 \). The lemma now follows at once from the definitions of \( \tilde{N}(M_1) \) and \( \tilde{N}(M_2) \).

**Lemma 3.** The pairs \( (\tilde{N}(M_1), \tilde{N}(M_2)) \) and \( (|M_1|, |M_2|) \) have the same homotopy type.

**Proof.** We define a deformation

\[
\psi_r : (\tilde{N}(M_1), \tilde{N}(M_2)) \rightarrow (\tilde{N}(M_1), \tilde{N}(M_2)) \quad 0 \leq r \leq 1,
\]

by

\[
\psi_r(x, t, y) = (x, r + t - tr, y)
\]

for \((x, t, y) \in \tilde{N}(M_1)\). Then \( \psi_0 \) = identity, \( \psi_r(x) = x \) if \( x \in |M_1| \), and \( \psi_1 \) is a retraction of \((\tilde{N}(M_1), \tilde{N}(M_2))\) onto \(|M_1|, |M_2|\). Hence \(|M_1|, |M_2|\) is a deformation retract of \((\tilde{N}(M_1), \tilde{N}(M_2))\) and has the same homotopy type.

**Lemma 4.** If \( L \) contains the \( m \)-dimensional skeleton of \( K \), and \( \dim K = n \), then \( \dim (K + L) \leq n - m - 1 \).

This follows easily from the definitions.

### 3. A normalization process for certain triad maps

Let \((X; A, B)\) be a triad which satisfies the conditions (a) and (b) of Section 1, and assume that \((A, A \cap B)\) is \( m \)-connected, \( m \geq 1 \). Let \( \alpha \in \pi_q(X; A, B) \), \( q \geq 2 \), and let

\[
f_0 : (E^s; E^{s-1}_+, E^{s-1}_-) \rightarrow (X; A, B)
\]

represent \( \alpha \). We are going to define a homotopy.

\[
f_t : (E^s; E^{s-1}_+, E^{s-1}_-) \rightarrow (X; A, B), \quad 0 \leq t \leq 2,
\]

of \( f_0 \), called the **normalization process**, and \( f_2 \) will be called a representative of \( \alpha \) in ordinary form. It is not asserted that this normal form is unique.

Let \( U = f_0^{-1}(\text{Int } A) \) and \( V = f_0^{-1}(\text{Int } B) \). Then \( \{U, V\} \) is an open covering of \( E^s \). Let \( \varepsilon \) be the Lebesgue number of this covering. Choose a rectilinear triangulation \([1, \text{Section 4.2}]\) of the triad \((E^s; E^{s-1}_+, E^{s-1}_-)\) so fine that every simplex has diameter \( < \varepsilon \). Let \( P_0 \) be the subcomplex of \( E^s \) spanned by all the simplexes contained in \( U \), and let \( Q_0 \) be the subcomplex spanned by all the simplexes contained in \( V \); then \( E^s = P_0 \cup Q_0 \). Let

\[
P_t = P_0 \cup E^{s-1}_+
\]

\[
Q_t = Q_0 \cup E^{s-1}_-
\]

\[
R_t = P_t \cap Q_t
\]
Consider the map \( g_0 : (P_1, R_1) \to (A, A \cap B) \) defined by \( f_0 \). This map is \( m \)-deformable, since \( (A, A \cap B) \) is \( m \)-connected. Let

\[
g_t : (P_1, R_1) \to (A, A \cap B), \quad 0 \leq t \leq 1,
\]

be a homotopy of \( g_0 \) such that \( g_t(x) = x \) if \( x \in R_1 \), and \( g_t(P_1^*) \subseteq A \cap B \). Let \( L \) denote the closed subcomplex of \( P_1 \) consisting of all simplexes \( \sigma \) such that \( g_t(\sigma) \subseteq A \cap B \). Then \( L \supseteq (R_1 \cup P_1^*) \). Let \( M = P_1 + L \), the supplementary subcomplex of \( L \) in \( P_1 \). Let

\[
\phi_t : P_1 \to P_1, \quad 1 \leq t \leq 2
\]

be the deformation of \( P_1 \) onto itself toward \( L \), as described in Section 2. Define

\[
g_t : (P_1, R_1) \to (A, A \cap B), \quad 1 \leq t \leq 2,
\]

by

\[
g_t = g_0 \phi_t,
\]

and define

\[
f_t : (E^q; E^q_{+}^{*-1}, E^q_{-}^{*-1}) \to (X; A, B), \quad 0 \leq t \leq 2,
\]

by

\[
f_t|_{P_1} = g_t, \quad 0 \leq t \leq 2,
\]

\[
f_t|_{Q_1} = f_0|_{Q_1}, \quad 0 \leq t \leq 2.
\]

