SURGERY TRANSFER

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Introduction

Given a Hurewicz fibration $F \longrightarrow E \xrightarrow{p} B$ with fibre an n-dimensional geometric Poincaré complex F we construct algebraic transfer maps in the Wall surgery obstruction groups

$$p^! : L_m(\mathbb{Z}[\pi_1(B)]) \longrightarrow L_{m+n}(\mathbb{Z}[\pi_1(E)]) \quad (m \geq 0)$$

and prove that they agree with the geometrically defined transfer maps. In subsequent work we shall obtain specific computations of the composites $p^! p_!$, $p_! p^!$ with $p_! : L_m(\mathbb{Z}[\pi_1(E)]) \longrightarrow L_m(\mathbb{Z}[\pi_1(B)])$ the change of rings maps, and some vanishing results.

The construction of $p^!$ is most straightforward in the case when F is finite, with L_* the free L-groups L^h_*. In §9 we shall extend the definition of $p^!$ to finitely dominated F and the projective L-groups L^p_*, as well as to simple F and the simple L-groups L^s_*, and also to the intermediate cases.

There are two main sources of applications of the surgery transfer. The equivariant surgery obstruction groups of Browder and Quinn [1] were defined in terms of the geometric surgery transfer maps of the normal sphere bundles of the fixed point sets. An algebraic version will necessarily involve the algebraic surgery transfer maps. (In this connection see Lück and Madsen [8].) The recent work of Hambleton, Milgram, Taylor and Williams [3] on the evaluation of the surgery obstructions of normal maps of closed manifolds with finite fundamental group depends on the factorization of the assembly map by twisted product formulae which are closely related to the algebraic surgery transfer.

Our construction of the quadratic L-theory transfer maps is by a combination of the algebraic

surgery theory of Ranicki [14],[19] and the method used by Lück [7] to define the algebraic K-theory transfer maps $p^!:K_m(\mathbb{Z}[\pi_1(B)])\longrightarrow K_m(\mathbb{Z}[\pi_1(E)])$ $(m=0,1)$ for a fibration with finitely dominated fibre F.

The algebraic surgery transfer maps $p^!$ for a fibration are a special case of transfer maps $(C,\alpha,U)^!:L_m(A)\longrightarrow L_{m+n}(B)$ $(m\geqslant 0)$ defined in abstract algebra. Here, A and B are rings with involution, C is an n-dimensional f.g. free B-module chain complex with a symmetric Poincaré duality chain equivalence $\alpha\simeq\alpha^*:C\longrightarrow C^{n-*}$, and $U:A\longrightarrow R=H_0(\mathrm{Hom}_B(C,C))^{op}$ is a morphism of rings with involution from A to the opposite of the ring of chain homotopy classes of B-module chain maps $f:C\longrightarrow C$, with the involution on R defined by $T(f)=\alpha^{-1}f^*\alpha$. An element of $L_{2i}(A)$ is represented by a nonsingular $(-)^i$-quadratic form $(M,\psi:M\longrightarrow M^*)$ on a f.g. free A-module $M=\bigoplus_k A$. We define

$(C,\alpha,U)^!(M,\psi)=(D,\theta)\in L_{n+2i}(B)$ to be the cobordism class of the $(n+2i)$-dimensional quadratic Poincaré complex (D,θ) given by

$$\theta_s = \begin{cases} U(\psi)(\bigoplus_k \alpha^{-1}) & \text{if } s=0 \\ 0 & \text{if } s\neq 0 \end{cases} :$$

$$D^{n+2i-r-s} = \bigoplus_k C^{n+i-r-s} \longrightarrow D_r = \bigoplus_k C_{r-i} .$$

There is a similar formula in the case $m=2i+1$, for which we refer to §4.

The algebraic transfer maps of fibration $F\longrightarrow E\xrightarrow{p}B$ with fibre an n-dimensional geometric Poincaré complex F are given by

$$p^! = (C(\tilde{F}),\alpha,U)^! : L_m(\mathbb{Z}[\pi_1(B)]) \longrightarrow L_{m+n}(\mathbb{Z}[\pi_1(E)])$$

with $C(\tilde{F})$ the cellular $\mathbb{Z}[\pi_1(E)]$-module chain complex of the cover $\tilde{F}$ of F induced from the universal cover $\tilde{E}$ of E, $\alpha=([F]\cap-)^{-1}:C(\tilde{F})\longrightarrow C(\tilde{F})^{n-*}$ the Poincaré duality

chain equivalence, and U determined by the fibre transport.

Here is the main idea in the identification of the algebraic and geometric surgery transfer. We know from the identification of the corresponding K-theory transfers in Lück [7] how to handle in algebra the lift of CW structures from the base to the total space of a fibration. We use the ultraquadratic L-theory of Ranicki [16,§7.8] both to encode the algebraic surgery data in the base spaces as CW structures, and to decode the algebraic surgery data from the lifted CW structures in the total spaces.

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The titles of the sections are:

§1. The algebraic K-theory transfer

We recall from Lück [7] the construction of the algebraic K-theory transfer maps, and the connection with topology.

Given a ring R let R^{op} denote the opposite ring, with the same elements and additive structure but with the opposite multiplication.

Definition 1.1 A representation (A,U) of a ring R in an additive category $\mathbb{A}$ is an object A in $\mathbb{A}$ together with a morphism of rings $U:R \longrightarrow \mathrm{Hom}_{\mathbb{A}}(A,A)^{op}$.

□

Given an associative ring R with 1 let $\mathbb{B}(R)$ be the additive category of based f.g. free R-modules R^n $(n \geqslant 0)$. A morphism $f:R^n \longrightarrow R^m$ is an R-module morphism, corresponding to the $m \times n$ matrix $(a_{ij})_{1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n}$ with entries $a_{ij} \in R$, such that

$$f = (a_{ij}) : R^n \longrightarrow R^m \; ; \; (x_j) \longrightarrow (\sum_{j=1}^{n} x_j a_{ij}) \; .$$

Example 1.2 The universal representation (R,U) of R in $\mathbb{B}(R)$ is defined by the ring isomorphism

$$U : R \longrightarrow \mathrm{Hom}_R(R,R)^{op} \; ; \; r \longrightarrow (s \longrightarrow sr) \; ,$$

which we shall use to identify $R = \mathrm{Hom}_R(R,R)^{op}$.

□

A functor of additive categories $F:\mathbb{A} \longrightarrow \mathbb{B}$ is required to preserve the additive structures.

Proposition 1.3 Given a ring R and an additive category $\mathbb{A}$ there is a natural one-one correspondence between functors $F:\mathbb{B}(R) \longrightarrow \mathbb{A}$ and representations (A,U) of R in $\mathbb{A}$.

Proof: Given a functor F define a representation (A,U) by

$$A = F(R) ,$$

$$U : R = \mathrm{Hom}_R(R,R)^{op} \longrightarrow \mathrm{Hom}_{\mathbb{A}}(A,A)^{op} ;$$

$$(\rho : R \longrightarrow R) \longrightarrow (F(\rho) : A \longrightarrow A) .$$

Conversely, given a representation (A,U) define a functor $F = -\otimes(A,U) : \mathbb{B}(R) \longrightarrow \mathbb{A}$ by

$$F(R^n) = A^n ,$$

$$F((a_{ij}) : R^n \longrightarrow R^m) = (U(a_{ij})) : A^n \longrightarrow A^m .$$

□

<u>Example</u> <u>1.4</u> A morphism of rings $f : R \longrightarrow S$ determines a representation (S,U) of R in $\mathbb{B}(S)$ with

$$U = f : R \longrightarrow \mathrm{Hom}_S(S,S)^{op} = S ,$$

such that $-\otimes(S,U) = f_! : \mathbb{B}(R) \longrightarrow \mathbb{B}(S)$ is the usual change of rings functor.

□

For any object A in an additive category $\mathbb{A}$ there is defined a representation $(A,1)$ of the ring $\mathrm{Hom}_{\mathbb{A}}(A,A)^{op}$ in $\mathbb{A}$. The corresponding functor is the full embedding

$$-\otimes(A,1) : \mathbb{B}(\mathrm{Hom}_{\mathbb{A}}(A,A)^{op}) \longrightarrow \mathbb{A} ;$$

$$\mathrm{Hom}_{\mathbb{A}}(A,A)^{op} \longrightarrow A .$$

The functor associated to a representation (A,U) of R in $\mathbb{A}$ is the composite

$$F = -\otimes(A,U) : \mathbb{B}(R) \xrightarrow{U_!} \mathbb{B}(\mathrm{Hom}_{\mathbb{A}}(A,A)^{op})$$

$$\xrightarrow{-\otimes(A,1)} \mathbb{A} \ .$$

Given chain complexes C,D in $\mathbb{A}$ let $\mathrm{Hom}_{\mathbb{A}}(C,D)$ be the abelian group chain complex defined by

$$\begin{aligned} d_{\mathrm{Hom}_{\mathbb{A}}(C,D)} \ : \ \mathrm{Hom}_{\mathbb{A}}(C,D)_r = & \sum_{q-p=r} \mathrm{Hom}_{\mathbb{A}}(C_p,D_q) \\ & \longrightarrow \mathrm{Hom}_{\mathbb{A}}(C,D)_{r-1} \ ; \ f \longrightarrow d_D f + (-)^q f d_C \ . \end{aligned}$$

There is a natural one-one correspondence between chain maps $f:C \longrightarrow D$ and 0-cycles $f' \in \mathrm{Hom}_{\mathbb{A}}(C,D)_0$, with

$$f' = (-)^n f \ : \ C_n \longrightarrow D_n \quad (n \in \mathbb{Z}) \ .$$

Similarly for chain homotopies and 1-chains. Thus $H_0(\mathrm{Hom}_{\mathbb{A}}(C,D))$ is isomorphic to the additive group of chain homotopy classes of chain maps $C \longrightarrow D$.

A chain complex C is finite if $C_r=0$ for $r<0$ and there exists $n \geqslant 0$ such that $C_r=0$ for $r>n$.

<u>Definition</u> <u>1.5</u> Given an additive category $\mathbb{A}$ let $\mathbb{D}(\mathbb{A})$ be the homotopy category of $\mathbb{A}$, the additive category of finite chain complexes in $\mathbb{A}$ and chain homotopy classes of chain maps with

$$\mathrm{Hom}_{\mathbb{D}(\mathbb{A})}(C,D) = H_0(\mathrm{Hom}_{\mathbb{A}}(C,D)) \ .$$

□

For a ring R we write $\mathbb{D}(\mathbb{B}(R))$ as $\mathbb{D}(R)$.

We refer to Ranicki [17],[18] for an account of the algebraic K-groups $K_m(\mathbb{A})$ $(m=0,1)$ of an additive category $\mathbb{A}$ with the split exact structure, and the application to chain complexes. In particular, the class of a finite chain complex C in $\mathbb{A}$ is defined by

$$[C] = \sum_{r=0}^{\infty} (-)^r [C_r] \in K_0(\mathbb{A}) \ ,$$

and the torsion of a self chain equivalence $f:C \longrightarrow C$ is

defined by

$$\tau(f) = \tau(d+\Gamma : C(f)_{odd} \longrightarrow C(f)_{even}) \in K_1(\mathbb{A})$$

for any chain contraction $\Gamma : 0 \simeq 1 : C(f) \longrightarrow C(f)$ of the algebraic mapping cone $C(f)$.

<u>Definition 1.6</u> The <u>generalized Morita maps</u> $\mu : K_m(\mathbb{D}(\mathbb{A})) \longrightarrow K_m(\mathbb{A})$ $(m=0,1)$ are defined for any additive category $\mathbb{A}$ by:

for $m=0$ μ sends the class $[C] \in K_0(\mathbb{D}(\mathbb{A}))$ of an object C in $\mathbb{D}(\mathbb{A})$ to the class $[C] \in K_0(\mathbb{A})$,

for $m=1$ μ sends the torsion $\tau(f) \in K_1(\mathbb{D}(\mathbb{A}))$ of an automorphism $f : C \longrightarrow C$ in $\mathbb{D}(\mathbb{A})$ to the torsion $\tau(f) \in K_1(\mathbb{A})$ of any representative self chain equivalence.

□

A morphism in $\mathbb{D}(\mathbb{A})$ is a chain homotopy class and the definition of μ involves a choice of representative chain map. The generalized Morita maps μ are therefore not induced by a functor $\mathbb{D}(\mathbb{A}) \longrightarrow \mathbb{A}$.

<u>Example 1.7</u> (Lück [7]) A Hurewicz fibration $F \longrightarrow E \xrightarrow{p} B$ with the fibre F a CW complex determines a ring morphism

$$U : \mathbb{Z}[\pi_1(B)] \longrightarrow H_0(\mathrm{Hom}_{\mathbb{Z}[\pi_1(E)]}(C(\tilde{F}), C(\tilde{F})))^{op}$$

with $C(\tilde{F})$ the cellular based free $\mathbb{Z}[\pi_1(E)]$-module chain complex of the pullback $\tilde{F}$ to F of the universal cover $\tilde{E}$ of E, and U the chain homotopy action of $H_0(\Omega B) = \mathbb{Z}[\pi_1(B)]$ on $C(\tilde{F})$ determined by the homotopy action of the loop space ΩB on F. For finite F this defines a representation $(C(\tilde{F}), U)$ of $\mathbb{Z}[\pi_1(B)]$ in $\mathbb{D}(\mathbb{Z}[\pi_1(E)])$. For the identity map $p = 1 : E \longrightarrow B = E$ with $F = \{*\}$ this is the universal representation (R, U) of 1.2 for $R = \mathbb{Z}[\pi_1(B)] = \mathbb{Z}[\pi_1(E)]$.

□

The transfer map in the torsion groups associated to a representation (C,U) of a ring R in $\mathbb{D}(\mathbb{A})$ is the composite

$$(C,U)^! : K_1(R) = K_1(\mathbb{B}(R)) \xrightarrow{U_!} K_1(\mathbb{D}(\mathbb{A})) \xrightarrow{\mu} K_1(\mathbb{A})$$

of the map $U_!$ induced by the functor $(C,U)\otimes - : \mathbb{B}(R) \longrightarrow \mathbb{D}(\mathbb{A})$ and the generalized Morita map μ. The torsion $\tau(f) \in K_1(R)$ of an automorphism $f : R^k \longrightarrow R^k$ is sent by $(C,U)^!$ to the torsion $\tau(U(f)) \in K_1(\mathbb{A})$ of the self chain equivalence $U(f) : \bigoplus_k C \longrightarrow \bigoplus_k C$.

The idempotent completion of an additive category $\mathbb{A}$ is the additive category $\hat{\mathbb{A}}$ with objects pairs

$$(A = \text{object of } \mathbb{A} \ , \ p = p^2 : A \longrightarrow A)$$

and morphisms $f : (A,p) \longrightarrow (A',p')$ defined by morphisms $f : A \longrightarrow A'$ in $\mathbb{A}$ such that $p'fp = f : A \longrightarrow A'$. The evident functor $\mathbb{D}(\hat{\mathbb{A}}) \longrightarrow \hat{\mathbb{D}}(\mathbb{A})$ is an equivalence of additive categories, since every chain homotopy projection in $\hat{\mathbb{A}}$ splits (Lück and Ranicki [9]).

For any ring R the additive category $\mathbb{P}(R)$ of f.g. projective R-modules is equivalent to the idempotent completion $\hat{\mathbb{B}}(R)$ of the additive category $\mathbb{B}(R)$ of based f.g. free R-modules, with an equivalence

$$\hat{\mathbb{B}}(R) \longrightarrow \mathbb{P}(R) \ ; \ (R^k,p) \longrightarrow \mathrm{im}(p) \ .$$

For any representation (C,U) of a ring R in $\mathbb{D}(\hat{\mathbb{A}})$ the functor $(C,U)\otimes - : \mathbb{B}(R) \longrightarrow \mathbb{D}(\hat{\mathbb{A}})$ extends to a functor $\mathbb{P}(R) \longrightarrow \mathbb{D}(\hat{\mathbb{A}})$ (cf. Lemma 9.3), and so determines a transfer map in the class groups

$$(C,U)^! : K_0(R) = K_0(\mathbb{P}(R)) \xrightarrow{U_!} K_0(\mathbb{D}(\hat{\mathbb{A}}))$$

$$\xrightarrow{\ \mu\ } K_0(\hat{\mathbb{A}}) \ .$$

The class $[\mathrm{im}(p)] \in K_0(R)$ of a projection $p=p^2 : R^k \longrightarrow R^k$ is sent by $(C,U)^!$ to the projective class $[\underset{k}{\oplus} C, U(p)] \in K_0(\hat{\mathbb{A}})$ of the chain homotopy projection $U(p) \simeq U(p)^2 : \underset{k}{\oplus} C \longrightarrow \underset{k}{\oplus} C$.

<u>Example 1.8</u> (Lück [7]) A representation (C,U) of a ring R in $\mathbb{D}(\mathbb{P}(S))$ induces algebraic K-theory transfer maps

$$(C,U)^! \ : \ K_m(R) = K_m(\mathbb{P}(R)) \longrightarrow K_m(S) = K_m(\mathbb{P}(S))$$

for $m=0,1$.

□

The algebraic K-theory transfer maps of a fibration $F \longrightarrow E \xrightarrow{p} B$ with finite (or finitely dominated) fibre F defined for $m=0,1$ by

$$p^! = (C(\tilde{F}),U)^! \ : \ K_m(\mathbb{Z}[\pi_1(B)]) \longrightarrow K_m(\mathbb{Z}[\pi_1(E)])$$

were shown in [7] to coincide with the geometric transfer maps using the following property of the functor

$$p^{\#} = -\otimes(C(\tilde{F}),U) \ : \ \mathbb{B}(\mathbb{Z}[\pi_1(B)]) \longrightarrow \mathbb{D}(\mathbb{Z}[\pi_1(E)]) \ .$$

<u>Proposition 1.9</u> Let (X',X) be a relative CW pair such that X' is obtained from X by adjoining cells in dimensions $r, r+1$

$$X' = X \cup \bigcup_I e^r \cup \bigcup_J e^{r+1} \ .$$

Given a map $X' \longrightarrow B$ to a connected space B let $(\tilde{X}',\tilde{X})$ be the pullback to (X',X) of the universal cover $\tilde{B}$ of B, and let

$$d \ : \ C(\tilde{X}',\tilde{X})_{r+1} = \underset{J}{\oplus}\mathbb{Z}[\pi_1(B)] \longrightarrow C(\tilde{X}',\tilde{X})_r = \underset{I}{\oplus}\mathbb{Z}[\pi_1(B)]$$

be the differential in the cellular based free

$\mathbb{Z}[\pi_1(B)]$-module chain complex. Let $F \longrightarrow E \xrightarrow{p} B$ be a Hurewicz fibration such that the fibre F is a CW complex. Let $F \longrightarrow (Y',Y) \longrightarrow (X',X)$ be the fibration obtained from p by pullback along the map $X' \longrightarrow B$, with $(\tilde{Y}',\tilde{Y})$ the pullback to (Y',Y) of the universal cover $\tilde{E}$ of E. Then (Y',Y) is homotopy equivalent to a relative CW pair (also denoted by (Y',Y)) with cellular based free $\mathbb{Z}[\pi_1(E)]$-module chain complex

$$C(\tilde{Y}',\tilde{Y}) = S^r C(p^{\#}(d) : \bigoplus_J C(\tilde{F}) \longrightarrow \bigoplus_I C(\tilde{F}))$$

the r-fold suspension of the algebraic mapping cone of a chain map in the chain homotopy class

$$p^{\#}(d) \ : \ p^{\#}(\bigoplus_J \mathbb{Z}[\pi_1(B)]) = \bigoplus_J C(\tilde{F})$$
$$\longrightarrow p^{\#}(\bigoplus_I \mathbb{Z}[\pi_1(B)]) = \bigoplus_I C(\tilde{F}) \ .$$

Proof: See Lück [7].

□

§2. Maps of L-groups

We refer to Ranicki [14],[19] for the definition of the quadratic L-groups $L_n(\mathbb{A})$ $(n \geq 0)$ of an additive category $\mathbb{A}$ with involution $* : \mathbb{A} \longrightarrow \mathbb{A}$, as the cobordism groups of n-dimensional quadratic Poincaré complexes $(C, \psi \in Q_n(C))$ in $\mathbb{A}$, and for the proof that these groups are 4-periodic, with $L_{2i}(\mathbb{A})$ (resp. $L_{2i+1}(\mathbb{A})$) the Witt group of nonsingular $(-)^i$-quadratic forms (resp. formations) in $\mathbb{A}$.

We now put an involution on the notions of §1.

Definition 2.1 An involution on an additive category $\mathbb{A}$ is a contravariant functor

$$* \ : \ \mathbb{A} \longrightarrow \mathbb{A} \ ; \ M \longrightarrow M^* \ ,$$

$$(f:M \longrightarrow N) \longrightarrow (f^*:N^* \longrightarrow M^*)$$

together with a natural equivalence

$$e : \mathrm{id}_{\mathbb{A}} \longrightarrow ** : \mathbb{A} \longrightarrow \mathbb{A} ;$$

$$M \longrightarrow (e(M):M \longrightarrow M^{**})$$

such that

$$e(M^*) = (e(M)^{-1})^* : M^* \longrightarrow M^{***} .$$

□

We shall use the natural isomorphisms $e(M):M \longrightarrow M^{**}$ to identify $M^{**}=M$.

<u>Example</u> <u>2.2</u> Given a ring R with involution

$$\bar{\ } : R \longrightarrow R ; r \longrightarrow \bar{r}$$

let the additive category

$$\mathbb{B}(R) = \{\text{based f.g. free } R\text{-modules}\}$$

have the duality involution

$$(R^n)^* = R^n \quad , \quad (a_{ij})^* = (\bar{a}_{ji}) ,$$

such that

$$L_n(\mathbb{B}(R)) = L_n(R) \quad (n \geqslant 0) .$$

By definition, a quadratic Poincaré complex over R is the same as a quadratic Poincaré complex in $\mathbb{B}(R)$.

□

<u>Notation</u> <u>2.3</u> Let $\mathbb{A}$ be an additive category with involution.

i) A chain complex C in $\mathbb{A}$ is <u>n-dimensional</u> if $C_r=0$ for $r<0$ and $r>n$.

ii) The <u>n-dual</u> of an n-dimensional chain complex C is the n-dimensional chain complex C^{n-*} in $\mathbb{A}$ with

$$d_{C^{n-*}} = (-)^r(d_C)^* :$$

$$(C^{n-*})_r = C^{n-r} = (C_{n-r})^* \longrightarrow (C^{n-*})_{r-1} .$$

iii) For $n\geqslant 0$ let $\mathbb{D}_n(\mathbb{A})$ be the additive category of n-dimensional chain complexes in $\mathbb{A}$ and chain homotopy classes of chain maps, with the n-duality involution $T=n-*:\mathbb{D}_n(\mathbb{A})\longrightarrow\mathbb{D}_n(\mathbb{A});C\longrightarrow C^{n-*}$.

□

A functor of additive categories with involution $F:\mathbb{A}\longrightarrow\mathbb{B}$ is a functor of the underlying additive categories together with a natural equivalence $G:F*\longrightarrow *F:\mathbb{A}\longrightarrow\mathbb{B}$, such that for any object M in $\mathbb{A}$ there is defined a commutative diagram in $\mathbb{B}$

$$\begin{array}{ccc}
F(M) & \xrightarrow{e_{\mathbb{B}}(F(M))} & F(M)^{**} \\
{\scriptstyle F(e_{\mathbb{A}}(M))}\Big\downarrow & & \Big\downarrow{\scriptstyle G(M)^*} \\
F(M^{**}) & \xrightarrow{G(M^*)} & F(M^*)^* .
\end{array}$$

<u>Notation</u> <u>2.4</u> A functor $F:\mathbb{A}\longrightarrow\mathbb{B}$ of additive categories with involution induces morphisms of the quadratic L-groups which we write as

$$F_! \ : \ L_n(\mathbb{A}) \longrightarrow L_n(\mathbb{B}) \ ;$$

$$(C,\psi) \longrightarrow (F(C),F(\psi)) \quad (n\geqslant 0) .$$

□

<u>Example</u> <u>2.5</u> A morphism of rings with involution $f:R\longrightarrow S$

determines functors of additive categories with involution $f_!:\mathbb{B}(R)\longrightarrow\mathbb{B}(S)$ which induces change of rings morphisms in the quadratic L-groups $f_!:L_n(R)\longrightarrow L_n(S)$ $(n\geqslant 0)$.

□

Definition 2.6 Given a nonsingular symmetric form $(A,\alpha=\alpha^*:A\longrightarrow A^*)$ in an additive category with involution $\mathbb{A}$ let the ring $\mathrm{Hom}_{\mathbb{A}}(A,A)^{op}$ have the involution

$$\bar{\ }:\ \mathrm{Hom}_{\mathbb{A}}(A,A)^{op}\longrightarrow \mathrm{Hom}_{\mathbb{A}}(A,A)^{op}\ ;$$

$$(f:A\longrightarrow A)\longrightarrow(\alpha^{-1}f^*\alpha:A\longrightarrow A^*\longrightarrow A^*\longrightarrow A)\ .$$

□

By analogy with Definition 1.1:

Definition 2.7 A symmetric representation (A,α,U) of a ring with involution R in an additive category with involution $\mathbb{A}$ is a nonsingular symmetric form (A,α) in $\mathbb{A}$ together with a morphism of rings with involution $U:R\longrightarrow \mathrm{Hom}_{\mathbb{A}}(A,A)^{op}$.

□

In particular, (A,U) is a representation of R in the additive category $\mathbb{A}$ in the sense of 1.1.

By analogy with Example 1.2:

Example 2.8 The universal symmetric representation (R,α,U) of a ring with involution R in $\mathbb{B}(R)$ is defined by

$$\alpha\ :\ R\longrightarrow R^*\ ;\ r\longrightarrow(\ s\longrightarrow s\bar{r}\ )$$

with U the isomorphism of rings with involution

$$U : R \longrightarrow \mathrm{Hom}_R(R,R)^{op} \ ; \ r \longrightarrow (\ s \longrightarrow sr\ ) \ .$$

We shall use U as an identification of rings with involution $R = \mathrm{Hom}_R(R,R)^{op}$.

□

By analogy with Proposition 1.3:

<u>Proposition</u> <u>2.9</u> Given a ring with involution R and an additive category with involution $\mathbb{A}$ there is a natural one-one correspondence between functors of pairs of additive categories with involution $F:\mathbb{B}(R)\longrightarrow\mathbb{A}$ and symmetric representations (A,α,U) of R in $\mathbb{A}$.

<u>Proof</u>: Given a functor F define a symmetric representation (A,α,U) by

$$A = F(R) \ ,$$

$$\alpha = G(R) \ : \ F(R^*) = A \longrightarrow F(R)^* = A^* \ ,$$

$$U \ : \ R = \mathrm{Hom}_R(R,R)^{op} \longrightarrow \mathrm{Hom}_{\mathbb{A}}(A,A)^{op} \ ;$$

$$(\rho : R \longrightarrow R) \longrightarrow (F(\rho) : A \longrightarrow A) \ .$$

Conversely, given a symmetric representation (A,α,U) define a functor $F = -\otimes(A,\alpha,U):\mathbb{B}(R)\longrightarrow\mathbb{A}$ by

$$F(R) = A \ ,$$

$$G(R) = \alpha \ : \ F(R^*) = A \longrightarrow F(R)^* = A^* \ ,$$

$$F((a_{ij}):R^n \longrightarrow R^m) = (U(a_{ij})) \ : \ A^n \longrightarrow A^m \ .$$

□

By definition, a nonsingular symmetric form (C,α) in $\mathbb{D}_n(\mathbb{A})$ is an n-dimensional symmetric complex C in $\mathbb{A}$ together with a self dual chain homotopy class of chain equivalences $\alpha \simeq T\alpha : C \longrightarrow C^{n-*}$.

Proposition 2.10 A symmetric representation (C,α,U) of a ring with involution R in $\mathbb{D}_n(\mathbb{A})$ determines a functor $F=-\otimes(C,\alpha,U):\mathbb{B}(R)\longrightarrow\mathbb{D}_n(\mathbb{A})$ inducing morphisms in the quadratic L-groups

$$F_! = -\otimes(C,\alpha,U) : L_m(R) \longrightarrow L_m(\mathbb{D}_n(\mathbb{A})) \quad (m\geqslant 0) .$$

Proof: Immediate from 2.4 and 2.9.

□

Given a ring with involution S let $\mathbb{D}_n(S)=\mathbb{D}_n(\mathbb{B}(S))$, the additive category of n-dimensional chain complexes of based f.g. free S-modules and chain homotopy classes of chain maps with the n-duality involution $C\longrightarrow C^{n-*}$. A symmetric representation (C,α,U) of a ring with involution R in $\mathbb{D}_n(S)$ determines by 2.10 a functor $F=-\otimes(C,\alpha,U):\mathbb{B}(R)\longrightarrow\mathbb{D}_n(S)$ inducing morphisms in the quadratic L-groups

$$F_! = -\otimes(C,\alpha,U) : L_m(R) \longrightarrow L_m(\mathbb{D}_n(S)) \quad (m\geqslant 0) .$$

§3. The generalized Morita maps in L-theory

By analogy with the algebraic K-theory generalized Morita maps $\mu:K_m(\mathbb{D}(\mathbb{A}))\longrightarrow K_m(\mathbb{A})$ $(m=0,1)$ of §1 we define generalized Morita maps in the quadratic L-groups $\mu:L_m(\mathbb{D}_n(\mathbb{A}))\longrightarrow L_{m+n}(\mathbb{A})$ $(m,n\geqslant 0)$ by passing from nonsingular quadratic forms and formations in $\mathbb{D}_n(\mathbb{A})$ to quadratic Poincaré complexes in $\mathbb{A}$. The L-theory μ is the identity for $n=0$, since $\mathbb{D}_0(\mathbb{A})=\mathbb{A}$. For $n\geqslant 1$ the maps μ are not isomorphisms and are not induced by functors of additive categories with involution: a morphism in $\mathbb{D}_n(\mathbb{A})$ is a chain homotopy class and as in K-theory the definition of μ involves a choice of representative chain map.

Proposition 3.1 i) A nonsingular $(-)^i$-quadratic form in

$\mathbb{D}_n(\mathbb{A})$

$$(M,\theta\in\mathrm{coker}(1-(-)^iT:\mathrm{Hom}_{\mathbb{D}_n(\mathbb{A})}(M,M^*)\longrightarrow\mathrm{Hom}_{\mathbb{D}_n(\mathbb{A})}(M,M^*)))$$

is represented by an n-dimensional chain complex M in $\mathbb{A}$ together with a chain map $\theta:M\longrightarrow M^{n-*}$ such that $(1+(-)^iT)\theta=\theta+(-)^{n+i}\theta^*:M\longrightarrow M^{n-*}$ is a chain equivalence.

ii) The cobordism class $(C,\psi)\in L_{n+2i}(\mathbb{A})$ of the $(n+2i)$-dimensional quadratic Poincare complex in $\mathbb{A}$ (C,ψ) defined by $C=M^{n+i-*}$ and

$$\psi_s = \begin{cases} \theta & \text{if } s=0 \\ 0 & \text{if } s\geqslant 1 \end{cases} :$$

$$C^{n+2i-r-s} = M_{r-i-s} \longrightarrow C_r = M^{n+i-r}$$

depends only on the class

$$\theta \in \mathrm{coker}(1-(-)^iT:H_0(\mathrm{Hom}_{\mathbb{A}}(M,M^{n-*}))\longrightarrow H_0(\mathrm{Hom}_{\mathbb{A}}(M,M^{n-*})))$$

$$= \mathrm{coker}(1-(-)^iT:\mathrm{Hom}_{\mathbb{D}_n(\mathbb{A})}(M,M^{n-*})\longrightarrow\mathrm{Hom}_{\mathbb{D}_n(\mathbb{A})}(M,M^{n-*}))\ .$$

iii) Suppose given (C,ψ) as in ii), an n-dimensional chain complex L, a chain map $j:L\longrightarrow M$ and $\{\chi\in\mathrm{Hom}_A(L_r,L^{n+1-r})\,|\,r\geqslant 0\}$ defining a chain homotopy

$$\chi+(-)^{n+i+1}\chi^* \ :\ j^*\psi j \simeq 0 \ :\ L \longrightarrow L^{n-*}$$

such that the chain map $(j^*(1+T)\psi_0\ \ 0):C(j)\longrightarrow L^{n-*}$ is a chain equivalence, with $C(j)$ the algebraic mapping cone of j. Then $(C,\psi)=0\in L_{n+2i}(\mathbb{A})$.

<u>Proof</u>: i) Trivial.

ii) The isomorphism of abelian groups

$$Q_{(-)^i}(M) = \mathrm{coker}(1-(-)^iT:\mathrm{Hom}_{\mathbb{D}_n(\mathbb{A})}(M,M^*)$$
$$\longrightarrow\mathrm{Hom}_{\mathbb{D}_n(\mathbb{A})}(M,M^*))$$
$$\longrightarrow Q_{n+2i}(C)\ ;$$

$$[\theta : M \longrightarrow M^{n-*}] \longrightarrow \{\psi_s \in \mathrm{Hom}_{\mathbb{A}}(C^{n+2i-r-s}, C_r) \mid r, s \geqslant 0\}$$

defined by

$$\psi_0 = \theta \ , \ \psi_s = 0 \text{ for } s \geqslant 1$$

sends the class of θ to the quadratic structure $\psi \in Q_{n+2i}(C)$.

iii) Define an $(n+2i+1)$-dimensional quadratic Poincaré pair in $\mathbb{A}$ $(f : C \longrightarrow D, (\delta\psi, \psi))$ by

$$f = j^* : C = M^{n+i-*} \longrightarrow D = L^{n+i-*} ,$$

$$\delta\psi_0 = \chi : D^{n+2i+1-r} = L_{r-i-1} \longrightarrow D_r = L^{n+i-r} ,$$

$$\delta\psi_s = 0 \text{ for } s \geqslant 1 .$$

$\square$

We refer to §2 of Ranicki [19] for the definition of a nonsingular $(-)^i$-quadratic formation $(F,G) = (F, \begin{bmatrix} \gamma \\ \mu \end{bmatrix} G)$ in an additive category with involution $\mathbb{A}$, and for the result that $(F,G) = 0 \in L_{2i+1}(\mathbb{A})$ if and only if there there exist a $(-)^{i+1}$-quadratic form in $\mathbb{A}$ (H, ζ) and a morphism $j : F \longrightarrow H^*$ such that the morphism defined in $\mathbb{A}$ by

$$\begin{bmatrix} \mu^* & \gamma^* j^* \\ j & \zeta + (-)^{i+1} \zeta^* \end{bmatrix} : F \oplus H \longrightarrow G^* \oplus H^*$$

is an isomorphism.

<u>Proposition</u> <u>3.2</u> i) A nonsingular $(-)^i$-quadratic formation $(F, \begin{bmatrix} \gamma \\ \mu \end{bmatrix} G)$ in $\mathbb{D}_n(\mathbb{A})$ is represented by

n-dimensional chain complexes F,G in $\mathbb{A}$ together with chain maps $\gamma:G\longrightarrow F$, $\mu:G\longrightarrow F^{n-*}$, $\theta:G\longrightarrow G^{n-*}$ and a chain homotopy

$$\chi\ :\ \gamma^{*}\mu\ \simeq\ \theta+(-)^{n+i+1}\theta^{*}\ :\ G\longrightarrow G^{n-*}$$

such that the chain map

$$\begin{pmatrix} \chi+(-)^{n+i}\chi^{*} & \gamma^{*} \\ \gamma & 0 \end{pmatrix}\ :\ C(\mu^{*})^{n+1-*}\longrightarrow C(\mu^{*}:F\longrightarrow G^{n-*})$$

is a chain equivalence.

ii) The cobordism class $(C,\psi)\in L_{n+2i+1}(\mathbb{A})$ of the $(n+2i+1)$-dimensional quadratic Poincaré complex $(C=S^{i}C(\mu^{*}),\psi)$ in $\mathbb{A}$ with

$$d_C\ =\ \begin{pmatrix} d_G^{*} & (-)^{r-1}T\mu \\ 0 & d_F \end{pmatrix}\ :$$

$$C_r\ =\ G^{n-r+i}\oplus F_{r-i-1}\longrightarrow C_{r-1}\ =\ G^{n-r-i+1}\oplus F_{r-i-2}\ ,$$

$$\psi_0\ =\ \begin{pmatrix} (-)^{(n+1)(r-1)}\chi & 0 \\ (-)^{n(r-1)}\gamma & 0 \end{pmatrix}\ :$$

$$C^{n+2i+1-r}\ =\ G_{r-i-1}\oplus F^{n-r+i}$$

$$\longrightarrow C_r\ =\ G^{n-r+i}\oplus F_{r-i-1}\ ,$$

$$\psi_1\ =\ \begin{pmatrix} (-)^{(n+1)r+1}\theta & 0 \\ 0 & 0 \end{pmatrix}\ :$$

$$C^{n+2i-r}\ =\ G_{r-i}\oplus F^{n-r+i-1}$$

$$\longrightarrow C_{r-i}\ =\ G^{n-r+i}\oplus F_{r-i-1}\ ,$$

$$\psi_s = 0 : C^{n+2i+1-r-s} \longrightarrow C_r \text{ for } s \geqslant 2 ,$$

depends only on the chain homotopy classes

$$\gamma \in H_0(\mathrm{Hom}_{\mathbb{A}}(G,F)) = \mathrm{Hom}_{\mathbb{D}_n(\mathbb{A})}(G,F) ,$$

$$\mu \in H_0(\mathrm{Hom}_{\mathbb{A}}(G,F^{n-*})) = \mathrm{Hom}_{\mathbb{D}_n(\mathbb{A})}(G,F^{n-*}) .$$

iii) Suppose given (C,ψ) as in i), an n-dimensional chain complex H in $\mathbb{A}$ and chain maps $j:F \longrightarrow H^{n-*}$, $\zeta : H \longrightarrow H^{n-*}$ such that the chain map

$$\begin{bmatrix} \mu^* & \gamma^* j^* \\ j & \zeta + (-)^{n+i+1}\zeta^* \end{bmatrix} : F \oplus H \longrightarrow G^{n-*} \oplus H^{n-*}$$

is a chain equivalence. Then $(C,\psi) = 0 \in L_{n+2i+1}(\mathbb{A})$.