From Lemma 4 of Section 2 it follows, since \( L \supseteq P_1^* \), that

(3.1) \quad \text{dim } M \leq q - m - 1.

We now define

\[
P = \overline{N(M) \cup E^q_{+}^{*-1}},
\]

\[
Q = \text{Cl} \{ E^q - N(M) \},
\]

\[
R = P \cap Q.
\]

Then \( P \cup Q = E^q \), and

\[
E^q_{+1} \subseteq P \subseteq f_2^{-1}(A),
\]

\[
E^q_{-1} \subseteq Q \subseteq f_2^{-1}(B).
\]

The map

\[
f = f_2 : (E^q; E^q_{+}^{*-1}, E^q_{-}^{*-1}) \to (X; A, B)
\]

is the desired normal form for a representative of \( \alpha \).

4. Proof of Theorem I when \( m \geq n \geq 2 \)

In this section we give the proof of the main theorem when \( m \geq n \geq 2 \). The modifications in the proof which are necessary when \( n = 1 \) will be described in the next section.
We assume, then, that the triad \((X; A, B)\) satisfies the hypotheses (a), (b), and (c) of Section 1, and that \(m \geq n \geq 2\). Let \(a \in \pi_0 (X; A, B)\), \(q \leq m + n\), and let
\[
f : (E^q; E^{q-1}_+, E^{q-1}_-) \to (X; A, B)
\]
be a representative of \(a\) in normal form. Let \(P, Q, R\) and \(M\) denote the subsets of \(E^q\) described at the end of Section 3. Clearly, the spaces \(Q\) and \(E^q - M\) have the same homotopy type. Since \(\dim M \leq q - m - 1 \leq q - 3\), it follows from Lemma 4.2.1 of [1] that \(Q\) is simply connected.

To prove that \(a = 0\), it suffices, by Lemma 4.7.1 of [1], to show that the map
\[
h : (Q, R) \to (B, A \cap B)
\]
defined by \(f\), is deformable. We will do this by using Theorem 4.4.3 of [1]. We need to consider the cohomology groups \(H^j(Q, R, \pi_j), 2 \leq j \leq q\), where \(\pi_j = \pi_j(B, A \cap B)\). By the excision axiom,
\[
H^j(Q, R) \approx H^j(E^q, P).
\]
It follows from the exactness of the cohomology sequence of the triple \((E^q, P, E^{q-1}_-)\) that
\[
H^j(E^q, P) \approx H^{j-1}(P, E^{q-1}_+),
\]
and by the excision axiom again,
\[
H^{j-1}(P, E^{q-1}_+) \approx H^{j-1}(\tilde{N}(M), \tilde{N}(M) \cap E^{q-1}_{}).
\]
It follows from Section 2 that the pairs \((\tilde{N}(M), \tilde{N}(M) \cap E^{q-1}_-\)) and \((M, M \cap E^{q-1}_+)\) are of the same homotopy type. Therefore
\[
H^{j-1}(\tilde{N}(M), \tilde{N}(M) \cap E^{q-1}_-) \approx H^{j-1}(M, M \cap E^{q-1}_{}).
\]
Combining these isomorphisms, we have,
\[
H^j(Q, R, \pi_j) \approx H^{j-1}(M, M \cap E^{q-1}_+, \pi_j).
\]
Now \(H^j(Q, R, \pi_j) = 0\) for \(j \leq n\), because \(\pi_j = 0\) for \(j \leq n\). If \(j > q - m\), then \(H^{j-1}(M, M \cap E^{q-1}_+, \pi_j) = 0\) because \(\dim M \leq q - m - 1\), by (3.1). Hence if \(q \leq m + n\), \(H^j(Q, R, \pi_j) = 0\) for all values of \(j\), and since \(Q\) is simply connected it follows from Theorem 4.4.3 of [1] that \(h\) is deformable and therefore that \(a = 0\).