<u>Proof</u>: i) The inclusion of the lagrangian $(G,0) \longrightarrow H_{(-)^i}(F)$ extends to an isomorphism of $(-)^i$-quadratic forms in $\mathbb{D}_n(\mathbb{A})$

$$\begin{bmatrix} \gamma & \tilde{\gamma} \\ \mu & \tilde{\mu} \end{bmatrix} : H_{(-)^i}(G) \longrightarrow H_{(-)^i}(F)$$

which is represented by a chain equivalence in $\mathbb{A}$

$$\begin{bmatrix} \gamma & \tilde{\gamma} \\ \mu & \tilde{\mu} \end{bmatrix} : G \oplus G^{n-*} \longrightarrow F \oplus F^{n-*}$$

with chain homotopy inverse

$$\begin{bmatrix} \gamma & \tilde{\gamma} \\ \mu & \tilde{\mu} \end{bmatrix}^{-1} = \begin{bmatrix} \tilde{\mu}^* & (-)^i\tilde{\gamma}^* \\ (-)^i\mu^* & \gamma^* \end{bmatrix} :$$

$$F \oplus F^{n-*} \longrightarrow G \oplus G^{n-*} .$$

For any representative chain map $\tilde{\mu}:G^{n-*}\longrightarrow F^{n-*}$ there exist a chain map $\nu:F\longrightarrow F^{n-*}$ and a chain homotopy

$$\eta \ : \ \mu\tilde{\mu}^{*} \simeq \nu + (-)^{n+i+1}\nu^{*} \ : \ F \longrightarrow F^{n-*} \ .$$

The chain maps in $\mathbb{A}$ defined by

$$f = \begin{bmatrix} \chi+(-)^{n+i}\chi^{*} & \gamma^{*} \\ \gamma & 0 \end{bmatrix} : C(\mu^{*})^{n+1-*} \longrightarrow C(\mu^{*})$$

$$g = \begin{bmatrix} 0 & \tilde{\mu}^{*} \\ \tilde{\mu} & \eta+(-)^{n+i}\eta^{*} \end{bmatrix} : C(\mu^{*}) \longrightarrow C(\mu^{*})^{n+1-*}$$

are such that there are defined chain homotopies

$$\begin{bmatrix} \epsilon & \tilde{\gamma}^{*} \\ 0 & \delta \end{bmatrix} : gf \simeq \begin{bmatrix} 1 & 0 \\ \beta & 1 \end{bmatrix} = \text{automorphism}$$

$$: C(\mu^{*})^{n+1-*} \longrightarrow C(\mu^{*})^{n+1-*} ,$$

$$\begin{bmatrix} \epsilon^{*} & 0 \\ \tilde{\gamma} & \delta^{*} \end{bmatrix} : fg \simeq \begin{bmatrix} 1 & \beta^{*} \\ 0 & 1 \end{bmatrix} = \text{automorphism}$$

$$: C(\mu^{*}) \longrightarrow C(\mu^{*})$$

with $\beta=\tilde{\mu}(\chi+(-)^{n+i}\chi^{*})+(\eta+(-)^{n+i}\eta^{*})\gamma$, and δ,ϵ chain homotopies

$$\delta \ : \ \tilde{\mu}\gamma^{*}+(-)^{i}\mu\tilde{\gamma}^{*} \simeq 1 \ : \ F^{n-*} \longrightarrow F^{n-*} ,$$

$$\epsilon \ : \ \tilde{\mu}^{*}\gamma+(-)^{i}\tilde{\gamma}^{*}\mu \simeq 1 \ : \ G \longrightarrow G \ .$$

Thus both fg and gf are chain equivalences, and f is a chain equivalence with chain homotopy inverse $(gf)^{-1}g\simeq g(fg)^{-1}$.

ii) With $\begin{bmatrix} \gamma & \tilde{\gamma} \\ \mu & \tilde{\mu} \end{bmatrix}$ as in i) there exist chain maps in $\mathbb{A}$

$$\tilde{\gamma} : G^{n-*} \longrightarrow F \ , \ \tilde{\mu} : G^{n-*} \longrightarrow F^{n-*} \ ,$$

$$\tilde{\theta} : G^{n-*} \longrightarrow G$$

and a chain homotopy

$$\tilde{\chi} : \tilde{\gamma}^*\tilde{\mu} \simeq \tilde{\theta} + (-)^{n+i+1}\tilde{\theta}^* \ : \ G^{n-*} \longrightarrow G$$

such that the chain map

$$\begin{bmatrix} \tilde{\chi}+(-)^{n+i}\tilde{\chi}^* & \tilde{\gamma}^* \\ \tilde{\gamma} & 0 \end{bmatrix} : C(\tilde{\mu}^*)^{n+1-*} \longrightarrow C(\tilde{\mu}^*)$$

is a chain equivalence in $\mathbb{A}$. Let $(\tilde{C},\tilde{\psi})$ be the $(n+2i+1)$-dimensional quadratic Poincaré complex derived from $(F,G^{n-*},\tilde{\gamma},\tilde{\mu},\tilde{\theta},\tilde{\chi})$ in the way (C,ψ) is derived from $(F,G,\gamma,\mu,\theta,\chi)$. Define an $(n+2i+2)$-dimensional quadratic Poincaré cobordism $((f\ \tilde{f}):C\oplus\tilde{C}\longrightarrow D,(\delta\psi,\psi\oplus-\tilde{\psi}))$ by

$$D = S^{i+1}F \ , \ \delta\psi = 0 \ ,$$

$$f = (0\ \ 1) \ : \ C_r = G^{n-r+i}\oplus F_{r-i-1} \longrightarrow D_r = F_{r-i-1}$$

$$\tilde{f} = (0\ \ 1) \ : \ \tilde{C}_r = G_{r-i}\oplus F_{r-i-1} \longrightarrow D_r = F_{r-i-1} \ .$$

Thus $(C,\psi)=(\tilde{C},\tilde{\psi})\in L_{n+2i+1}(\mathbb{A})$. Since $\tilde{\theta}$ and $\tilde{\chi}$ can be chosen independently of θ and χ it follows that the cobordism is independent of these choices also. Given $(F,G,\gamma,\mu,\theta,\chi)$ and chain equivalences $h:F\longrightarrow F'$, $k:G\longrightarrow G'$ it is possible to define $(F',G',\gamma',\mu',\theta',\chi')$ such that the corresponding quadratic Poincaré complex (C',ψ') is homotopy equivalent to (C,ψ), and so $(C',\psi')=(C,\psi)\in L_{n+2i+1}(\mathbb{A})$.

iii) Define an $(n+2i+2)$-dimensional quadratic Poincaré pair $(f:C\longrightarrow D,(\delta\psi,\psi))$ by

$D = H^{n+i+1-*}$,

$f = (0 \ \ j)$:

$$C_r = G^{n-r+i} \oplus F_{r-i-1} \longrightarrow D_r = H^{n+i+1-r} ,$$

$\delta\psi_0 = \zeta$:

$$D^{n+2i+2-r} = H_{r-i-1} \longrightarrow D_r = H^{n+i+1-r} ,$$

$\delta\psi_s = 0$ for $s \geq 1$.

This is a quadratic Poincaré null-cobordism of (C,ψ), so that $(C,\psi)=0 \in L_{n+2i+1}(\mathbb{A})$.

□

<u>Definition 3.3</u> For any additive category with involution $\mathbb{A}$ define the <u>generalized Morita maps</u>

$$\mu : L_m(\mathbb{D}_n(\mathbb{A})) \longrightarrow L_{m+n}(\mathbb{A}) \quad (m,n \geq 0)$$

for $m=2i$ (resp. $2i+1$) by sending a nonsingular $(-)^i$-quadratic form (M,θ) (resp. formation (F,G)) in $\mathbb{D}_n(\mathbb{A})$ to the cobordism class of the $(m+n)$-dimensional quadratic Poincaré complex (C,ψ) in $\mathbb{A}$ defined in Proposition 3.1 ii) (resp. 3.2 ii)). The verification that the maps μ are well-defined is contained in Propositions 3.1 iii) (resp. 3.2 iii)).

□

For a ring with involution R apply 3.3 to $\mathbb{A}=\mathbb{B}(R)$ to obtain generalized Morita maps $\mu : L_m(\mathbb{D}_n(R)) \longrightarrow L_{m+n}(R)$ $(m,n \geq 0)$.

§4. <u>The quadratic L-theory transfer</u>

As before, let $\mathbb{A}$ be an additive category with

involution, and let $\mathbb{D}_n(\mathbb{A})$ be the chain homotopy category of n-dimensional chain complexes in $\mathbb{A}$ with the n-duality involution.

<u>Definition 4.1</u> The <u>quadratic</u> <u>L-theory</u> <u>transfer</u> <u>maps</u> of a symmetric representation (C,α,U) of a ring with involution R in $\mathbb{D}_n(\mathbb{A})$

$$(C,\alpha,U)^! : L_m(R) \longrightarrow L_{m+n}(\mathbb{A}) \quad (m\geqslant 0)$$

are the composites

$$(C,\alpha,U)^! : L_m(R) = L_m(\mathbb{B}(R)) \xrightarrow{-\otimes(C,\alpha,U)} L_m(\mathbb{D}_n(\mathbb{A})) \xrightarrow{\mu} L_{m+n}(\mathbb{A})$$

of the maps $-\otimes(C,\alpha,U)$ of 2.10 and the generalized Morita maps μ of 3.3.

□

<u>Example</u> <u>4.2</u> Let $\mathbb{A}$ be the additive category $\mathbb{B}(S)$ of based f.g. free S-modules with the duality involution, for a ring with involution S. The transfer maps determined by an n-dimensional symmetric representation (C,α,U) of a ring with involution R in $\mathbb{D}_n(\mathbb{A})=\mathbb{D}_n(S)$ are morphisms of quadratic L-groups

$$(C,\alpha,U)^! : L_m(R) \longrightarrow L_{m+n}(\mathbb{A}) = L_{m+n}(S) \quad (m,n\geqslant 0) .$$

□

<u>Example</u> <u>4.3</u> Given a Hurewicz fibration $F \longrightarrow E \xrightarrow{p} B$ with the fibre F a finite n-dimensional geometric Poincaré complex we shall define in §5 below a symmetric representation $(C(\tilde{F}),\alpha,U)$ of $\mathbb{Z}[\pi_1(B)]$ in $\mathbb{D}_n(\mathbb{Z}[\pi_1(E)])$, with $\tilde{F}$ the pullback to F of the universal cover $\tilde{E}$ of E and $\alpha=([F]\cap -)^{-1} : C(\tilde{F}) \longrightarrow C(\tilde{F})^{n-*}$ the Poincaré duality chain equivalence. The algebraic surgery transfer maps will be defined in §5 to be

$$p^!_{alg} = (C(\tilde{F}),\alpha,U)^! :$$

$$L_m(\mathbb{Z}[\pi_1(B)]) \longrightarrow L_{m+n}(\mathbb{Z}[\pi_1(E)]) \quad (m\geqslant 0) .$$

In §6 we shall recall the definition via the lifting of normal maps of the geometric surgery transfer maps $p^!_{geo}$, which will be identified with $p^!_{alg}$ in §8.

□

<u>Example</u> <u>4.4</u> Given a morphism of rings with involution $f:R\longrightarrow S$ define a symmetric representation (C,α,U) of R in $\mathbb{D}_0(S)=\mathbb{B}(S)$ by

$$\alpha_0 : C_0 = S \longrightarrow C^0 = S^* ;$$

$$s \longrightarrow (\ t \longrightarrow t\bar{s}\ ) ,$$

$$C_r = 0 \text{ for } r\neq 0 ,$$

$$U = f : R \longrightarrow H_0(\mathrm{Hom}_S(C,C))^{op} = S .$$

In this case the transfer maps are just the change of rings morphisms $(C,\alpha,U)^! = f_! : L_m(R)\longrightarrow L_m(S)$. For $f=1:R\longrightarrow S=R$ (C,α,U) is the universal symmetric representation (2.8) of R in $\mathbb{B}(R)$.

□

<u>Example</u> <u>4.5</u> Given a ring with involution S and an integer $k\geqslant 1$ let $R=M_k(S)$ be the ring of $k\times k$ matrices $(s_{ij})_{1\leqslant i,j\leqslant k}$ with entries $s_{ij}\in S$, with the involution

$$\bar{\ } : R \longrightarrow R ; (s_{ij}) \longrightarrow (\bar{s}_{ji}) .$$

Define a symmetric representation (C,α,U) of R in $\mathbb{D}_0(S)=\mathbb{B}(S)$ by

$$C_0 = \sum_1^k S \quad , \quad C_r = 0 \text{ for } r \neq 0 \ ,$$

$$\alpha : C_0 = \sum_1^k S \longrightarrow C^0 = (\sum_1^k S)^* \ ;$$

$$(s_1, s_2, \ldots, s_k) \longrightarrow$$

$$((t_1, t_2, \ldots, t_k) \longrightarrow t_1\bar{s}_1 + t_2\bar{s}_2 + \ldots + t_k\bar{s}_k) \ ,$$

$$U = 1 : R = M_k(S) \longrightarrow H_0(\mathrm{Hom}_S(C,C))^{op} = M_k(S) \ .$$

The generalized Morita maps $\mu : L_*(R) \longrightarrow L_*(S)$ in this case are just the usual Morita maps, which are isomorphisms for the projective and round L-groups. See Hambleton, Taylor and Williams [5] and Hambleton, Ranicki and Taylor [4] for Morita maps in quadratic L-theory.

□

<u>Example 4.6</u> Let $F = \bigcup_k \{*\} \longrightarrow E \xrightarrow{p} B$ be a k-sheeted finite covering, so that $\pi_1(E)$ is a subgroup of $\pi_1(B)$ of index k. There are evident identifications of spaces

$$\tilde{F} = \pi_1(B) = \bigcup_k \pi_1(E) \subset \tilde{B} = \tilde{E} \ ,$$

and also of $\mathbb{Z}$-module chain complexes

$$C(\tilde{F}) = \mathbb{Z}[\pi_1(B)] = \bigoplus_k \mathbb{Z}[\pi_1(E)] \ .$$

The symmetric representation $(C(\tilde{F}), \alpha, U)$ of $\mathbb{Z}[\pi_1(B)]$ in $\mathbb{D}_0(\mathbb{Z}[\pi_1(E)]) = \mathbb{B}(\mathbb{Z}[\pi_1(E)])$ associated to $p : E \longrightarrow B$ (as in 4.3) is given by

$$U : \mathbb{Z}[\pi_1(B)] = H_0(\mathrm{Hom}_{\mathbb{Z}[\pi_1(B)]}(C(\tilde{F}), C(\tilde{F})))^{op}$$

$$\xrightarrow{\text{restriction}}$$

$$H_0(\mathrm{Hom}_{\mathbb{Z}[\pi_1(E)]}(C(\tilde{F}),C(\tilde{F})))^{op} = M_k(\mathbb{Z}[\pi_1(E)]) \ ,$$

$$\alpha = \bigoplus_k 1 \ : \ C(\tilde{F}) = \bigoplus_k \mathbb{Z}[\pi_1(E)] \longrightarrow$$

$$\mathrm{Hom}_{\mathbb{Z}[\pi_1(E)]}(C(\tilde{F}),\mathbb{Z}[\pi_1(E)]) = \bigoplus_k \mathbb{Z}[\pi_1(E)]^* \ .$$

The algebraic transfer maps in this case are the composites

$$p^!_{alg} \ : \ L_m(\mathbb{Z}[\pi_1(B)]) \xrightarrow{U_!}$$

$$L_m(M_k(\mathbb{Z}[\pi_1(E)])) \xrightarrow{\mu} L_m(\mathbb{Z}[\pi_1(E)])$$

with $U_!$ induced by U as in 2.5 and μ the Morita maps of 4.5. In this case $p^!_{alg}$ can be described more directly by the restrictions of $\mathbb{Z}[\pi_1(B)]$-module actions to $\mathbb{Z}[\pi_1(E)]$-module actions, and it is clear that $p^!_{alg}=p^!_{geo}$.

□

<u>Example</u> <u>4.7</u> The algebraic S^1-bundle transfer maps of Munkholm and Pedersen [10] and Ranicki [16,§7.8] $p^!_{alg}:L_m(R)\longrightarrow L_{m+1}(S)$ are defined for any ring with involution S, with $R=S/(t-1)$ for a central element $t\in S$ such that $\bar{t}=t^{-1}$. (We are only dealing with the orientable case here). From our point of view these are the quadratic L-theory transfer maps $p^!_{alg}=(C,\alpha,U)^!$ of 4.1 with (C,α,U) the symmetric representation of R in $\mathbb{D}_1(S)$ given by

$$d = 1-t \ : \ C_1 = S \longrightarrow C_0 = S \ ,$$

$$\alpha = \begin{cases} -t : C_1 = S \longrightarrow C^0 = S \\ 1 : C_0 = S \longrightarrow C^1 = S . \end{cases}$$

For an S^1-bundle $S^1 \longrightarrow E \xrightarrow{p} B$ one takes $R = \mathbb{Z}[\pi_1(B)]$, $S = \mathbb{Z}[\pi_1(E)]$, $t = \text{fibre} \in \pi_1(E)$.

□

§5. <u>The algebraic surgery transfer</u>

A map $p: E \longrightarrow B$ of connected spaces with homotopy fibre of the homotopy type of a finite (or finitely dominated) CW complex F determines a representation of $\mathbb{Z}[\pi_1(B)]$ in $\mathbb{D}(\mathbb{Z}[\pi_1(E)])$

$$(C(\tilde{F}), U: \mathbb{Z}[\pi_1(B)] \longrightarrow H_0(\mathrm{Hom}_{\mathbb{Z}[\pi_1(E)]}(C(\tilde{F}), C(\tilde{F})))^{op})$$

as in 1.7. We shall now show that if F is a finite n-dimensional geometric Poincaré complex then for any choice of orientation map $w(B): \pi_1(B) \longrightarrow \mathbb{Z}_2$ in the base there is defined a symmetric representation $(C(\tilde{F}), \alpha, U)$ of $\mathbb{Z}[\pi_1(B)]$ in $\mathbb{D}_n(\mathbb{Z}[\pi_1(E)])$, and hence obtain from §4 quadratic L-theory transfer maps

$$p^!_{alg} = (C(\tilde{F}), \alpha, U)^! :$$

$$L_m(\mathbb{Z}[\pi_1(B)]) \longrightarrow L_{m+n}(\mathbb{Z}[\pi_1(E)]) \quad (m \geqslant 0) .$$

In §8 below we shall identify these algebraic surgery transfer maps with the geometric surgery transfer maps.

There is no loss of generality in assuming that $F \longrightarrow E \xrightarrow{p} B$ is a Hurewicz fibration with the fibre $F = p^{-1}(*)$ a finite CW complex F. If F is disconnected then $p: E \longrightarrow B$ is the composite of a Hurewicz fibration

$p':E \longrightarrow B'$ with connected fibre $p'^{-1}(*)$ and a finite covering $B' \longrightarrow B$. Since transfer theory is well-known for finite covers (cf. 4.6) there is no loss of generality in taking F to be connected. In fact, the algebraic transfer maps are defined in exactly the same way for disconnected F, and only the geometric treatment of the orientation maps has to be modified by using groupoids instead of groups.

Transport of the fibre along paths in the base space gives a map $\Omega B \longrightarrow F^F$ which on π_0 induces a group morphism $U:\pi_1(B) \longrightarrow [F,F]$ to the monoid of homotopy classes of self-maps of F (Whitehead [24,p.186]). Analogously, one has the pointed transport of the pointed fibre along paths in E, defining a morphism $U^+:\pi_1(E) \longrightarrow [F,F]^+$ to the monoid of pointed homotopy classes of pointed self-maps of F. Homotopy along a path defines a morphism $\pi_1(F) \longrightarrow [F,F]^+$ (Whitehead [24,p.98ff]).

<u>Proposition</u> <u>5.1</u> The transport maps define a morphism from an exact sequence of groups to an exact sequence of pointed sets

$$\begin{tikzcd}
\pi_1(F) \arrow[r] \arrow[d, equal] & \pi_1(E) \arrow[r, "p_*"] \arrow[d, "U^+"] & \pi_1(B) \arrow[r] \arrow[d, "U"] & \{1\} \\
\pi_1(F) \arrow[r] & {[F,F]^+} \arrow[r] & {[F,F]} \arrow[r] & \{1\}
\end{tikzcd}$$

□

We shall now use 5.1 in the case when F is a geometric Poincaré complex to lift an orientation map w(B) for $\pi_1(B)$ to an orientation map w(E) for $\pi_1(E)$.

__Definition 5.2__ An __orientation map__ for a group π is a morphism $w:\pi \longrightarrow \mathbb{Z}_2=\{\pm 1\}$. Let $\mathbb{Z}[\pi]^w$ denote the ring $\mathbb{Z}[\pi]$ with the __w-twisted involution__

$$- : \mathbb{Z}[\pi] \longrightarrow \mathbb{Z}[\pi] \ ; \ \sum_{g\in\pi} n_g g \longrightarrow \sum_{g\in\pi} n_g w(g) g^{-1} \ .$$

Given a chain complex C in $\mathbb{B}(\mathbb{Z}[\pi])$ let ${}^wC^{n-*}$ denote the n-dual chain complex C^{n-*} in $\mathbb{B}(\mathbb{Z}[\pi])$ defined using the w-twisted involution on $\mathbb{Z}[\pi]$. If w is trivial ${}^wC^{n-*}$ is written as C^{n-*}. Let $\mathbb{Z}^w$ denote the right $\mathbb{Z}[\pi]$-module with additive group $\mathbb{Z}$ and

$$\mathbb{Z}^w \times \mathbb{Z}[\pi] \longrightarrow \mathbb{Z}^w \ ; \ (m, \sum_{g\in\pi} n_g g) \longrightarrow m(\sum_{g\in\pi} w(g) n_g) \ .$$

Let ${}^w\mathbb{Z}$ denote the left $\mathbb{Z}[\pi]$-module defined in the same way.

□

When w is clear we abbreviate $\mathbb{Z}[\pi]^w$ to $\mathbb{Z}[\pi]$.

An n-dimensional geometric Poincaré complex X is a (connected) finite CW complex together with an orientation map $w(X):\pi_1(X) \longrightarrow \mathbb{Z}_2$ and a fundamental class

$$[X] \in H_n(X;\mathbb{Z}^{w(X)}) = H_n(\mathbb{Z}^w \otimes_{\mathbb{Z}[\pi_1(X)]} C(\tilde{X}))$$

such that the $\mathbb{Z}[\pi_1(X)]$-module chain map $[X] \cap - : {}^{w(X)}C(\tilde{X})^{n-*} \longrightarrow C(\tilde{X})$ is a chain equivalence, with $\tilde{X}$ the universal cover. See Wall [21] for the general theory.

The orientation map $w=w(X):\pi=\pi_1(X) \longrightarrow \mathbb{Z}_2$ of an

n-dimensional geometric Poincaré complex X is determined by the topology of X, since the cap product with a fundamental class $[X]\in H_n(X;\mathbb{Z}^w)$ defines an isomorphism of $\mathbb{Z}[\pi]$-modules

$$[X]\cap - \ :\ H_0({}^wC(\tilde{X})^{n-*}) \longrightarrow H_0(\tilde{X}) = \mathbb{Z}\ .$$

If $H^n(\tilde{X})$ is defined to be $H_0(C(\tilde{X})^{n-*})$ using the untwisted involution $(\bar{g}=g^{-1})$ on $\mathbb{Z}[\pi]$ then we get $H^n(\tilde{X})\cong{}^w\mathbb{Z}$.

<u>Definition</u> <u>5.3</u> Let X be an n-dimensional geometric Poincaré complex.

i) The <u>degree</u> of a pointed self-map $f:X\longrightarrow X$ is the number $d(f)\in\mathbb{Z}$ such that

$$\tilde{f}^*\ :\ H^n(\tilde{X}) \longrightarrow H^n(\tilde{X})\ ;\ 1 \longrightarrow d(f)\ ,$$

with $\tilde{f}:\tilde{X}\longrightarrow\tilde{X}$ a lift of f to a self map of the universal cover $\tilde{X}$.

ii) The <u>homotopy</u> <u>orientation</u> of X is the monoid morphism

$$\hat{w} = \hat{w}(X)\ :\ [X,X]^+ \longrightarrow \mathbb{Z}^\times\ ;\ f \longrightarrow d(f)\ ,$$

with $\mathbb{Z}^\times$ the monoid defined by $\mathbb{Z}$ and multiplication.

□

Let $f:X\longrightarrow X$ be a pointed self homotopy equivalence, inducing an automorphism $f_*:\pi\longrightarrow\pi$ of the fundamental group $\pi=\pi_1(X)$. A lift $\tilde{f}:\tilde{X}\longrightarrow\tilde{X}$ of f to the universal cover $\tilde{X}$ induces a $\mathbb{Z}$-module chain equivalence $\tilde{f}:C(\tilde{X})\longrightarrow C(\tilde{X})$ which is f_*-equivariant

$$\tilde{f}(gx) = f_*(g)(x) \in C(\tilde{X}) \quad (g\in\pi, x\in C(\tilde{X}))\ .$$

The induced isomorphism of additive groups $\tilde{f}^*: H^n(\tilde{X}) = {}^w\mathbb{Z} \longrightarrow H^n(\tilde{X}) = {}^w\mathbb{Z}$ is also f_*-equivariant. Hence we have

$$w = wf_* : \pi \xrightarrow{f_*} \pi \xrightarrow{w} \mathbb{Z}_2$$

and f_* defines an automorphism $f_*: \mathbb{Z}[\pi]^w \longrightarrow \mathbb{Z}[\pi]^w$ of the ring with involution $\mathbb{Z}[\pi]^w$. The $\mathbb{Z}$-module automorphism $f_*: H_n(X;\mathbb{Z}^w) = \mathbb{Z} \longrightarrow H_n(X;\mathbb{Z}^w) = \mathbb{Z}$ is such that $f_*([X]) = d(f)[X]$, with $d(f) = \hat{w}(f) \in \{\pm 1\} = \mathbb{Z}_2 \subset \mathbb{Z}^\times$. In particular, it follows that the orientation map w and the homotopy orientation $\hat{w}$ are related by a commutative diagram of monoid morphisms

$$\begin{tikzcd}
\pi_1(X) \arrow[r] \arrow[d, "w"'] & {[X,X]^+} \arrow[d, "\hat{w}"] \\
\{\pm 1\} \arrow[r] & \mathbb{Z}^\times\ .
\end{tikzcd}$$

<u>Proposition</u> <u>5.4</u> For any pointed self homotopy equivalence $f: X \longrightarrow X$ there is defined a chain homotopy commutative diagram of $\mathbb{Z}$-module chain complexes and chain equivalences

$$\begin{tikzcd}
{}^wC(\tilde{X})^{n-*} \arrow[r, "d(f)(\tilde{f}^{-1})^*"] \arrow[d, "{[X]\cap -}"'] & {}^wC(\tilde{X})^{n-*} \arrow[d, "{[X]\cap -}"] \\
C(\tilde{X}) \arrow[r, "\tilde{f}"] & C(\tilde{X})
\end{tikzcd}$$

with the horizontal chain maps f_*-equivariant, and the

vertical chain maps $\pi_1(X)$-equivariant.

□

<u>Definition</u> <u>5.5</u> An <u>n-dimensional</u> <u>Poincaré</u> <u>fibration</u> $F \longrightarrow E \xrightarrow{p} B$ is a Hurewicz fibration with the fibre F an n-dimensional geometric Poincaré complex, together with an orientation map $w(B):\pi_1(B) \longrightarrow \mathbb{Z}_2$. The <u>lift</u> of $w(B)$ is the orientation map

$$p^! w(B) = w(E) : \pi_1(E) \longrightarrow \mathbb{Z}_2 \ ;$$

$$g \longrightarrow w(B)(p_*(g)).\hat{w}(F)(U^+(g))$$

with U^+ as in 5.1 and $\hat{w}$ as in 5.3.

□

<u>Proposition</u> <u>5.6</u> An n-dimensional Poincaré fibration $F \longrightarrow E \xrightarrow{p} B$ determines a symmetric representation $(C(\tilde{F}),\alpha,U)$ of $\mathbb{Z}[\pi_1(B)]^{w(B)}$ in $\mathbb{D}_n(\mathbb{Z}[\pi_1(E)]^{w(E)})$ with $\alpha = ([F]\cap -)^{-1} : C(\tilde{F}) \longrightarrow C(\tilde{F})^{n-*}$ the Poincaré duality chain equivalence and $(C(\tilde{F}),U)$ the representation of $\mathbb{Z}[\pi_1(B)]$ in $\mathbb{D}_n(\mathbb{Z}[\pi_1(E)])$ associated to p.

<u>Proof</u>: We have to show that

$$U : \mathbb{Z}[\pi_1(B)]^{w(B)} \longrightarrow H_0(\mathrm{Hom}_{\mathbb{Z}[\pi_1(E)]}(C(\tilde{F}),C(\tilde{F})))^{op}$$

is a morphism of rings with involution, or equivalently that for every $g \in \pi_1(B)$ there is defined a chain homotopy commutative diagram of $\mathbb{Z}[\pi_1(E)]$-module chain complexes

$$\begin{tikzcd} w(E)_{C(\tilde{F})^{n-*}} \arrow[r, "U(g)^*"] \arrow[d, "{[F]\cap -}"'] & w(E)_{C(\tilde{F})^{n-*}} \arrow[d, "{[F]\cap -}"] \\ C(\tilde{F}) \arrow[r, "w(B)(g)U(g^{-1})"] & C(\tilde{F}) \ . \end{tikzcd}$$

This follows from 5.4 and the $\pi_1(E)$-equivariant transport along the fibre $U:\pi_1(B) \longrightarrow [\tilde{F},\tilde{F}]_{\pi_1(E)}$ used to define the ring morphism U in Lück [7].

□

<u>Definition</u> <u>5.7</u> The <u>algebraic</u> <u>surgery</u> <u>transfer</u> <u>maps</u> of an n-dimensional Poincaré fibration $F \longrightarrow E \xrightarrow{p} B$ are the quadratic L-theory transfer maps of 4.1 associated to the symmetric representation $(C(\tilde{F}),\alpha,U)$ of $\mathbb{Z}[\pi_1(B)]^{w(B)}$ in $\mathbb{D}_n(\mathbb{Z}[\pi_1(E)]^{w(E)})$ given by 5.6

$$p^!_{alg} = (C(\tilde{F}),\alpha,U)^! :$$

$$L_m(\mathbb{Z}[\pi_1(B)]) \longrightarrow L_{m+n}(\mathbb{Z}[\pi_1(E)]) \quad (m \geqslant 0) \ .$$

□

By definition, the algebraic surgery transfer maps are the composites

$$p^!_{alg} : L_m(\mathbb{Z}[\pi_1(B)]) \xrightarrow{(p^{\#})_!} L_m(\mathbb{D}_n(\mathbb{Z}[\pi_1(E)]))$$

$$\xrightarrow{\mu} L_{m+n}(\mathbb{Z}[\pi_1(E)]) \quad (m \geqslant 0)$$

of the maps induced as in §2 by the functor of additive categories with involution

$$p^{\#} = -\otimes(C(\tilde{F}),\alpha,U) \; : \; \mathbb{B}(\mathbb{Z}[\pi_1(B)]) \longrightarrow \mathbb{D}_n(\mathbb{Z}[\pi_1(E)])$$

and the generalized Morita maps μ of §3.

§6. The geometric surgery transfer

Wall [22] defined the rel∂ surgery obstruction $\sigma_*(f,b)\in L_m(\mathbb{Z}[\pi_1(X)])$ for a normal map $(f,b):(M,\partial M)\longrightarrow(X,\partial X)$ from a compact m-manifold with boundary $(M,\partial M)$ to a finite m-dimensional geometric Poincaré pair $(X,\partial X)$ with $\partial f=f|:\partial M\longrightarrow\partial X$ a homotopy equivalence, and $b:\nu_M\longrightarrow\tilde{\nu}_X$ a map from the stable normal bundle of M to a topological reduction of the Spivak normal fibration ν_X of X, with the w(X)-twisted involution on $\mathbb{Z}[\pi_1(X)]$. The surgery obstruction has the property that $\sigma_*(f,b)=0$ if (and for $m\geqslant 5$ only if) (f,b) is normal bordant rel∂ to a homotopy equivalence of pairs. Given a connected space B with finitely presented $\pi_1(B)$, and given an orientation map $w(B):\pi_1(B)\longrightarrow\mathbb{Z}_2$, it is possible to realize every element $x\in L_m(\mathbb{Z}[\pi_1(B)])$ ($m\geqslant 5$) as the surgery obstruction of an m-dimensional normal map $(f,b):(M,\partial M)\longrightarrow(X,\partial X)$ with a π_1-isomorphism reference map $X\longrightarrow B$ and orientation map $w(X):\pi_1(X)\longrightarrow\pi_1(B)\xrightarrow{w(B)}\mathbb{Z}_2$

$$x = \sigma_*(f,b) \in L_m(\mathbb{Z}[\pi_1(B)]) \; .$$

The total space E of an n-dimensional Poincaré fibration $F\longrightarrow E\xrightarrow{p}B$ over an m-dimensional geometric Poincaré complex B is homotopy equivalent to an (m+n)-dimensional geometric Poincaré complex, with the orientation map the lift $w(E)=p^!w(B):\pi_1(E)\longrightarrow\mathbb{Z}_2$ in the sense of 5.5 of the orientation map $w(B):\pi_1(B)\longrightarrow\mathbb{Z}_2$ (Quinn [12], Gottlieb [2]).

Quinn [11] used the realization theorem for

surgery obstructions to define geometric transfer maps in the quadratic L-groups for a fibre bundle (or even a block fibration) $F \longrightarrow E \xrightarrow{p} B$ with the fibre F a compact n-manifold

$$p^{!}_{geo} : L_m(\mathbb{Z}[\pi_1(B)]) \longrightarrow L_{m+n}(\mathbb{Z}[\pi_1(E)]) \ ;$$

$$\sigma_*((f,b):(M,\partial M) \longrightarrow (X,\partial X))$$

$$\longrightarrow \sigma_*((g,c):(N,\partial N) \longrightarrow (Y,\partial Y)) \ .$$

Here, $(g,c):(N,\partial N) \longrightarrow (Y,\partial Y)$ the $(m+n)$-dimensional normal map equipped with a reference map $Y \longrightarrow E$ obtained from the n-dimensional normal map $(f,b):M \longrightarrow X$ by the pullback of p along a reference map $X \longrightarrow B$.

The surgery obstruction of Wall [22] was defined using geometric intersection numbers on the homology remaining after surgery below the middle dimension. The theory of Ranicki [14],[15] associates an invariant in $L_m(\mathbb{Z}[\pi_1(X)])$ to a normal map $(f,b):(M,\partial M) \longrightarrow (X,\partial X)$ of m-dimensional geometric Poincaré pairs, with $b:\nu_M \longrightarrow \nu_X$ a map of the Spivak normal fibrations and $\partial f:\partial M \longrightarrow \partial X$ a homotopy equivalence. The quadratic kernel of (f,b) is an m-dimensional quadratic Poincaré complex $(C(f^!),\psi)$ over $\mathbb{Z}[\pi_1(X)]$. Here, $C(f^!)$ is the algebraic mapping cone of the Umkehr $\mathbb{Z}[\pi_1(X)]$-module chain map

$$f^! \ : \ C(\tilde{X},\partial\tilde{X}) \xrightarrow{([X]\cap -)^{-1}} C(\tilde{X})^{m-*} \xrightarrow{\tilde{f}^*} C(\tilde{M})^{m-*}$$

$$\xrightarrow{[M]\cap -} C(\tilde{M},\partial\tilde{M})$$

with $\tilde{X}$ the universal cover of X, $\tilde{f}:\tilde{M} \longrightarrow \tilde{X}$ a $\pi_1(X)$-equivariant lift of f to the pullback cover $\tilde{M}=f^*\tilde{X}$ of M. The Poincaré duality chain equivalence is given up to chain homotopy by the composite

$$(1+T)\psi_0 \ : \ C(f^!)^{m-*} \xrightarrow{e^*} C(\tilde{M},\partial\tilde{M})^{m-*}$$

$$\xrightarrow{[M]\cap -} C(\tilde{M}) \longrightarrow C(\tilde{M},\partial\tilde{M}) \xrightarrow{e} C(f^!)$$

with $e:C(\tilde{M},\partial\tilde{M})\longrightarrow C(f^!)$ the inclusion. The quadratic signature of (f,b) is the cobordism class

$$\sigma_*(f,b) = (C(f^!),\psi) \in L_m(\mathbb{Z}[\pi_1(X)]) \ .$$

A normal map from a manifold to a geometric Poincaré complex determines a normal map of geometric Poincaré complexes with quadratic signature the surgery obstruction.

<u>Definition 6.1</u> The <u>geometric surgery transfer maps</u> of an n-dimensional Poincaré fibration $F\longrightarrow E\xrightarrow{p} B$ with finitely presented $\pi_1(B)$

$$p^!_{geo} \ : \ L_m(\mathbb{Z}[\pi_1(B)]) \longrightarrow L_{m+n}(\mathbb{Z}[\pi_1(E)]) \ ;$$

$$\sigma_*((f,b):M\longrightarrow X) \longrightarrow \sigma_*((g,c):N\longrightarrow Y) \quad (m\geqslant 5)$$

are defined using the quadratic signature of normal maps of geometric Poincaré complexes. Here, $(g,c):N\longrightarrow Y$ is the $(m+n)$-dimensional normal map obtained from an m-dimensional normal map $(f,b):M\longrightarrow X$ by the pullback of p along a reference map $X\longrightarrow B$.

□

<u>Theorem 6.2</u> The geometric surgery transfer maps of an n-dimensional Poincaré fibration $F\longrightarrow E\xrightarrow{p} B$ coincide with the algebraic surgery transfer maps

$p^!_{geo} = p^!_{alg}$:

$$L_m(\mathbb{Z}[\pi_1(B)]) \longrightarrow L_{m+n}(\mathbb{Z}[\pi_1(E)]) \quad (m\geqslant 5) .$$

□

The proof of 6.2 is deferred to §8. The ideal proof would express the quadratic kernel of the pullback normal map of the total $(m+n)$-dimensional geometric Poincaré complexes $(g,c):N\longrightarrow Y$ as a twisted tensor product of the quadratic kernel of the normal map of the base m-dimensional geometric Poincaré complexes $(f,b):M\longrightarrow X$ and the symmetric Poincaré complex $(C(\tilde{F}),\phi)$. This would generalize the chain level proof of the surgery product formula in Ranicki [15] in the untwisted case $p=\text{projection}:E=B\times F\longrightarrow B$

$$\sigma_*((f,b)\times 1:M\times F\longrightarrow X\times F) = \sigma_*(f,b)\otimes\sigma^*(F)$$

$$\in L_{m+n}(\mathbb{Z}[\pi_1(B)\times\pi_1(F)])$$

which expressed the quadratic signature of a product $(f,b)\times 1$ as the tensor product of the quadratic signature of (f,b) and the symmetric signature $\sigma^*(F)=(C(\tilde{F}),\phi)\in L^n(\mathbb{Z}[\pi_1(F)])$. However, this would require the development of a fair amount of new technology, translating the homotopy action of ΩB on the geometric Poincaré complex F into a chain homotopy action of $C(\Omega B)$ on the symmetric Poincaré complex $(C(\tilde{F}),\phi)$ over $\mathbb{Z}[\pi_1(E)]$. For the purpose at hand we can assume by the realization theorem that the m-dimensional normal map $(f,b):M\longrightarrow X$ is $[(m-2)/2]$-connected. In the highly-connected case we can give a chain level geometric interpretation of both the element $U_!\sigma_*(f,b)\in L_m(\mathbb{D}_n(\mathbb{Z}[\pi_1(E)]))$ and its image

under the generalized Morita map $\mu : L_m(\mathbb{D}_n(\mathbb{Z}[\pi_1(E)])) \longrightarrow L_{m+n}(\mathbb{Z}[\pi_1(E)])$. For a fibre bundle $F \longrightarrow E \xrightarrow{p} B$ it is possible to dispense with some of the algebra, using instead the fibred intersection theory of Hatcher and Quinn [6] as outlined in Appendix 1 below.

§7. Ultraquadratic L-theory

Ultraquadratic L-theory was developed in §7.8 of Ranicki [16] in connection with the algebraic theory of codimension 2 surgery. We use it here to recognize quadratic Poincaré complexes in the image of the generalized Morita maps $\mu : L_m(\mathbb{D}_n(\mathbb{A})) \longrightarrow L_{m+n}(\mathbb{A})$ of §3, providing a tool for the identification in §8 below of the algebraic and geometric surgery transfer maps.