5. Modifications necessary when \(n = 1\)

If \(m > n = 1\), and \(\pi_2(B, A \cap B)\) is abelian, the proof goes through exactly as before, since all hypotheses of Theorem 4.4.3 of [1] remain satisfied.

We examine next the case where \(m = n = 1\) and \((B, A \cap B)\) is simple in dimension 2. We can no longer apply Lemma 4.2.1 of [1] to conclude that \(Q\) is simply connected. However it follows that \(\pi_3(B, A \cap B)\) is abelian. Hence we can use Theorem 4.4.2 of [1] instead of Theorem 4.4.3, to show that the map \(h : (Q, R) \to (B, A \cap B)\) is 2-deformable, and therefore \((X; A, B)\) is 2-connected.
Consider, finally, the case where \( m > n = 1 \), and it is assumed that \((A, A \cap B)\) is simple in dimension \((m + 1)\). By a simple change of notation we may assume the equivalent hypotheses; \( n > m = 1 \) and \((B, A \cap B)\) is simple in dimension \( n + 1 \). The proof now follows that of the case considered immediately above, using theorem (4.4.2) of [1].

6. On condition (b) of Theorem I

In many cases in which one actually wishes to apply Theorem I, there is given a triad \((X; A, B)\) such that \( X = A \cup B \) and the conditions (a), (c), and (d) of Theorem I are satisfied, but condition (b) is not satisfied. Then it is not possible to apply Theorem I directly. However, it sometimes happens that there exists a subset \( A' \subset X \) such that \( A' \supseteq A \), the triads \((X; A, B)\) and \((X; A', B)\) have isomorphic homotopy sequences, and condition (b) does apply to the triad \((X; A', B)\). Then we may apply Theorem I to the triad \((X; A', B)\) to conclude that certain of its homotopy groups vanish, and hence that the homotopy groups of the triad \((X; A, B)\) vanish in the corresponding dimensions. One such case is described in the following lemma:

**Lemma 5.** Let \((X; A, B)\) be a triad with \( X = A \cup B \), with \( A \) and \( B \) closed subsets of \( X \), and such that there exists an open neighborhood \( N \) of \( A \cap B \) in \( B \) with \( A \cap B \) a deformation retract of \( N \) in the strong sense. Let \( A' = A \cup N \). Then the triad \((X; A', B)\) satisfies condition (b) of Theorem I, and the triads \((X; A, B)\) and \((X; A', B)\) have isomorphic homotopy sequences.

**Proof.** It readily follows from the hypotheses that \( A' \) and \( X - A \) are open subsets of \( X \), that \((X - A) \subset \text{Int} \ B \), and that \( A' \cup (X - A) = X \). Hence the triad \((X; A', B)\) satisfies condition (b) of Theorem I.

Next, we observe that our hypotheses imply that \( A \) is a deformation retract of \( A' \), since the deformation retraction which is defined over \( N \) can be extended to all of \( A' \) in the obvious way. The continuity of this extension follows from the fact that both \( A \) and \( N \) are closed in \( A' \). This deformation retraction is a homotopy equivalence between the pairs \( (A, A \cap B) \) and \( (A', A' \cap B) \), and therefore the inclusion map \( (A, A \cap B) \to (A', A' \cap B) \) induces isomorphisms of the homotopy sequences of these two pairs. Next we look at the homomorphism induced by the inclusion map \( (B, A \cap B) \to (B, A' \cap B) \) on the homotopy sequences of these pairs. Since the homomorphisms \( \pi_n(B) \to \pi_n(B) \) and \( \pi_n(A \cap B) \to \pi_n(A' \cap B) \) thus induced are isomorphisms onto in all dimensions, it follows from the purely algebraic “five lemma” [2, Lemma 3, p. 435] that the injection \( \pi_n(B, A \cap B) \to \pi_n(B, A' \cap B) \) is also an isomorphism onto. This proof even goes through with minor modifications to prove that the injection \( \pi_1(B, A \cap B) \to \pi_1(B, A' \cap B) \) is \( 1 \to 1 \) and onto. Another application of the “five lemma” to the homotopy sequences of the triads \((X; A, B)\) and \((X; A', B)\) enables us to prove that these triads have isomorphic homotopy sequences, as was to be proved. Here again minor modifications are necessary in the lowest dimension.

**Corollary.** Theorem I remains true when condition (b) in its hypothesis is replaced by the hypothesis of Lemma 5.
Obviously, the symmetric statement obtained by interchanging the roles of \( A \) and \( B \) in Lemma 5 is also true.

A particular case where the hypothesis of Lemma 5 is satisfied occurs when we have a triad \((X; A, B)\) such that \(X = A \cup B\), \(A\) and \(B\) are closed in \(X\), and at least one of the pairs \((A, A \cap B)\) and \((B, A \cap B)\) is triangulable. For example, if \((B, A \cap B)\) is triangulable, we can choose \(N = N(A \cap B)\) as defined in Section 2.

7. Shrinking a subcomplex to a point

Let \((X, A)\) be a pair consisting of a CW-complex, \(X\), and a closed subcomplex, \(A\), and let \((\bar{X}, x_0)\) be the pair obtained by identifying all of \(A\) to a single point \(x_0\); then \(\bar{X}\) is a CW-complex and \(x_0\) is a vertex of \(A\). Assume that \((\bar{X}, \bar{A})\) is \(m\)-connected, \(m \geq 1\), and that \(A\) is \(n\)-connected, \(n \geq 1\). Let \(\phi: (X, A) \to (\bar{X}, x_0)\) denote the identification map and \(\phi_p : \pi_p(X, A) \to \pi_p(\bar{X}, x_0)\) the homomorphism induced by \(\phi\).

**Theorem II.** \(\phi_p\) is an isomorphism onto for \(p \leq m + n\), and is a homomorphism onto for \(p = m + n + 1\).

Before proceeding with the proof, we prove a lemma. Let \((X, A)\) be a pair consisting of a space \(X\) and closed subspace \(A\). Let \(A^*\) be the join of \(A\) with a single point \(a_0\), (i.e., the identification space of \(A \times I\) when all of \(A \times 1\) is identified to the single point \(a_0\)) and let \(X^*\) be the identification space resulting from \(X \cup A^*\) by identifying each point \(a \times 0 \in A^*\) with the corresponding point \(a \in A\). Assume that \((X, A)\) is \(m\)-connected, \(m \geq 1\), and that \(A\) is \(n\)-connected, \(n \geq 1\).

**Lemma 6.** With the above hypotheses, the triad \((X^*; X, A^*)\) is \((m + n + 1)\)-connected.

**Proof.** Since \(A\) is \(n\)-connected and \(A^*\) is contractible, \((A^*, A)\) is \((n + 1)\)-connected. Let \(N \subset A^*\) be the set \(\eta(A \times [0 \leq t < \frac{1}{2}])\) where \(\eta\) denotes the identification map \(\eta: A \times I \to A^*\). Then \(N\) is open in \(A^*\) and \(A\) is a deformation retract of \(N\). Moreover \(X \cap A^* = A\) is simply connected and \(X, A^*\) are closed in \(X^*\). The result now follows from the corollary to Lemma 5.

**Proof of Theorem II.** Let us extend the map \(\phi\) to a map \(\psi: (X^*, A^*) \to (\bar{X}, x_0)\), by defining \(\psi(x) = x_0\) for all points \(x \in A^*\). It is readily verified that \(\psi\) is continuous. Let \(\psi^*: X^* \to \bar{X}\) be the map defined by \(\psi\). Since \(A^*\) is contractible, it follows that \(\psi^*\) is a homotopy equivalence [6, Theorem 12], so that \(\psi^*_p : \pi_p(X^*) \to \pi_p(\bar{X})\) is an isomorphism onto for all \(p\). Also, the injection \(j: \pi_p(X^*) \to \pi_p(X^*, A^*)\) is an isomorphism onto for all \(p\). Consider the following diagram:

\[
\begin{array}{ccc}
\pi_p(X^*) & \xrightarrow{j} & \pi_p(X^*, A^*) \\
\downarrow \psi^* & & \downarrow \psi^*_p \\
\pi_p(\bar{X}, x_0) & & \\
\end{array}
\]