Let $\mathbb{A}$ be an additive category with involution. As in Ranicki [15],[19] define for any finite chain complex C in $\mathbb{A}$ and $\epsilon = \pm 1$ the $\mathbb{Z}$-module chain complex

$$W_\% C = W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} \mathrm{Hom}_{\mathbb{A}}(C^*, C) \ ,$$

with the generator $T \in \mathbb{Z}_2$ acting on $\mathrm{Hom}_{\mathbb{A}}(C^*, C)$ by the ϵ-transposition involution $T_\epsilon = \epsilon T$ and W the standard free $\mathbb{Z}[\mathbb{Z}_2]$-module resolution of $\mathbb{Z}$

$$W : \ \ldots \longrightarrow \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1+T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}_2] \ .$$

An m-chain $\psi \in (W_\% C)_m$ is a collection of morphisms

$$\psi = \{\psi_s \in \mathrm{Hom}_{\mathbb{A}}(C^*, C)_{m-s} \mid s \geqslant 0\}$$

such that for a cycle there is defined a chain map

$(1+T_\epsilon)\psi_0 : C^{m-*} \longrightarrow C$. An m-dimensional ϵ-quadratic (Poincaré) complex (C,ψ) in $\mathbb{A}$ is an m-dimensional chain complex C in $\mathbb{A}$ together with an element $\psi \in Q_m(C,\epsilon) = H_m(W_\% C)$ (such that $(1+T_\epsilon)\psi_0 : C^{m-*} \longrightarrow C$ is a chain equivalence). The skew-suspension isomorphisms

$$\bar{S} \; : \; Q_m(C,\epsilon) \longrightarrow Q_{m+2}(SC,-\epsilon) \; ; \; \psi \longrightarrow \bar{S}\psi$$

are defined by $(\bar{S}\psi)_s = \pm\psi_s$ $(s \geqslant 0)$, for any finite chain complex C in $\mathbb{A}$. The skew-suspension maps $\bar{S} : L_m(\mathbb{A},\epsilon) \longrightarrow L_{m+2}(\mathbb{A},-\epsilon)$ $(m \geqslant 0)$ in the $\pm\epsilon$-quadratic L-groups are also isomorphisms, so that

$$L_m(\mathbb{A},\epsilon) = L_{m+2}(\mathbb{A},-\epsilon) = L_{m+4}(\mathbb{A},\epsilon) \quad (m \geqslant 0) \; .$$

For $\epsilon = 1$ we write $Q_m(C,1) = Q_m(C)$, $L_m(\mathbb{A},1) = L_m(\mathbb{A})$, and 1-quadratic = quadratic.

Ultraquadratic complexes are ϵ-quadratic complexes (C,ψ) with $\psi_s = 0$ for $s \geqslant 1$.

For any finite chain complex C in $\mathbb{A}$ define the abelian group

$$\hat{Q}_m(C) = H_m(\mathrm{Hom}_{\mathbb{A}}(C^*,C)) = H_0(\mathrm{Hom}_{\mathbb{A}}(C^{m-*},C))$$

of chain homotopy classes of chain maps $\hat{\psi} : C^{m-*} \longrightarrow C$.

<u>Definition</u> <u>7.1</u> An <u>m-dimensional</u> ϵ<u>-ultraquadratic</u> <u>(Poincaré)</u> <u>complex</u> <u>in</u> $\mathbb{A}$ $(C,\hat{\psi})$ is an m-dimensional chain complex C in $\mathbb{A}$ together with an element $\hat{\psi} \in \hat{Q}_m(C)$ (such that $(1+T_\epsilon)\hat{\psi} : C^{m-*} \longrightarrow C$ is a chain equivalence).

□

There is a corresponding notion of cobordism of

ϵ-ultraquadratic Poincaré complexes in $\mathbb{A}$, with the m-dimensional cobordism group denoted by $\hat{L}_m(\mathbb{A},\epsilon)$, and by $\hat{L}_m(\mathbb{A})$ for $\epsilon=+1$. The ϵ-ultraquadratic L-groups are 4-periodic, with

$$\hat{L}_m(\mathbb{A},\epsilon) = \hat{L}_{m+2}(\mathbb{A},-\epsilon) = \hat{L}_{m+4}(\mathbb{A},\epsilon) \quad (m\geqslant 0)$$

by skew-suspension isomorphisms, just like for the ϵ-quadratic L-groups L_*. If $\mathbb{A}=\mathbb{B}(R)$ for a ring with involution R we write $\hat{L}_m(\mathbb{A})$ as $\hat{L}_m(R)$.

Define a map $\hat{Q}_m(C)\longrightarrow Q_m(C,\epsilon);\hat{\psi}\longrightarrow\psi$ by $\psi_0=\hat{\psi}$, $\psi_s=0$ $(s\geqslant 1)$. An m-dimensional ϵ-ultraquadratic (Poincaré) complex $(C,\hat{\psi})$ determines an m-dimensional quadratic (Poincaré) complex (C,ψ). The forgetful maps in the cobordism groups

$$\hat{L}_m(\mathbb{A},\epsilon) \longrightarrow L_m(\mathbb{A},\epsilon) \ ; \ (C,\hat{\psi}) \longrightarrow (C,\psi) \quad (m\geqslant 0)$$

are surjective for even m and injective for odd m.

The ultraquadratic L-group $\hat{L}_m(\mathbb{Z})$ was identified in §7.8 of [16] with the cobordism group C_{m-1} of knots $k:S^{m-1}\subset S^{m+1}$ $(m\geqslant 4)$. A Seifert surface for a knot $k:S^{m-1}\subset S^{m+1}$ is a codimension 1 framed submanifold $M^m\subset S^{m+1}$ with boundary $\partial M=k(S^{m-1})$. Inclusion defines an m-dimensional normal map $(f,b):(M,\partial M)\longrightarrow(D^{m+2},S^{m-1})$ with quadratic kernel $\sigma_*(f,b)=(C,\psi)$ such that $H_*(C)=H_{*+1}(D^{m+2},M)=\tilde{H}_*(M)$. The framing determines a map $M\longrightarrow S^{m+1}-M$ which induces a chain map $\hat{\psi}:C^{m-*}\longrightarrow C$, defining an m-dimensional ultraquadratic Poincaré complex $(C,\hat{\psi})$ over $\mathbb{Z}$. The knot complement $U=S^{m+1}-$(open nbhd. of $k(S^{m-1})$) has boundary $\partial U=S^{m-1}\times S^1$, and there is defined an $(m+1)$-dimensional normal map $(U,\partial U)\longrightarrow(D^{m+2},S^{m-1})\times S^1$ which is a $\mathbb{Z}$-homology

equivalence. Let $(L^{m+1};M^m,zM^m)$ be the fundamental domain for the infinite cyclic cover $\bar{U}$ of U obtained by cutting U along M, and let

$$((e;f,zf),(a;b,zb)) :$$

$$(L^{m+1};M^m,zM^m) \longrightarrow D^{m+2}\times([0,1];\{0\},\{1\})$$

be the corresponding $(m+1)$-dimensional normal map of triads. The inclusions $j:M\longrightarrow L$, $k:zM\longrightarrow L$ induce $\mathbb{Z}$-module chain maps $j,k:C=C(f^!)\longrightarrow D=C(g^!)$ such that $j-k:C\longrightarrow D$ is a chain equivalence. The ultraquadratic structure $\hat{\psi}\in\hat{Q}_m(C)$ is determined by the symmetric structure $(1+T)\hat{\psi}:C^{m-*}\longrightarrow C$ and j,k, since up to chain homotopy

$$(j-k)^{-1}j = \hat{\psi}((1+T)\hat{\psi})^{-1} : C \longrightarrow C\ ,$$

$$(j-k)^{-1}k = -T\hat{\psi}((1+T)\hat{\psi})^{-1} : C \longrightarrow C\ .$$

More generally:

<u>Proposition</u> <u>7.2</u> Let (C,ψ) be an m-dimensional ϵ-quadratic Poincaré complex in $\mathbb{A}$. A cobordism $((j\ k):C\oplus C\longrightarrow D,(\delta\psi,\psi\oplus-\psi))$ with $j-k:C\longrightarrow D$ a chain equivalence determines an ϵ-ultraquadratic structure $\hat{\psi}\in\hat{Q}_m(C)$ with image $\psi\in Q_m(C,\epsilon)$, such that

$$(C,\psi) = \mu(C^{m-*},\hat{\psi}) \in \mathrm{im}(\mu:L_0(\mathbb{D}_m(\mathbb{A}),\epsilon)\longrightarrow L_m(\mathbb{A},\epsilon))$$

with $(C^{m-*},\hat{\psi})$ a nonsingular ϵ-quadratic form in $\mathbb{D}_m(\mathbb{A})$.
<u>Proof</u>: Define a morphism in $\mathbb{D}_m(\mathbb{A})$

$$h = (j-k)^{-1}j : C \xrightarrow{\ j\ } D \xrightarrow{\ (j-k)^{-1}\ } C\ .$$

By the chain homotopy invariance of the Q-groups we can replace $((j\ k),(\delta\psi,\psi\oplus-\psi))$ by a homotopy equivalent

cobordism $((h\ h-1):C\oplus C\longrightarrow C,(\delta\psi,\psi\oplus-\psi)\in Q_{m+1}((h\ h-1),\epsilon))$. On the chain level

$$h_{\%}(\psi) - (h-1)_{\%}(\psi) = d(\delta\psi) \in (W_{\%}C)_m ,$$

so that there is defined a chain homotopy

$$(1+T_{\epsilon})\delta\psi_0 \ : \ h(1+T_{\epsilon})\psi_0 \simeq (1+T_{\epsilon})\psi_0(1-h^*) \ : \ C^{m-*} \longrightarrow C \ .$$

The m-dimensional ϵ-ultraquadratic Poincaré complex $(C,\hat{\psi})$ in $\mathbb{A}$ defined by the chain map

$$\hat{\psi} = h(1+T_{\epsilon})\psi_0 \ : \ C^{m-*} \xrightarrow{(1+T_{\epsilon})\psi_0} C \xrightarrow{h} C$$

is such that $\hat{\psi}+\epsilon\hat{\psi}^*\simeq(1+T_{\epsilon})\psi_0:C^{m-*}\longrightarrow C$. Define a chain $\chi\in(W_{\%}C)_{m+1}$ such that $\hat{\psi}-\psi=d(\chi+\delta\psi)\in(W_{\%}C)_m$ by

$$\chi_s = \begin{cases} 0 & \text{if } s=0 \\ hT_{\epsilon}\psi_{s-1} & \text{if } s\geqslant 1 \end{cases} : C^{m+1-r-s} \longrightarrow C_r \ .$$

Thus $\psi=\hat{\psi}\in Q_m(C,\epsilon)$ and

$$(C,\psi) = (C,\hat{\psi}) = \mu(C^{m-*},\hat{\psi}) \in L_m(\mathbb{A},\epsilon) \ .$$

$\square$

<u>Corollary</u> <u>7.3</u> Let $(f,b):M\longrightarrow X$ be an $(i-1)$-connected normal map of $(n+2i)$-dimensional geometric Poincaré complexes, and let

$$((e;f,zf),(a;b,zb)) \ : \ (L;M,zM) \longrightarrow X\times([0,1];\{0\},\{1\})$$

be an $(i-1)$-connected normal bordism between (f,b) and a disjoint copy (zf,zb). If the $(i-1)$-connected normal

map of $(n+2i+1)$-dimensional geometric Poincaré complexes

$$(e/(f=zf),a/(b=zb)) :$$

$$L/(M=zM) \longrightarrow X\times([0,1]/0=1) = X\times S^1$$

is a $\mathbb{Z}[\pi_1(X)]$-homology equivalence then the bordism determines an $(-)^i$-ultraquadratic structure $\hat{\psi}\in\hat{Q}_n(S^{-i}C(f^!))$ with image the quadratic kernel structure $\psi\in Q_n(S^{-i}C(f^!),(-)^i)=Q_{n+2i}(C(f^!))$. The nonsingular $(-)^i$-quadratic form $(S^{-i}C(f^!)^{n+2i-*},\hat{\psi})$ in $\mathbb{D}_n(\mathbb{Z}[\pi_1(X)])$ is such that

$$\sigma_*(f,b) = (C(f^!),\psi) = \mu(S^{-i}C(f^!)^{n+2i-*},\hat{\psi})$$

$$\in \mathrm{im}(\mu:L_0(\mathbb{D}_n(\mathbb{Z}[\pi_1(X)]),(-)^i)\longrightarrow L_n(\mathbb{Z}[\pi_1(X)],(-)^i))$$

$$= \mathrm{im}(\mu:L_{2i}(\mathbb{D}_n(\mathbb{Z}[\pi_1(X)]))\longrightarrow L_{n+2i}(\mathbb{Z}[\pi_1(X)])) .$$

<u>Proof</u>: The kernel $\mathbb{Z}[\pi_1(X)]$-module chain complexes

$$C = C(f^!:C(\tilde{X})\longrightarrow C(\tilde{M})) \ , \ D = C(g^!:C(\tilde{X}\times[0,1])\longrightarrow C(\tilde{L}))$$

are i-fold suspensions of n-dimensional chain complexes (up to chain equivalence). The inclusions $M\longrightarrow L$, $zM\longrightarrow L$ induce $\mathbb{Z}[\pi_1(X)]$-module chain maps $j:C\longrightarrow D$, $k:C\longrightarrow D$ such that $j-k:C\longrightarrow D$ is a chain equivalence. Let $h=(j-k)^{-1}j:C\longrightarrow C$ for any chain homotopy inverse $(j-k)^{-1}:D\longrightarrow C$. The quadratic kernel

$$\sigma_*((e;f,zf),(a;b,zb)) = ((j\ \ k):C\oplus C\longrightarrow D,(\delta\psi,\psi\oplus-\psi))$$

is the i-fold skew-suspension of a cobordism of n-dimensional $(-)^i$-quadratic Poincaré complexes over

$\mathbb{Z}[\pi_1(X)]$ satisfying the hypothesis of 7.2. It follows that $\psi \in Q_{n+2i}(C)$ is the image of the element $\hat{\psi} \in H_0(\mathrm{Hom}_{\mathbb{A}}(C^{n+2i-*},C))$ defined by the composite chain map

$$\hat{\psi} : C^{n+2i-*} \xrightarrow{\phi_0} C \xrightarrow{j} D \xrightarrow{(j-k)^{-1}} C$$

with $\phi_0 = [M] \cap - : C^{n+2i-*} \longrightarrow C$ the Poincaré duality chain equivalence. The nonsingular $(-)^i$-quadratic form $(S^{-i}C^{n+2i-*},\hat{\psi})$ in $\mathbb{D}_n(\mathbb{Z}[\pi_1(X)])$ is such that

$$\begin{aligned}
\sigma_*(f,b) &= (C,\psi) = \mu(S^{-i}C^{n+2i-*},\hat{\psi}) \\
&\in \mathrm{im}(\mu : L_0(\mathbb{D}_n(\mathbb{Z}[\pi_1(X)]),(-)^i) \longrightarrow L_n(\mathbb{Z}[\pi_1(X)],(-)^i)) \\
&= \mathrm{im}(\mu : L_{2i}(\mathbb{D}_n(\mathbb{Z}[\pi_1(X)])) \longrightarrow L_{n+2i}(\mathbb{Z}[\pi_1(X)])) .
\end{aligned}$$

□

<u>Proposition</u> <u>7.4</u> Let $((j\ \ j') : C \oplus C' \longrightarrow D,(\delta\psi,\psi \oplus -\psi'))$ be a cobordism of m-dimensional ϵ-quadratic Poincaré complexes in $\mathbb{A}$, such that D, $C(j)$ and $C(j')$ are the suspensions of $(m-1)$-dimensional chain complexes (up to chain equivalence), with $m \geqslant 1$. The chain homotopy classes of the chain maps

$$\gamma = \text{inclusion} : G = S^{-1}D \longrightarrow S^{-1}C(j') = F$$

$$\mu = \text{inclusion} :$$

$$G = S^{-1}D \longrightarrow S^{-1}C(j) \simeq C(j')^{m-*} = F^{m-1-*}$$

are the components of a morphism of ϵ-symmetric forms in $\mathbb{D}_{m-1}(\mathbb{A})$

$$\begin{pmatrix}\gamma\\ \mu\end{pmatrix} : (G,0) \longrightarrow H^{\epsilon}(F) = (F\oplus F^{m-1-*}, \begin{pmatrix} 0 & 1\\ \epsilon & 0\end{pmatrix})$$

such that $\gamma^*\mu=(1+T_{-\epsilon})\theta_0 : G\longrightarrow G^{m-1-*}$ for a certain element $\theta\in Q_{m-1}(G^{m-1-*},-\epsilon)$ determined by $(\delta\psi,\psi\oplus-\psi)$.

If the morphism $\begin{pmatrix}\gamma\\ \mu\end{pmatrix} : G\longrightarrow F\oplus F^{m-1-*}$ is a split injection in $\mathbb{D}_{m-1}(\mathbb{A})$ and if $\theta\in \mathrm{im}(\hat{Q}_{m-1}(G^{m-1-*})\longrightarrow Q_{m-1}(G^{m-1-*},-\epsilon))$ then G is a lagrangian of the hyperbolic ϵ-quadratic form

$$H_{\epsilon}(F)=(F\oplus F^{m-1-*}, \begin{pmatrix} 0 & 1\\ 0 & 0\end{pmatrix})$$

and (F,G) is a nonsingular ϵ-quadratic formation in $\mathbb{D}_{m-1}(\mathbb{A})$ such that

$$(C,\psi) = \mu(F,G) \in \mathrm{im}(\mu : L_1(\mathbb{D}_{m-1}(\mathbb{A}),\epsilon)\longrightarrow L_m(\mathbb{A},\epsilon)) \ .$$

<u>Proof</u>: Let (D^{m+1-*},θ) be the $(m+1)$-dimensional ϵ-quadratic complex in $\mathbb{A}$ (not in general Poincaré) defined by the algebraic Thom construction, the image of $(\delta\psi/\psi\oplus-\psi')\in Q_{m+1}(C(j\ j'),\epsilon)$ under the isomorphism

$$((1+T_{\epsilon})(\delta\psi_0,\psi_0\oplus-\psi_0')_{\%})^{-1} :$$

$$Q_{m+1}(C(j\ j'),\epsilon) \longrightarrow$$

$$Q_{m+1}(D^{m+1-*},\epsilon) = Q_{m-1}(G^{m-1-*},-\epsilon) \ .$$

Up to chain homotopy

$$\gamma^*\mu : G = S^{-1}D \xrightarrow{\text{inclusion}}$$

$$S^{-1}C(j\ j') \simeq D^{m-*} = G^{m-1-*} \ ,$$

so that there exists a chain homotopy

$$\gamma^*\mu \simeq (1+T_{-\epsilon})\theta_0 \ : \ G \longrightarrow G^{m-1-*} \ ,$$

and

$$(\gamma^* \ \ \mu^*)\begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix}\begin{pmatrix} \gamma \\ \mu \end{pmatrix} = \gamma^*\mu + \epsilon\mu^*\gamma \simeq 0 \ : \ G \longrightarrow G^{m-1-*}$$

as required for G to be a lagrangian in $H^{\epsilon}(F)$. If $\theta \in Q_{m-1}(G^{m-1-*},-\epsilon)$ is the image of $\hat{\theta}\in\hat{Q}_{m-1}(G^{m-1-*})$ then $(G,\hat{\theta})$ is the hessian $(-\epsilon)$-quadratic form in $\mathbb{D}_{m-1}(\mathbb{A})$ required for G to be a lagrangian in $H_{\epsilon}(F)$. The algebraic Thom construction defines a one-one correspondence between the homotopy equivalence classes of $(m+1)$-dimensional ϵ-quadratic Poincaré pairs in $\mathbb{A}$ and $(m+1)$-dimensional ϵ-quadratic complexes in $\mathbb{A}$ (Proposition 3.4 of Ranicki [14]). Thus $((j \ \ j'):C\oplus C'\longrightarrow D,(\delta\psi,\psi\oplus\psi'))$ is homotopy equivalent to the $(m+1)$-dimensional ϵ-quadratic Poincaré pair $((0 \ \ \pm 1):\partial D\longrightarrow D,(0,\partial\hat{\theta}))$ defined by

$$d_{\partial D} = \begin{bmatrix} d_D^* & (-)^r(1+T_{-\epsilon})\hat{\theta} \\ 0 & (-)^r d_D \end{bmatrix} :$$

$$\partial D_r = D^{m-r}\oplus D_r \longrightarrow \partial D_{r-1} = D^{m-r+1}\oplus D_{r-1} \ ,$$

$$\partial\hat{\theta}_0 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} :$$

$$\partial D^{m-r} = D_r\oplus D^{m-r} \longrightarrow \partial D_r = D^{m-r}\oplus D_r \ ,$$

$$\partial\hat{\theta}_1 = \begin{bmatrix} (-)^{m-r+s}\hat{\theta} & 0 \\ 0 & 0 \end{bmatrix} :$$

$$\partial D^{m-r-1} = D_{r+1} \oplus D^{m-r-1} \longrightarrow \partial D_r = D^{m-r} \oplus D_r \quad ,$$

$$\partial\hat{\theta}_s = 0 : \partial D^{m-r-s} \longrightarrow \partial D_r \quad (s \geqslant 2) \ .$$

Up to chain homotopy

$$\mu^* : F = S^{-1}C(j') \xrightarrow{\text{inclusion}}$$

$$S^{-1}C(j\ j') \simeq D^{m-*} = G^{m-1-*} \ ,$$

so that there is defined a chain equivalence $f : C \longrightarrow C(\mu^*)$. Choosing a representative chain map $\hat{\theta} : D \longrightarrow D^{m+1-*}$ and a chain homotopy $\chi : \Upsilon^* \mu \simeq (1+T_\epsilon)\hat{\theta} : D \longrightarrow D^{m+1-*}$ define a chain map $g : \partial D \longrightarrow C(\mu^*)$ by

$$g = \begin{bmatrix} 1 & \chi \\ 0 & \Upsilon \end{bmatrix} :$$

$$\partial D_r = D^{m-r} \oplus D_r \longrightarrow C(\mu^*)_r = G^{m-r-1} \oplus F_{r-1}$$

such that

$$f_{\%}(\psi) = g_{\%}(\partial\hat{\theta}) \in Q_m(C(\mu^*), \epsilon) \ .$$

Now $(C(\mu^*), g_{\%}(\partial\hat{\theta}))$ is the m-dimensional ϵ-quadratic Poincaré complex in $\mathbb{A}$ constructed in 3.2 from the nonsingular ϵ-quadratic formation (F,G) in $\mathbb{D}_{m-1}(\mathbb{A})$, so that

$$(C,\psi) = (C(\mu^*), f_{\%}(\psi)) = (C(\mu^*), g_{\%}(\partial\hat{\theta}))$$

$$= \mu(F,G) \in \mathrm{im}(\mu : L_1(\mathbb{D}_{m-1}(\mathbb{A}),\epsilon) \longrightarrow L_m(\mathbb{A},\epsilon)) .$$

□

§8. The connection

We now connect the algebra and the geometry, verifying the claim of Theorem 6.2 that the geometric surgery transfer maps for an n-dimensional Poincaré fibration $F \longrightarrow E \xrightarrow{p} B$ coincide with the algebraic surgery transfer maps

$$p^!_{geo} = p^!_{alg} :$$

$$L_m(\mathbb{Z}[\pi_1(B)]) \longrightarrow L_{m+n}(\mathbb{Z}[\pi_1(E)]) \quad (m \geqslant 0) .$$

We know from 1.9 how a CW complex structure behaves under transfer on the cellular chain level. The strategy is to encode the L-theory data in CW complex structures, and to decode the lifted L-theory data from the CW lifts using the ultraquadratic L-theory of §7.