---

\(^3\) For the definition and properties of a CW-complex, see [6].
Since commutativity holds, it follows that $\psi_*$ is an isomorphism onto in all dimensions. Consider next the following diagram:

$$
\begin{array}{c}
\pi_p(X, A) \xrightarrow{i_*} \pi_p(X^*, A^*) \\
\downarrow \phi_p \quad \quad \quad \quad \quad \downarrow \psi_* \\
\pi_p(\bar{X}, x_0)
\end{array}
$$

Commutativity again holds. Since $\psi_*$ is an isomorphism onto in all dimensions, it follows that the injection $i_*$ and $\phi_p$, are equivalent homomorphisms. The theorem now follows from Lemma 8 and consideration of the homotopy sequence of the triad $\langle X^*, X, A^* \rangle$.

8. An application of Theorem II

Let $(X; A, B)$ be a triad such that $X = A \cup B$, $A$ and $B$ are closed subsets of $X$, and $(A, A \cap B)$ is a pair consisting of a CW-complex and closed subcomplex. Let

$$i_n : \pi_p(A, A \cap B) \to \pi_p(X, B), \quad p = 1, 2, \ldots,$$

denote the injection.

**Theorem III.** If $(A, A \cap B)$ is $m$-connected, $m \geq 1$, and $A \cap B$ is $r$-connected, $r \geq 1$, then $i_p$ is an isomorphism into for $p \leq m + r$, and the image subgroup is a direct summand of $\pi_p(X, B)$.

**Proof.** Let $(\bar{A}, x_0)$ denote the pair obtained from $(A, A \cap B)$ by identifying all of $A \cap B$ to a single point, $x_0$, and let $\phi : (A, A \cap B) \to (\bar{A}, x_0)$ be the identification map. Define a map $\psi : (X, B) \to (\bar{A}, x_0)$ by

$$\psi(x) = \begin{cases} x_0, & x \in B, \\ \phi(x), & x \in A. \end{cases}$$

The map $\psi$ thus defined is continuous, since $A$ and $B$ are closed subsets of $X$.

Consider now the following diagram:

$$
\begin{array}{c}
\pi_p(A, A \cap B) \xrightarrow{i_p} \pi_p(X, B) \\
\downarrow \phi_p \quad \quad \quad \quad \quad \downarrow \psi_p \\
\pi_p(\bar{A}, x_0)
\end{array}
$$

Commutativity clearly holds. It follows from Theorem II that $\phi_p$ is an isomorphism onto for $p \leq m + r$, and from this the theorem follows at once.

The authors wish to acknowledge that the main idea for this proof is due to P. Hilton [5].

A closely related result is the following:

**Theorem III'.** Let $(X; A, B)$ be a triad such that $X = (\text{Int} \ A) \cup (\text{Int} \ B)$,
\((A, A \cap B)\) is \(m\)-connected, \(m \geq 1\), and \(A \cap B\) is \(r\)-connected, \(r \geq 1\). Then the injection
\[i_p : \pi_p(A, A \cap B) \to \pi_p(X, B)\]

is an isomorphism into, for \(p \leq m + r\).

The proof of this result is similar to the proof of the main theorem, except that Lemma 4.7.2 of [1] is used in place of Lemma 4.7.1; cf. the proof given in Section 4.10 of [1].

It seems probable that under the hypotheses of Theorem III' the image subgroup is a direct summand of \(\pi_p(X, B)\), but the authors are unable to prove this fact.

9. Geometric proof of an algebraic theorem of Eilenberg and MacLane

For each abelian group \(\Pi\) and integer \(n \geq 2\), Eilenberg and MacLane [3, p. 507] have defined an abstract complex \(K(\Pi, n)\). In a recent paper [4] they have defined a "suspension" operation, which is a chain transformation (raising dimensions by 1) of \(K(\Pi, n)\) into \(K(\Pi, n + 1)\), and hence induces homomorphisms,
\[S_q : H_q[K(\Pi, n), G] \to H_{q+1}[K(\Pi, n + 1), G]\]
of the corresponding homology groups with \(G\) as coefficient group. Concerning these homomorphisms, they have stated a theorem which is equivalent to the following:

**Theorem IV.** The suspension homomorphism, \(S_q\), is an isomorphism onto for \(q < 2n\), and is a homomorphism onto for \(q = 2n\).