We consider first the case m=2i. By Chapter 5 of Wall [22] every element $x \in L_{2i}(\mathbb{Z}[\pi_1(B)])$ $(i \geqslant 3)$ is the Witt class of the nonsingular $(-)^i$-quadratic form in $\mathbb{B}(\mathbb{Z}[\pi_1(B)])$

$$(K_i(M) \ , \ \lambda : K_i(M) \times K_i(M) \longrightarrow \mathbb{Z}[\pi_1(B)] \ ,$$

$$\mu : K_i(M) \longrightarrow \mathbb{Z}[\pi_1(B)]/\langle a-(-)^i\bar{a} \mid a \in \mathbb{Z}[\pi_1(B)]\rangle \ )$$

on the kernel $\mathbb{Z}[\pi_1(B)]$-module

$$K_i(M) = \pi_{i+1}(f) = H_i(f^!) = \ker(\tilde{f}_* : H_i(\tilde{M}) \longrightarrow H_i(\tilde{X}))$$

of an (i-1)-connected normal map $(f,b):(M,\partial M) \longrightarrow (X,\partial X)$

from a $2i$-dimensional manifold with boundary $(M,\partial M)$ to a $2i$-dimensional geometric Poincaré pair $(X,\partial X)$, with $\partial f:\partial M\longrightarrow\partial X$ a homotopy equivalence, and with a π_1-isomorphism reference map $X\longrightarrow B$ such that $w(X):\pi_1(X)\longrightarrow\pi_1(B)\xrightarrow{w(B)}\mathbb{Z}_2$. The adjoint of λ defines an isomorphism in $\mathbb{B}(\mathbb{Z}[\pi_1(B)])$

$$\lambda\ :\ K_i(M)\longrightarrow K_i(M)^*\ ;$$

$$u\longrightarrow (\ v\longrightarrow \lambda(u,v)\ )\ .$$

$(K_i(M),\lambda,\mu)$ can be viewed as a nonsingular $(-)^i$-quadratic form $(K_i(M)^*,\psi)$ over $\mathbb{Z}[\pi_1(B)]$, with ψ an equivalence class of $\mathbb{Z}[\pi_1(B)]$-module morphisms $\hat{\psi}:K_i(M)^*\longrightarrow K_i(M)$ such that

$$\hat{\psi}+(-)^i\hat{\psi}^*\ =\ \lambda^{-1}\ :\ K_i(M)^*\longrightarrow K_i(M)\quad ,$$

$$\hat{\psi}(\lambda(v))(\lambda(v))\ =\ \mu(v)\quad (v\in K_i(M))\ ,$$

with $\hat{\psi}$ equivalent to $\hat{\psi}+\chi+(-)^{i+1}\chi^*$ for any $\mathbb{Z}[\pi_1(B)]$-module morphism $\chi:K_i(M)^*\longrightarrow K_i(M)$. The surgery obstruction is thus given by

$$x\ =\ \sigma_*(f,b)\ =\ (K_i(M),\lambda,\mu)\ =\ (K_i(M)^*,\hat{\psi})$$

$$\in L_{2i}(\mathbb{Z}[\pi_1(B)])\ .$$

We shall be regarding modules as 0-dimensional chain complexes, and for any $q\in\mathbb{Z}$ we write S^qC for the q-fold suspension of a chain complex C, with

$$d_{S^qC}\ =\ d_C\ :\ (S^qC)_r\ =\ C_{r-q}\longrightarrow (S^qC)_{r-1}\ =\ C_{r-q-1}\ .$$

The quadratic kernel $\sigma_*(f,b)=(C(f^!),\psi)$ of the

$(i-1)$-connected $2i$-dimensional normal map $(f,b):M\longrightarrow X$ is an $(i-1)$-connected $2i$-dimensional quadratic Poincaré complex over $\mathbb{Z}[\pi_1(B)]$ which is homotopy equivalent to $(S^iK_i(M),\hat{\psi})$. Thus we can identify $\psi_0=\hat{\psi}$, and up to chain homotopy

$$(1+T)\psi_0 = \hat{\psi}+(-)^i\hat{\psi}^* = \lambda^{-1} :$$

$$C(f^!)^{2i-*} = S^iK_i(M)^* \longrightarrow C(f^!) = S^iK_i(M) .$$

The quadratic structure $\psi\in Q_{2i}(C(f^!))$ is the equivalence class of $\mathbb{Z}[\pi_1(B)]$-module morphisms $\hat{\psi}:K_i(M)^*\longrightarrow K_i(M)$ described above. A choice of representative $\hat{\psi}$ is a choice of ultraquadratic structure $\hat{\psi}\in\hat{Q}_{2i}(C(f^!))$ for the quadratic structure $\psi\in Q_{2i}(C(f^!))$. We now fix a choice of $\hat{\psi}$.

Let $\langle v_1,v_2,\ldots,v_k\rangle$ be a base for the f.g. free $\mathbb{Z}[\pi_1(B)]$-module $K_i(M)$, and use the dual to define a base for $K^i(M)=K_i(M)^*$. The functor of additive categories with involution

$$p^{\#} = -\otimes(C(\tilde{F}),\alpha,U) :$$

$$\mathbb{B}(\mathbb{Z}[\pi_1(B)]) \longrightarrow \mathbb{D}_n(\mathbb{Z}[\pi_1(E)])$$

sends the morphisms in $\mathbb{B}(\mathbb{Z}[\pi_1(B)])$

$$\hat{\psi}\ ,\ \hat{\psi}^* :$$

$$K_i(M)^* = \bigoplus_k\mathbb{Z}[\pi_1(B)] \longrightarrow K_i(M) = \bigoplus_k\mathbb{Z}[\pi_1(B)]$$

to chain homotopy classes of $\mathbb{Z}[\pi_1(E)]$-module chain maps

$p^{\#}(\hat{\psi})$, $p^{\#}(\hat{\psi}^*)$:

$$p^{\#}(K_i(M)^*) = \bigoplus_k C(\tilde{F}) \longrightarrow p^{\#}(K_i(M)) = \bigoplus_k C(\tilde{F})$$

such that there is defined a chain homotopy commutative diagram

$$\begin{tikzcd}
\bigoplus_k C(\tilde{F})^{n-*} \arrow[r, "p^{\#}(\hat{\psi})^*"] \arrow[d, "\oplus[F]\cap-"'] & \bigoplus_k C(\tilde{F})^{n-*} \arrow[d, "\oplus[F]\cap-"] \\
\bigoplus_k C(\tilde{F}) \arrow[r, "p^{\#}(\hat{\psi}^*)"] & \bigoplus_k C(\tilde{F})
\end{tikzcd}$$

and such that

$$p^{\#}(\hat{\psi}) + (-)^i p^{\#}(\hat{\psi}^*) = p^{\#}(\lambda^{-1}) : \bigoplus_k C(\tilde{F}) \longrightarrow \bigoplus_k C(\tilde{F})$$

is a chain equivalence.

In Lemmas 8.1,8.3 below we shall show that the quadratic kernel $\sigma_*(g,c)=(C(g^!),\eta)$ of the pullback $(n+i-1)$-connected $(n+2i)$-dimensional normal map $(g,c):(N,\partial N)\longrightarrow(Y,\partial Y)$ is homotopy equivalent to the $(n+2i)$-dimensional quadratic Poincaré complex (D,η) defined by the $\mathbb{Z}[\pi_1(E)]$-module chain complex

$$D = S^i p^{\#}(K_i(M)) = \bigoplus_k S^i C(\tilde{F})$$

with the (ultra)quadratic structure

$$\eta_0 : D^{n+2i-*} = \bigoplus_k S^i C(\tilde{F})^{n-*} \xrightarrow{\oplus[F]\cap-} \bigoplus_k S^i C(\tilde{F})$$

$$\xrightarrow{p^{\#}(\hat{\psi})} D = \bigoplus_k S^i C(\tilde{F}) ,$$

$$\eta_s = 0 : D^{n+2i-r-s} \longrightarrow D_r \quad (s \geq 1) .$$

It will follow that the nonsingular $(-)^i$-quadratic form $(p^{\#}(K_i(M)^*), (\oplus[F]\cap-)^{-1}p^{\#}(\hat{\psi}))$ in $\mathbb{D}_n(\mathbb{Z}[\pi_1(E)])$ defined by

$$(\oplus[F]\cap-)^{-1}p^{\#}(\hat{\psi}) :$$

$$p^{\#}(K_i(M)^*) = \bigoplus_k C(\tilde{F}) \xrightarrow{p^{\#}(\hat{\psi})} p^{\#}(K_i(M)) = \bigoplus_k C(\tilde{F})$$

$$\xrightarrow{\oplus([F]\cap-)^{-1}} \bigoplus_k C(\tilde{F})^{n-*} = (p^{\#}(K_i(M)^*))^{n-*}$$

is such that

$$\begin{aligned} p^!_{geo}\sigma_*(f,b) &= \sigma_*(g,c) = \\ &\quad \mu(p^{\#}(K_i(M)^*), (\oplus[F]\cap-)^{-1}p^{\#}(\hat{\psi})) \\ &\qquad = p^!_{alg}(K_i(M)^*, \hat{\psi}) = p^!_{alg}\sigma_*(f,b) \\ &\in \mathrm{im}(\mu : L_0(\mathbb{D}_n(\mathbb{Z}[\pi_1(E)]), (-)^i) \longrightarrow L_n(\mathbb{Z}[\pi_1(E)], (-)^i)) \\ &\quad = \mathrm{im}(\mu : L_{2i}(\mathbb{D}_n(\mathbb{Z}[\pi_1(E)])) \longrightarrow L_{n+2i}(\mathbb{Z}[\pi_1(E)])) , \end{aligned}$$

verifying that $p^!_{geo} = p^!_{alg}$ in the case $m=2i$.

For the symmetric structure of $\sigma_*(g,c) = (C(g^!), \eta)$ we have:

<u>Lemma</u> <u>8.1</u> The symmetric kernel $\sigma^*(g,c) = (C(g^!), (1+T)\eta)$ is such that up to chain homotopy

$$(1+T)\eta_0 :$$

$$C(g^!)^{n+2i-*} = S^i p^\#(K_i(M)^*)^{n-*} = \underset{k}{\oplus} S^i C(\tilde{F})^{n-*}$$

$$\xrightarrow{\oplus[F]\cap -} \underset{k}{\oplus} S^i C(\tilde{F}) = S^i p^\#(K_i(M)^*)$$

$$\xrightarrow{p^\#((1+T)\psi_0)} C(g^!) = S^i p^\#(K_i(M)) = \underset{k}{\oplus} S^i C(\tilde{F}) \ .$$

Proof: Represent the base elements $v_j \in K_i(M)$ $(1 \leqslant j \leqslant k)$ by framed immersions $v_j : S^i \longrightarrow \mathrm{int}(M^{2i})$ with nullhomotopies in X, and with $\pi_1(B)$-equivariant lifts $\tilde{v}_j : \tilde{S}^i = \pi_1(B) \times S^i \longrightarrow \tilde{M}$. Replace $f : M \longrightarrow X$ by the inclusion of M in the CW complex $M \cup \underset{k}{\vee} e^{i+1}$ homotopy equivalent to X (which is also denoted by X), so that

$$C(f^!) \simeq S^{-1} C(\tilde{X}, \tilde{M}) = S^i K_i(M) = \underset{k}{\oplus} S^i \mathbb{Z}[\pi_1(B)] \ .$$

In the total spaces of the pullbacks $g : N \longrightarrow Y$ is replaced by the inclusion of N in the CW complex $N \cup \underset{k}{\vee} F \times e^{i+1}$, so that

$$C(g^!) \simeq S^{-1} C(\tilde{Y}, \tilde{N}) = S^i p^\#(K_i(M)) = \underset{k}{\oplus} S^i C(\tilde{F}) \ .$$

The Poincaré duality chain equivalence is given up to chain homotopy by the composite

$$(1+T)\eta_0 \ : \ C(g^!)^{n+2i-*} \xrightarrow{e^*} C(\tilde{N}, \partial\tilde{N})^{n+2i-*}$$

$$\xrightarrow{[N]\cap -} C(\tilde{N}) \longrightarrow C(\tilde{N}, \partial\tilde{N}) \xrightarrow{e} C(g^!)$$

with e the inclusion.

For a sufficiently large number $q \geqslant 0$ the framed immersions can be approximated by framed embeddings

$v_j:S^i \longrightarrow \mathrm{int}(M^{2i}\times D^q)$ with nullhomotopies in X. Let V_j be a regular neighbourhood of $v_j(S^i)$ in $M\times D^q$, and let $P_j=\mathrm{closure}(M\times D^q-V_j)$, so that

$$M\times D^q = V_j\cup_{\partial V_j}P_j \quad , \quad \partial P_j = \partial V_j\cup\partial(M\times D^q) \ ,$$

$$(V_j,\partial V_j) = v_j(S^i)\times(D^{i+q},S^{i+q-1}) \ .$$

The intersection number

$$\lambda_{j,j'} = \lambda(\tilde{v}_j,\tilde{v}_{j'}) \in \mathbb{Z}[\pi_1(B)] \quad (1\leqslant j,j'\leqslant k)$$

is the image of $1\in\mathbb{Z}[\pi_1(B)]$ under the composite $\mathbb{Z}[\pi_1(B)]$-module morphism

$$\begin{aligned}
&H_i(\tilde{S}^i) = \mathbb{Z}[\pi_1(B)] \\
&\quad \xrightarrow{\tilde{v}_{j'}} H_i(\tilde{M}) \cong H_{i+q}(\tilde{M}\times D^q,\tilde{M}\times S^{q-1}) \\
&\quad \longrightarrow H_{i+q}(\tilde{M}\times D^q,\tilde{P}_j) \cong H_{i+q}(\tilde{V}_j,\partial\tilde{V}_j) \\
&\qquad\qquad\qquad\qquad = H_0(\tilde{S}^i) = \mathbb{Z}[\pi_1(B)] \ ,
\end{aligned}$$

which can also be expressed as

$$H_i(\tilde{S}^i) = \mathbb{Z}[\pi_1(B)] \xrightarrow{\tilde{v}_{j'}} H_i(\tilde{M},\partial\tilde{M})$$

$$\xrightarrow{([M]\cap-)^{-1}} H^i(\tilde{M}) \xrightarrow{\tilde{v}_j^*} H^i(\tilde{S}^i) = \mathbb{Z}[\pi_1(B)] \ .$$

The pullbacks from the n-dimensional Poincaré fibration $F\longrightarrow E\xrightarrow{p}B$ define framed Poincaré immersions $w_j:F^n\times S^i\longrightarrow N^{n+2i}$ with nullhomotopies in Y, and with $\pi_1(E)$-equivariant lifts $\tilde{w}_j:\tilde{F}\times S^i\longrightarrow\tilde{N}$ $(1\leqslant j\leqslant k)$. Let

$W_j, Q_j \subset N\times D^q$ be the total spaces of the fibrations over $V_j, P_j \subset M\times D^q$, so that

$$N\times D^q = W_j \cup_{\partial W_j} Q_j \quad , \quad \partial Q_j = \partial W_j \cup \partial(N\times D^q) \ ,$$

$$(W_j, \partial W_j) = w_j(F\times S^i)\times(D^{i+q}, S^{i+q-1}) \ .$$

For any embedding $D^{2i+q} \subset \mathrm{int}(V_j) \subset M^{2i}\times D^q$ the pair

$$(\ (M\times D^q - \mathrm{int}(D^{2i+q})) \cup_{v_{j'}\times 1} D^{i+1}\times D^q \ , \ P_j \cup D^{i+1}\times S^{q-1}\ )$$

has a relative CW structure with one $(i+q)$-cell and one $(i+q+1)$-cell, such that the cellular chain complex in $\mathbb{B}(\mathbb{Z}[\pi_1(B)])$ is $\lambda_{j,j'}:\mathbb{Z}[\pi_1(B)]\longrightarrow\mathbb{Z}[\pi_1(B)]$. By 1.9 the chain homotopy class of the $\mathbb{Z}[\pi_1(E)]$-module chain map $p^{\#}(\lambda_{j,j'}):C(\widetilde{F})\longrightarrow C(\widetilde{F})$ coincides with the composite

$$C(\widetilde{F}) \longrightarrow S^{-i}C(\widetilde{F}\times S^i) \xrightarrow{\widetilde{w}_{j'}}$$

$$S^{-i}C(\widetilde{N},\partial\widetilde{N}) \simeq S^{-i-q}C(\widetilde{N}\times D^q, \partial(\widetilde{N}\times D^q))$$

$$\longrightarrow S^{-i-q}C(\widetilde{N}\times D^q, \widetilde{Q}_j) \simeq S^{-i-q}C(\widetilde{W}_j, \partial\widetilde{W}_j) \simeq C(\widetilde{F}\times S^i)$$

$$\longrightarrow C(\widetilde{F}) \ ,$$

and hence also with the composite

$$C(\widetilde{F}) \longrightarrow S^{-i}C(\widetilde{F}\times S^i) \xrightarrow{\widetilde{w}_{j'}} S^{-i}C(\widetilde{N},\partial\widetilde{N})$$

$$\xrightarrow{([N]\cap -)^{-1}} S^i C(\widetilde{N})^{n-*} \xrightarrow{\widetilde{w}_j^*} S^i C(\widetilde{F}\times S^i)^{n-*}$$

$$\longrightarrow C(\widetilde{F})^{n-*} \xrightarrow{[F]\cap -} C(\widetilde{F}) \ .$$

The (j,j')-component of the $\mathbb{Z}[\pi_1(E)]$-module chain equivalence

$$((1+T)\eta_0)^{-1} :$$

$$C(g^!) \simeq \bigoplus_k S^i C(\tilde{F}) \longrightarrow C(g^!)^{n+2i-*} \simeq \bigoplus_k S^i C(\tilde{F})^{n-*}$$

is thus the composite

$$S^i C(\tilde{F}) \xrightarrow{p^{\#}(\lambda_{j,j'})} S^i C(\tilde{F})$$

$$\xrightarrow{([F]\cap -)^{-1}} S^i C(\tilde{F})^{n-*}$$

and up to chain homotopy

$$(1+T)\eta_0 : C(g^!)^{n+2i-*} = \bigoplus_k S^i C(\tilde{F})^{n-*} \xrightarrow{\oplus [F]\cap -}$$

$$\bigoplus_k S^i C(\tilde{F}) \xrightarrow{p^{\#}(\lambda^{-1})} C(g^!) = \bigoplus_k S^i C(\tilde{F}) .$$

$\square$

We extend the description of the symmetric structure of $\sigma_*(g,c)$ given by 8.1 to the quadratic structure, using the ultraquadratic L-theory of §7. A choice of ultraquadratic structure $\hat{\psi}:K_i(M)^* \longrightarrow K_i(M)$ for $\sigma_*(f,b)$ is used to construct a normal bordism between $(f,b):M \longrightarrow X$ and a copy $(zf,zb):zM \longrightarrow zX$ which encodes the quadratic self-intersection form μ in the CW structure. The quadratic structure of $\sigma_*(g,c)$ is then decoded from the CW structure of the pullback normal bordism between $(g,c):N \longrightarrow Y$ and a copy $(zg,zc):zN \longrightarrow zY$, using 1.9 and 7.3. The construction of the bordism is motivated by the way in which the infinite cyclic cover

of a knot complement can be obtained by cutting along a Seifert surface.

Lemma 8.2 A choice of ultraquadratic structure $\hat{\psi}$ for $(K_i(M),\lambda,\mu)$ can be realized by an $(i-1)$-connected $(2i+1)$-dimensional normal bordism

$$((e;f,zf),(a;b,zb)) : (L;M,zM) \longrightarrow X\times([0,1];\{0\},\{1\})$$

between $(f,b):M\longrightarrow X$ and a disjoint copy $(zf,zb):zM\longrightarrow zX$, such that the difference of the $\mathbb{Z}[\pi_1(B)]$-module morphisms $j,k:K_i(M)\longrightarrow K_i(L)$ induced by the inclusions $j:M\longrightarrow L$, $k:zM\longrightarrow L$ is an isomorphism $j-k:K_i(M)\longrightarrow K_i(L)$ with

$$(j-k)^{-1}j = \hat{\psi}\lambda : K_i(M) \longrightarrow K_i(M) ,$$

$$(j-k)^{-1}k = (-)^{i+1}\hat{\psi}^*\lambda : K_i(M) \longrightarrow K_i(M) .$$

The $(i-1)$-connected $(2i+1)$-dimensional normal map

$$(e/(f=zf),a/(b=zb)) : (L/(M=zM),\partial M\times S^1)$$

$$\longrightarrow (X,\partial X)\times([0,1]/0=1) = (X,\partial X)\times S^1$$

is a $\mathbb{Z}[\pi_1(B)]$-homology equivalence, with the homotopy equivalence $\partial f\times 1:\partial M\times S^1\longrightarrow\partial X\times S^1$ on the boundary.

Proof: Every based f.g. free lagrangian of the $(-)^i$-quadratic form $(K_i(M),\lambda,\mu)\oplus(K_i(M),-\lambda,-\mu)$ can be realized by disjoint framed embeddings of S^i in $M\cup_{\partial M\times[0,1]}zM$ with nullhomotopies in X, such that the trace of the surgeries on these framed embedded i-spheres defines a normal bordism between (f,b) and (zf,zb). The realization of the lagrangian

$$\mathrm{im}\left(\begin{bmatrix}\hat{\psi}\lambda \\ (-)^{i+1}\hat{\psi}^*\lambda\end{bmatrix} : K_i(M) \longrightarrow K_i(M)\oplus K_i(M)\right)$$

has the required properties. (This lagrangian is a direct complement of the diagonal lagrangian $\mathrm{im}(\begin{bmatrix}1\\1\end{bmatrix}:K_i(M)\longrightarrow K_i(M)\oplus K_i(M))$. The realization of the diagonal lagrangian is the product $(2i+1)$-dimensional normal map

$$(f,b)\times 1 \ : \ M\times([0,1];\{0\},\{1\}) \longrightarrow X\times([0,1];\{0\},\{1\}) \ .$$

The required normal map (e,a) can also be obtained from $(f,b)\times 1$ by surgeries on i-spheres in the interior of $M\times[0,1]$ representing a base of $K_i(M\times[0,1])=K_i(M)$.)

□

We can now extend Lemma 8.1 to the quadratic structure:

<u>Lemma</u> <u>8.3</u> The quadratic kernel $\sigma_*(g,c)=(C(g^!),\eta)$ is such that up to chain homotopy

$$\begin{aligned}\eta_0 \ : \ C(g^!)^{n+2i-*} &= S^i p^{\#}(K_i(M)^*)^{n-*} = \bigoplus_k S^i C(\tilde{F})^{n-*} \\ &\xrightarrow{\oplus[F]\cap -} \bigoplus_k S^i C(\tilde{F}) = S^i p^{\#}(K_i(M)^*) \\ &\xrightarrow{p^{\#}(\hat{\psi})} C(g^!) = S^i p^{\#}(K_i(M)) = \bigoplus_k S^i C(\tilde{F}) \ ,\end{aligned}$$

$$\eta_s = 0 \ : \ C(g^!)^{n+2i-r-s} \longrightarrow C(g^!)_r \quad (s\geqslant 1) \ .$$

<u>Proof</u>: Let $\hat{\psi},(e,a),(f,b),j,k$ be as in 8.2, and let

$$((h;g,zg),(d;c,zc)) :$$

$$(P;N,zN) \longrightarrow Y\times([0,1];\{0\},\{1\})$$

be the $(n+i-1)$-connected $(n+2i+1)$-dimensional normal bordism between $(g,c):N\longrightarrow Y$ and a disjoint copy $(zg,zc):zN\longrightarrow zY$ obtained from $((e;f,zf),(a;b,zb))$ by pullback from $F\longrightarrow E\xrightarrow{p}B$ along the reference map $X\longrightarrow B$. The $(n+i-1)$-connected $(n+2i+1)$-dimensional normal map

$$(h/(g=zg),d/(c=zc)) :$$

$$P/(N=zN) \longrightarrow Y\times([0,1]/0=1) = Y\times S^1$$

is a $\mathbb{Z}[\pi_1(E)]$-homology equivalence. By 7.3 the quadratic kernel $\sigma_*(g,c)$ is determined by the chain homotopy classes of the $\mathbb{Z}[\pi_1(E)]$-module chain maps $C(g^!)\longrightarrow C(h^!)$, $C(zg^!)\longrightarrow C(h^!)$ and the Poincaré duality chain equivalence $C(g^!)^{n+2i-*}\longrightarrow C(g^!)$. We shall now arrange CW structures for (e,a) in such a way that only cells in dimensions $i,i+1$ occur in the relevant pairs, and 1.9 applies to obtain the $\mathbb{Z}[\pi_1(E)]$-module chain homotopy data in the total spaces of the pullbacks from $F\longrightarrow E\xrightarrow{p}B$ as the algebraic transfers of $\mathbb{Z}[\pi_1(B)]$-module data.

L is the trace of surgeries on $(i-1)$- and i-spheres in M, so that (L,M) has a relative CW structure with i- and $(i+1)$-cells, with the cellular chain complex in $\mathbb{B}(\mathbb{Z}[\pi_1(B)])$ given by

$$d = j : C(\tilde{L},\tilde{M})_{i+1} = K_i(M) \longrightarrow C(\tilde{L},\tilde{M})_i = K_i(L) .$$

Replacing $e:L\longrightarrow X\times[0,1]$ by the inclusion of L in the mapping cylinder it may be assumed that L is a subcomplex of X, such that (X,L) and (X,M) have cellular chain complexes in $\mathbb{B}(\mathbb{Z}[\pi_1(B)])$

$$C(\tilde{X},\tilde{L}) = S^{i+1}K_i(L) ,$$

$$d = (j \;\; 1) : C(\tilde{X},\tilde{M})_{i+1} = K_i(M)\oplus K_i(L)$$

$$\longrightarrow C(\tilde{X},\tilde{M})_i = K_i(L) \ .$$

The kernel chain complexes $C(f^!)$, $C(e^!)$ are chain equivalent to $S^{-1}C(\tilde{X},\tilde{M})$, $S^{-1}C(\tilde{X},\tilde{L})$ respectively. Replacing the inclusion $C(f^!)\longrightarrow C(e^!)$ by the inclusion $S^{-1}C(\tilde{X},\tilde{M})\longrightarrow S^{-1}C(\tilde{X},\tilde{L})$ of the chain equivalent complexes corresponds in the pullbacks to replacing $C(g^!)\longrightarrow C(h^!)$ by $S^{-1}C(\tilde{Y},\tilde{N})\longrightarrow S^{-1}C(\tilde{Y},\tilde{P})$, and by 1.9

$$C(\tilde{Y},\tilde{N}) = p^{\#}C(\tilde{X},\tilde{M}) \quad , \quad C(\tilde{Y},\tilde{P}) = p^{\#}C(\tilde{X},\tilde{L}) \ .$$

Thus up to chain homotopy the inclusion $C(g^!)\longrightarrow C(h^!)$ may be identified with the $\mathbb{Z}[\pi_1(E)]$-module chain map

$$p^{\#}(j) \ : \ C(g^!) = S^i p^{\#}(K_i(M)) \longrightarrow$$

$$C(h^!) = S^i p^{\#}(K_i(L)) \ .$$

Similarly, up to chain homotopy $C(zg^!)\longrightarrow C(h^!)$ may be identified with

$$p^{\#}(k) \ : \ C(zg^!) = S^i p^{\#}(K_i(M)) \longrightarrow$$

$$C(h^!) = S^i p^{\#}(K_i(L)) \ .$$

The $\mathbb{Z}[\pi_1(B)]$-module isomorphism $j-k:K_i(M)\longrightarrow K_i(L)$ such that $(j-k)^{-1}j=\hat{\psi}\lambda:K_i(M)\longrightarrow K_i(M)$ lifts to (the chain homotopy class of) a $\mathbb{Z}[\pi_1(E)]$-module chain equivalence

$$p^{\#}(j-k) = p^{\#}(j) - p^{\#}(k) \ : \ C(g^!) \longrightarrow C(h^!)$$

such that up to chain homotopy

$$p^{\#}(j-k)^{-1}p^{\#}(j) = p^{\#}(\hat{\psi}\lambda) \ :$$

$$C(g^!) = \bigoplus_k S^i C(\tilde{F}) \longrightarrow \bigoplus_k S^i C(\tilde{F}) \ .$$

Applying 8.1 and 7.3 we have that the quadratic kernel $\sigma_*(g,c)$ is homotopy equivalent to the

$(n+2i)$-dimensional quadratic Poincaré complex $(\underset{k}{\oplus} S^i C(\tilde{F}), \eta)$ over $\mathbb{Z}[\pi_1(E)]$ with

$$(1+T)\eta_0 \ : \ C(g^!)^{n+2i-*} = \underset{k}{\oplus} S^i C(\tilde{F})^{n-*} \xrightarrow{\oplus[F]\cap-}$$

$$\underset{k}{\oplus} S^i C(\tilde{F}) \xrightarrow{p^{\#}(\lambda^{-1})} C(g^!) = \underset{k}{\oplus} S^i C(\tilde{F}) \ ,$$

$$\eta_0 = p^{\#}(\hat{\psi}\lambda)(1+T)\eta_0 \ :$$

$$C(g^!)^{n+2i-*} = \underset{k}{\oplus} S^i C(\tilde{F})^{n-*} \xrightarrow{\oplus[F]\cap-} \underset{k}{\oplus} S^i C(\tilde{F})$$

$$\xrightarrow{p^{\#}(\hat{\psi})} C(g^!) = \underset{k}{\oplus} S^i C(\tilde{F}) \ ,$$

$$\eta_s = 0 \ : \ C(g^!)^{n+2i-r-s} \longrightarrow C(g^!)_r \quad (s\geq 1) \ .$$

□

This completes the proof of Theorem 6.2 in the case $m=2i$, and we proceed to the case $m=2i+1$.

By Chapter 6 of Wall [22] every element $x\in L_{2i+1}(\mathbb{Z}[\pi_1(B)])$ $(i\geq 2)$ is the Witt class of the kernel nonsingular $(-)^i$-quadratic formation over $\mathbb{Z}[\pi_1(B)]$

$$(F,G) = (K_{i+1}(U,\partial U), K_{i+1}(M_0,\partial U))$$

of an $(i-1)$-connected $(2i+1)$-dimensional normal map $(f,b):(M,\partial M)\longrightarrow(X,\partial X)$ with $\partial f:\partial M\longrightarrow\partial X$ a homotopy equivalence, and with a π_1-isomorphism reference map $X\longrightarrow B$ such that $w(X):\pi_1(X)\longrightarrow\pi_1(B)\xrightarrow{w(B)}\mathbb{Z}_2$. Here, U is the connected sum of a sufficiently large number $k\geq 0$ of framed embeddings $S^i\subset \mathrm{int}(M)$ with nullhomotopies in X to generate the f.g. $\mathbb{Z}[\pi_1(B)]$-module $K_i(M)$, and $M_0=\mathrm{closure}(M-U)$. Thus $F=K_{i+1}(U,\partial U)$ is a based f.g. free $\mathbb{Z}[\pi_1(B)]$-module, and $G=K_{i+1}(M_0,\partial U)$ is a based f.g. free

lagrangian of the hyperbolic $(-)^i$-quadratic form $H_{(-)^i}(F)=(F\oplus F^*,\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix})$, with

$$F = G = \bigoplus_k \mathbb{Z}[\pi_1(B)] \ .$$

The inclusion $\begin{bmatrix}\gamma\\ \mu\end{bmatrix}:G\longrightarrow F\oplus F^*$ extends to an isomorphism of hyperbolic $(-)^i$-quadratic forms

$$\begin{bmatrix}\gamma & \tilde{\gamma}\\ \mu & \tilde{\mu}\end{bmatrix} : H_{(-)^i}(G) \longrightarrow H_{(-)^i}(F) \ .$$

Surgery on the framed embedded i-spheres in U defines an (i-1)-connected (2i+2)-dimensional normal map of triads

$$((e;f,f'),(a;b,b')) \ : \ (L^{2i+2};M^{2i+1},M'^{2i+1})$$
$$\longrightarrow X\times([0,1];\{0\},\{1\})$$

with (F^*,G) the kernel nonsingular $(-)^i$-quadratic formation of (f',b'), and

$$G = K_{i+1}(L) \ , \ F = K_{i+1}(L,M') \ , \ F^* = K_{i+1}(L,M) \ ,$$

$$C(e^!) = S^{i+1}G \ , \ C(e^!,f'^!) = S^{i+1}F \ ,$$

$$C(e^!,f^!) = S^{i+1}F^* \ .$$

The quadratic kernel $\sigma_*((e;f,f'),(a;b,b'))$ is an (i-1)-connected (2i+2)-dimensional quadratic Poincaré cobordism over $\mathbb{Z}[\pi_1(B)]$ with algebraic Thom complex homotopy equivalent to the i-connected (2i+2)-dimensional quadratic complex $(S^{i+1}G^*,\theta)$ corresponding to the $(-)^{i+1}$-quadratic hessian form (G,θ) such that $\gamma^*\mu=\theta+(-)^{i+1}\theta^*:G\longrightarrow G^*$. The base elements of the f.g. free $\mathbb{Z}[\pi_1(B)]$-module $G=\pi_{i+1}(e)$ can be represented by immersed (i+1)-spheres in $\mathrm{int}(L^{2i+2})$ with nullhomotopies in X, so that the form (G,θ) can be

expressed in terms of geometric intersection and self-intersection numbers exactly as in Chapter 5 of Wall [22].

The pullback of $((e;f,f'),(a;b,b'))$ from $F \longrightarrow E \xrightarrow{p} B$ along the reference map $X \longrightarrow B$ is an $(n+i-1)$-connected normal map of $(n+2i+2)$-dimensional geometric Poincaré triads

$$((h;g,g'),(d;c,c')) \ : \ (P^{n+2i+2};N^{n+2i+1},N'^{n+2i+1})$$

$$\longrightarrow X\times([0,1];\{0\},\{1\}) \ .$$

The $\mathbb{Z}[\pi_1(E)]$-module chain maps $C(h^!) \longrightarrow C(h^!,g'^!)$, $C(h^!) \longrightarrow C(h^!,g^!)$ defined by projections are given up to chain homotopy by

$$p^{\#}(\gamma) \ : \ C(h^!) = S^{i+1}p^{\#}G = \bigoplus_k S^{i+1}C(\tilde{F})$$

$$\longrightarrow C(h^!,g'^!) = S^{i+1}p^{\#}F = \bigoplus_k S^{i+1}C(\tilde{F}) \ ,$$

$$(\oplus[F]\cap-)^{-1}p^{\#}(\mu) \ : \ C(h^!) = S^{i+1}p^{\#}G = \bigoplus_k S^{i+1}C(\tilde{F})$$

$$\xrightarrow{p^{\#}(\mu)} S^{i+1}p^{\#}(F^*) = \bigoplus_k S^{i+1}C(\tilde{F})$$

$$\xrightarrow{(\oplus[F]\cap-)^{-1}}$$

$$\bigoplus_k S^{i+1}C(\tilde{F})^{n-*} = C(h^!,g'^!)^{n+2i+2-*} \simeq C(h^!,g^!) \ .$$

The quadratic kernel $\sigma_*((h;g,g'),(d;c,c'))$ is the i-fold skew-suspension of an $(n+1)$-dimensional $(-)^i$-quadratic Poincaré cobordism over $\mathbb{Z}[\pi_1(E)]$ satisfying the hypotheses of Proposition 7.4, with

$$\begin{bmatrix} p^{\#}(\gamma) \\ (\oplus[F]\cap-)^{-1}p^{\#}(\mu) \end{bmatrix} \ : \ (p^{\#}G,0) \longrightarrow H_{(-)^i}(p^{\#}F)$$

the inclusion of a lagrangian of the $(-)^i$-quadratic hyperbolic form $H_{(-)^i}(p^\# F)$ in $\mathbb{D}_n(\mathbb{Z}[\pi_1(E)])$ because it extends to an isomorphism of $(-)^i$-quadratic forms

$$\begin{pmatrix} p^\#(\gamma) & p^\#(\tilde{\gamma})(\oplus[F]\cap -) \\ (\oplus[F]\cap -)^{-1}p^\#(\mu) & (\oplus[F]\cap -)^{-1}p^\#(\tilde{\mu})(\oplus[F]\cap -) \end{pmatrix}$$

$$: H_{(-)^i}(p^\# G) \longrightarrow H_{(-)^i}(p^\# F) \ .$$

Working as in the proof of Lemma 8.1 the hessian $(-)^{i+1}$-quadratic form in $\mathbb{D}_n(\mathbb{Z}[\pi_1(E)])$ may be expressed as $(p^\# G, (\oplus[F]\cap -)^{-1}p^\#(\hat{\theta}))$, with $\hat{\theta}: G \longrightarrow G^*$ an $(-)^{i+1}$-ultraquadratic structure for (G,θ). By Proposition 7.4 the nonsingular $(-)^i$-quadratic formation $(p^\# F, p^\# G)$ in $\mathbb{D}_n(\mathbb{Z}[\pi_1(E)])$ is such that

$$p^!_{geo}\sigma_*(f,b) = \sigma_*(g,c) = \mu(p^\# F, p^\# G)$$

$$= p^!_{alg}(F,G) = p^!_{alg}\sigma_*(f,b)$$

$$\in \mathrm{im}(\mu: L_1(\mathbb{D}_n(\mathbb{Z}[\pi_1(E)]),(-)^i)$$

$$\longrightarrow L_{n+1}(\mathbb{Z}[\pi_1(E)],(-)^i))$$

$$= \mathrm{im}(\mu: L_{2i+1}(\mathbb{D}_n(\mathbb{Z}[\pi_1(E)]))$$

$$\longrightarrow L_{n+2i+1}(\mathbb{Z}[\pi_1(E)])) \ .$$

This verifies $p^!_{geo} = p^!_{alg}$ also in the case $m=2i+1$, completing the proof of Theorem 6.2.