We shall give a proof this result, based on Theorem I. Let \(L\) be a CW-complex with \(\pi_i(L) = 0\) for \(i < n\) and \(n < i < m, m > 2n\), and with \(\pi_n(L) \approx \Pi\). That this realization is possible follows from a theorem of J. H. C. Whitehead, [7]. Let \(K_n = K(\Pi, n), K_{n+1} = K(\Pi, n + 1), K_0 = K(0, n)\). \(K_0\) is the complex constructed on the group consisting of the identity element only, and is homologically trivial. It may be considered to be a subcomplex of both \(K_n\) and \(K_{n+1}\), and it is easily seen from the definition of \(S_q\) that we might equally well consider the equivalent homomorphisms induced by the suspension,
\[S : H_q(K_n, K_0) \to H_{q+1}(K_{n+1}, K_0)\].

Let a vertex \(p \in L\) be chosen, and let \(a_1, a_2\) be a pair of distinct points. Let \(L_1 = L \ast \{a_1\}\) (the join of \(L\) with \(a_1\)), \(L_2 = L \ast \{a_2\} \cup L_1\), \(p_1 = \{p\} \ast \{a_1\}\), \(p_2 = \{p\} \ast \{a_2\}\), \(\tilde{p} = p_1 \cup p_2\). Let \(\bar{L}\) be the space obtained from \(L\) by identifying the segment \(\bar{p}\) to a single point, which we will also denote by \(p\). Since \(\tilde{p}\) is contractible, the identification map is a homotopy equivalence [6, theorem 12], and induces isomorphisms
\[\psi : H_q(\bar{L}, \tilde{p}) \approx H_q(\bar{L}, p)\]
of the singular homology groups in all dimensions.
Consider now the homomorphisms indicated in the following diagram (singular homology groups):

\[
\begin{array}{ccc}
H_q(S^0(L), \mathcal{S}(p)) \xrightarrow{\kappa} H_q(K_n, K_0) \xrightarrow{\eta} H_{q+1}(K_{n+1}, K_0) & \xleftarrow{\kappa'} H_{q+1}(\mathcal{S}(L), \mathcal{S}(p)) \\
\downarrow \eta & & \downarrow \eta' \\
H_{q+1}(L, \mathcal{L}) & & H_{q+1}(L, \mathcal{L}) \\
\uparrow \phi & & \uparrow \psi \\
H_{q+1}(L_1, L) \xrightarrow{i_1} H_{q+1}(L_1, L \cup p_1) \xrightarrow{i} H_{q+1}(\mathcal{L}, L \cup \mathcal{L}) \xleftarrow{i_0} H_{q+1}(\mathcal{L}, \mathcal{L})
\end{array}
\]

The homomorphisms \(\kappa, \kappa', \eta, \eta'\), are defined in [3]. The homomorphisms \(i_1, i_2, j, \) and \(\phi\) are induced by inclusion maps, and \(\psi\) is the boundary operator. By applying Theorem I to the triad \((L; L_1, L_2)\), we find that this triad is \(2n\)-connected.

Hence \(\tau(L) \cong \tau(L_i)\) for \(i \leq 2n - 1\), and \(\tau(L_i)\) is a homomorphic image of \(\tau(L)\). Therefore \(\tau(L) = 0\) for \(i < n + 1\) and \(n + 1 < i \leq 2n\), while \(\tau(L_i) \cong \tau(L)\) for all \(i\). Hence \(\eta\) and \(\eta'\) are isomorphisms onto all dimensions, \(\kappa\) is an isomorphism onto for \(q \leq m\), and \(\kappa'\) is an isomorphism onto for \(q + 1 \leq 2n\), and is onto for \(q + 1 = 2n + 1\).

The homomorphisms \(\phi, i_1, i_2, j\) are isomorphisms onto in all dimensions. It can be verified that the commutativity relation

\[
\kappa' \eta^{-1} \psi j i^{-1} \phi^{-1} \eta = \kappa
\]

holds. Therefore \(S\) is an isomorphism onto for \(q < 2n\), and is onto for \(q = 2n\).

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**Bibliography**