We can now write the surgery transfer maps unambiguously as

$$p^! : L_m(\mathbb{Z}[\pi_1(B)]) \longrightarrow L_{m+n}(\mathbb{Z}[\pi_1(E)]) \quad (m \geqslant 0) \ .$$

§9. Change of K-theory

We now extend the definition of the algebraic surgery transfer maps $(C,\alpha,U)^!:L_m(R)\longrightarrow L_{m+n}(S)$ to the intermediate L-groups, and show that they are compatible with the Rothenberg exact sequences.

An involution $R\longrightarrow R;r\longrightarrow \bar{r}$ on a ring R determines a duality involution $*:\mathbb{P}(R)\longrightarrow \mathbb{P}(R);P\longrightarrow P^*=\mathrm{Hom}_R(P,R)$ on the additive category $\mathbb{P}(R)$ of f.g. projective R-modules by

$$R\times P^*\longrightarrow P^*\ ;\ (r,f)\longrightarrow (x\longrightarrow f(x).\bar{r})\ ,$$

$$e(P)\ :\ P\xrightarrow{\ \simeq\ } P^{**}\ ;\ x\longrightarrow (f\longrightarrow \overline{f(x)})\ .$$

The duality involution on $\mathbb{P}(R)$ determines involutions on the algebraic K-groups

$$*\ :\ K_0(R)\longrightarrow K_0(R)\ ;\ [P]\longrightarrow [P^*]\ ,$$

$$*\ :\ K_1(R)\longrightarrow K_1(R)\ ;$$

$$\tau(f:P\longrightarrow Q)\longrightarrow \tau(f^*:Q^*\longrightarrow P^*)$$

and also on the reduced K-groups

$$\tilde{K}_i(R)=\mathrm{coker}(K_i(\mathbb{Z})\longrightarrow K_i(R))\quad (i=0,1)\ .$$

The intermediate quadratic L-groups $L_*^X(R)$ of a ring with involution R are defined for $*$-invariant subgroups $X\subseteq\tilde{K}_i(R)$ $(i=0,1)$, such that $x^*\in X$ for all $x\in X$. The intermediate L-groups for $X=\{0\},\tilde{K}_i(R)$ are written as

$$L_*^{\tilde{K}_0(R)}(R)=L_*^p(R)\quad ,\quad L_*^{\{0\}\subseteq\tilde{K}_1(R)}(R)=L_*^s(R)\quad ,$$

$$L_*^{\{0\}\subseteq\tilde{K}_0(R)}(R)=L_*^{\tilde{K}_1(R)}(R)=L_*^h(R)=L_*(R)\ .$$

For $*$-invariant subgroups $X \subseteq X' \subseteq K_i(R)$ there is defined a Rothenberg exact sequence

$$\ldots \longrightarrow L_n^X(R) \longrightarrow L_n^{X'}(R) \longrightarrow \hat{H}^n(\mathbb{Z}_2; X'/X)$$
$$\longrightarrow L_{n-1}^X(R) \longrightarrow \ldots$$

with

$$\hat{H}^n(\mathbb{Z}_2; X'/X) =$$
$$\{a \in X'/X \mid a^* = (-)^n a\} / \{b + (-)^n b^* \mid b \in X'/X\} .$$

See Ranicki [13],[14] for further details.

We consider first the torsion case $X \subseteq \tilde{K}_1(R)$.

A representation (C,U) of R in $\mathbb{D}(S)$ determines a transfer map in the absolute torsion groups $(C,U)^! : K_1(R) \longrightarrow K_1(S)$ (Example 1.8), and also in the reduced torsion groups $(C,U)^! : \tilde{K}_1(R) \longrightarrow \tilde{K}_1(S)$. By definition, $\mathbb{D}(S)$ is the homotopy category of finite chain complexes of based f.g. free S-modules. We shall now make use of the bases.

<u>Proposition</u> <u>9.1</u> Let (C,α,U) be a symmetric representation of R in $\mathbb{D}_n(S)$, for some rings with involution R,S.

i) For any $*$-invariant subgroups $X \subseteq \tilde{K}_1(R)$, $Y \subseteq \tilde{K}_1(S)$ such that $(C,U)^!(X) \subseteq Y$ and $\tau(\alpha : C \longrightarrow C^{n-*}) \in Y$ there are defined transfer maps in the intermediate torsion L-groups

$$(C,\alpha,U)^! : L_m^X(R) \longrightarrow L_{m+n}^Y(S) \quad (n \geqslant 0) .$$

ii) For any $*$-invariant subgroups $X \subseteq X' \subseteq \tilde{K}_1(R)$, $Y \subseteq Y' \subseteq \tilde{K}_1(S)$ such that $(C,U)^!(X) \subseteq Y$, $(C,U)^!(X') \subseteq Y'$, $\tau(\alpha) \in Y$ there is defined a morphism of Rothenberg exact sequences

$$\begin{tikzcd}
\cdots \arrow[r] & L^X_m(R) \arrow[r] \arrow[d, "(C,\alpha,U)^!"] & L^{X'}_m(R) \arrow[r] \arrow[d, "(C,\alpha,U)^!"] & \hat{H}^m(\mathbb{Z}_2;X'/X) \arrow[r] \arrow[d, "(C,U)^!"] & \cdots \\
\cdots \arrow[r] & L^Y_{m+n}(S) \arrow[r] & L^{Y'}_{m+n}(S) \arrow[r] & \hat{H}^{m+n}(\mathbb{Z}_2;Y'/Y) \arrow[r] & \cdots .
\end{tikzcd}$$

Proof: The transfer map in the reduced torsion groups $(C,U)^!:\tilde{K}_1(R)\longrightarrow\tilde{K}_1(S)$ is such that

$$*(C,U)^! = (-)^n(C,U)^!* \; : \; \tilde{K}_1(R) \longrightarrow \tilde{K}_1(S) \; .$$

Let $m=2i$. For any nonsingular $(-)^i$-quadratic form (M,ψ) on a based f.g. free R-module $M=R^k$ the n-dimensional $(-)^i$-quadratic Poincaré complex $(\underset{k}{\oplus}C,\theta)$ representing $(C,U)^!(M,\psi)$ has reduced torsion

$$\begin{aligned}\tau((1+T)\theta_0:\underset{k}{\oplus}C^{n-*}\longrightarrow\underset{k}{\oplus}C)\\ = (C,U)^!\tau(\psi+(-)^i\psi^*:M\longrightarrow M^*) \in \tilde{K}_1(S)\end{aligned}$$

the image of $\tau(\psi+(-)^i\psi^*)\in\tilde{K}_1(R)$. Similarly for $m=2i+1$ and formations.

□

Next, we consider the projective case $X\subseteq\tilde{K}_0(R)$. It is more convenient to work with the preimage of X in $K_0(R)$, so we regard X as a $*$-invariant subgroup of $K_0(R)$ such that $[R]\in X$.

Given a ring S let $\mathbb{E}(S)=\mathbb{D}(\mathbb{P}(S))$, the homotopy category of finite-dimensional f.g. projective S-module chain complexes. A representation (C,U) of a ring R in $\mathbb{E}(S)$ determines transfer maps in the algebraic K-groups

$$(C,U)^! \; : \; K_i(R) = K_i(\mathbb{P}(R)) \longrightarrow$$

$$K_i(S) = K_i(\mathbb{P}(S)) \quad (i=0,1)$$

(Example 1.8). For $n\geq 0$ let $\mathbb{E}_n(S)=\mathbb{D}_n(\mathbb{P}(S))$, the full subcategory of $\mathbb{E}(S)$ with objects n-dimensional f.g. projective S-module chain complexes. An involution on S determines the n-duality involution $C \longrightarrow C^{n-*}$ on $\mathbb{E}_n(S)$.

<u>Proposition</u> <u>9.2</u> Let (C,α,U) be a symmetric representation of R in $\mathbb{E}_n(S)$, for some rings with involution R,S.

i) For any $*$-invariant subgroups $X\subseteq K_0(R)$, $Y\subseteq K_0(S)$ such that $[R]\in X$, $[S]\in Y$, $(C,U)^!(X)\subseteq Y\subseteq K_0(S)$ there are defined transfer maps in the intermediate class L-groups

$$(C,\alpha,U)^! \; : \; L^X_m(R) \longrightarrow L^Y_{m+n}(S) \quad (n\geq 0) \; .$$

ii) For any $*$-invariant subgroups $X\subseteq X'\subseteq K_0(R)$, $Y\subseteq Y'\subseteq K_0(S)$ such that $[R]\in X$, $[S]\in Y$, $(C,U)^!(X)\subseteq Y$, $(C,U)^!(X')\subseteq Y'$ there is defined a morphism of Rothenberg exact sequences

$$\begin{tikzcd}
\dots \arrow[r] & L^X_m(R) \arrow[r] \arrow[d, "(C,\alpha,U)^!"] & L^{X'}_m(R) \arrow[r] \arrow[d, "(C,\alpha,U)^!"] & \hat{H}^m(\mathbb{Z}_2;X'/X) \arrow[r] \arrow[d, "(C,U)^!"] & \dots \\
\dots \arrow[r] & L^Y_{m+n}(S) \arrow[r] & L^{Y'}_{m+n}(S) \arrow[r] & \hat{H}^{m+n}(\mathbb{Z}_2;Y'/Y) \arrow[r] & \dots \, .
\end{tikzcd}$$

□

The proof of 9.2 is somewhat more involved than that of 9.1.

A splitting (B,r,i) in $\mathbb{A}$ of an object (A,p) in the idempotent completion $\hat{\mathbb{A}}$ is an object B in $\mathbb{A}$ together with morphisms $r:A\longrightarrow B$, $i:B\longrightarrow A$ in $\mathbb{A}$ such that

$$ri = 1 : B \longrightarrow B \; , \; ir = p : A \longrightarrow A \; .$$

<u>Lemma</u> <u>9.3</u> A functor of additive categories $F:\mathbb{A}\longrightarrow\mathbb{B}$

extends to a functor $\hat{F}:\hat{\mathbb{A}}\longrightarrow\mathbb{B}$ if and only if for each object (A,p) in $\hat{\mathbb{A}}$ the object $(F(A),F(p))$ in $\hat{\mathbb{B}}$ has a splitting in $\mathbb{B}$. Any two such extensions of F are naturally equivalent.

Proof: It is clear that the splitting condition is necessary for F to extend to $\hat{F}$, so we need only prove that it is sufficient. For each object (A,p) in $\hat{\mathbb{A}}$ choose a splitting (B,r,i) of the object $(F(A),F(p))$ in $\hat{\mathbb{B}}$, and set $\hat{F}(A,p)=B$, with $(B,r,i)=(F(A),1,1)$ for $p=1:A\longrightarrow A$. For a morphism $f:(A,p)\longrightarrow(A',p')$ let

$$\hat{F}(f)\ :\ \hat{F}(A)\ =\ B\xrightarrow{\ i\ }A\xrightarrow{\ f\ }A'\xrightarrow{\ r'\ }\hat{F}(A')\ =\ B'\ .$$

$\square$

An additive category $\mathbb{A}$ is idempotent complete if the functor $\mathbb{A}\longrightarrow\hat{\mathbb{A}};A\longrightarrow(A,1)$ is an equivalence of categories. Applying 9.3 to $1:\mathbb{A}\longrightarrow\mathbb{A}$ we have that $\mathbb{A}$ is idempotent complete if and only if every object (A,p) in $\hat{\mathbb{A}}$ splits in $\mathbb{A}$. If $\mathbb{B}$ is idempotent complete every functor $F:\mathbb{A}\longrightarrow\mathbb{B}$ extends to a functor $\hat{F}:\hat{\mathbb{A}}\longrightarrow\mathbb{B}$, namely the composite of $\hat{F}:\hat{\mathbb{A}}\longrightarrow\hat{\mathbb{B}}$ and an equivalence $\hat{\mathbb{B}}\longrightarrow\mathbb{B}$.

For any ring S the additive category $\mathbb{P}(S)$ of f.g. projective S-modules is idempotent complete, with every object (A,p) in $\hat{\mathbb{P}}(S)$ split by the triple (B,r,i) defined by

$$r\ :\ A\longrightarrow B=\mathrm{im}(p)\ ;\ x\longrightarrow p(x)\ ,$$

$$i\ =\ \text{inclusion}\ :\ B\longrightarrow A\ .$$

This is the special case $n=0$ of:

Lemma 9.4 For any ring S and any $n\geqslant 0$ the homotopy category $\mathbb{E}_n(S)$ of n-dimensional f.g. projective S-module chain complexes is idempotent complete.

Proof: For every chain homotopy projection $p\simeq p^2:D\longrightarrow D$ of an object D in $\mathbb{E}_n(S)$ there exists by Lemma 3.4 of Lück [7] an $(n+1)$-dimensional infinitely generated

projective S-module chain complex C with chain maps $r:D\longrightarrow C$, $i:C\longrightarrow D$ and chain homotopies

$$ri \simeq 1 : C \longrightarrow C \quad , \quad ir \simeq p : D \longrightarrow D \; .$$

Since C is dominated by an object in $\mathbb{E}_n(S)$ (namely D) it is chain equivalent to an object in $\mathbb{E}_n(S)$, by Proposition 3.1 of Ranicki [17].

□

The idempotent completion of an additive category $\mathbb{A}$ with an involution $*:\mathbb{A}\longrightarrow\mathbb{A}$ is an additive category $\hat{\mathbb{A}}$ with the involution

$$* \; : \; \hat{\mathbb{A}} \longrightarrow \hat{\mathbb{A}} \; ; \; (A,p) \longrightarrow (A^*,p^*) \; .$$

For a ring with involution R the functor $\hat{\mathbb{B}}(R)\longrightarrow\mathbb{P}(R);(R^k,p)\longrightarrow \mathrm{im}(p)$ is an equivalence of additive categories with involution. Both 9.3 and 9.4 have evident versions for additive categories with involution.

<u>Definition</u> <u>9.5</u> Let R,S be rings with involution. The <u>projective</u> <u>surgery</u> <u>transfer</u> <u>maps</u> of a symmetric representation (C,α,U) of R in $\mathbb{E}_n(S)$

$$(C,\alpha,U)^! \; : \; L^p_m(R) \longrightarrow L^p_{m+n}(S) \quad (m\geqslant 0)$$

are the composites

$$(C,\alpha,U)^! \; : \; L^p_m(R) = L_m(\mathbb{P}(R)) \xrightarrow{\hat{F}} L_m(\mathbb{E}_n(S)) \xrightarrow{\mu} L^p_{m+n}(S)$$

with μ the generalized Morita maps of 3.3 for $\mathbb{A}=\mathbb{P}(S)$ and $\hat{F}$ induced by the functor of additive categories with involution $\hat{F}:\hat{\mathbb{B}}(R)\simeq\mathbb{P}(R)\longrightarrow\mathbb{E}_n(S)$ associated by 9.4 to the functor

$$F = -\otimes(C,\alpha,U) : \mathbb{B}(R) \longrightarrow \mathbb{E}_n(S) \; ; \; R \longrightarrow C \; .$$

□

The proof of 9.2 is now completed by observing that the transfer map in the projective class groups $(C,U)^! : K_0(R) \longrightarrow K_0(S)$ is such that

$$*(C,U)^! = (-)^n (C,U)^! * : K_0(R) \longrightarrow K_0(S) \; .$$

<u>Remark 9.6</u> Our methods also apply to construct algebraic surgery transfer maps in the round L-groups $L_*^{rX}(R)$ of Hambleton, Ranicki and Taylor [4], which are defined for $*$-invariant subgroups $X \subseteq K_1(R)$. For any symmetric representation (C,α,U) of R in $\mathbb{E}_n(S)$ and any $*$-invariant subgroup $X \subseteq K_1(R)$, $Y \subseteq K_1(S)$ such that $(C,U)^!(X) \subseteq Y$ there are defined round L-theory transfer maps

$$(C,\alpha,U)^! : L_m^{rX}(R) \longrightarrow L_{m+n}^{rY}(S) \quad (m \geqslant 0)$$

which are compatible with the round L-theory Rothenberg exact sequences.

□

<u>Remark 9.7</u> The connection established in §8 between the algebraic and geometric surgery transfer maps extends to the intermediate cases, and also to round L-theory.

□

<u>Remark 9.8</u> Our algebraic constructions apply also to the ϵ-quadratic L-groups $L_*(R,\epsilon)$, which are defined for a ring with involution R and a central unit $\epsilon \in R$ such that $\bar{\epsilon}\epsilon = 1$. $L_{2i}(R,\epsilon)$ (resp. $L_{2i+1}(R,\epsilon)$) is the Witt group of nonsingular $(-)^i\epsilon$-quadratic forms (resp. formations) over R. A symmetric representation (C,α,U) of R in $\mathbb{D}_n(S)$ such that $U(\epsilon) = \eta : C \longrightarrow C$ for a central unit

$\eta \in S$ with $\bar{\eta}\eta = 1$ induces transfer maps

$$(C,\alpha,U)^! \ : \ L_m(R,\epsilon) \longrightarrow L_{m+n}(S,\eta) \quad (m \geq 0) \ .$$

Hitherto we considered the case $\epsilon = 1 \in R$ for which $L_*(R,1) = L_*(R)$, with $\eta = 1 \in S$.

□

Appendix 1. Fibred intersections

The proof of $p^!_{geo} = p^!_{alg}$ in §8 makes heavy use of the algebraic properties of the L-groups. For a fibre bundle $F \longrightarrow E \xrightarrow{p} B$ with the fibre F a compact n-dimensional manifold it is possible to verify that the algebraic and geometric surgery transfer maps coincide more directly, using the bordism intersection theory of Hatcher and Quinn [6] to obtain fibred versions of the geometric intersection forms (resp. formations) used by Wall [22] to define the surgery obstruction of a highly-connected even (resp. odd-) dimensional normal map. The quadratic kernel of the pullback normal map is the fibred intersection form (resp. formation) both algebraically and geometrically. We now sketch the argument for the intersection pairing λ in the even-dimensional case, leaving the self-intersection function μ and the odd-dimensional case to the interested reader.

Given two maps $v_i : Q_i \longrightarrow M$ $(i = 1,2)$ let $E(v_1,v_2)$ be the pointed space of triples (x_1,x_2,ω) defined by points $x_i \in Q_i$ and a path $\omega : [0,1] \longrightarrow M$ from $\omega(0) = v_1(x_1)$ to $\omega(1) = v_2(x_2)$, so that there is defined a homotopy fibre square

$$\begin{array}{ccc} E(v_1,v_2) & \longrightarrow & Q_1 \\ \big\downarrow & & \big\downarrow{\scriptstyle v_1} \\ Q_2 & \xrightarrow{v_2} & M \ . \end{array}$$

Given a stable vector bundle η over a space M let $\Omega_n^{fr}(M,\eta)$ be the bordism group of n-manifolds N equipped with a map $N \longrightarrow M$ and a compatible stable bundle map $\nu_N \longrightarrow \eta$. For trivial η this is the usual framed cobordism group $\Omega_n^{fr}(M)=\pi_n^S(M\cup\{*\})$. For $v_1=v_2:Q_1=Q_2=\{*\}\longrightarrow M$ the homotopy pullback is the loop space, $E(*,*)=\Omega M$.

Now suppose that M is an m-manifold, and that $v_i:Q_i\longrightarrow M$ is an immersion of a q_i-manifold Q_i $(i=1,2)$ such that $v_1(Q_1)$ intersects $v_2(Q_2)$ in general position. Let $Q_1 \cap Q_2$ denote the corresponding (q_1+q_2-m)-dimensional submanifold of M. The bordism invariant of the intersection ([6,2.1]) is the bordism class

$$\lambda(v_1,v_2) = [Q_1 \cap Q_2]$$
$$\in \Omega^{fr}_{q_1+q_2-m}(E(v_1,v_2),\nu_{Q_1}\oplus\nu_{Q_2}\oplus\tau_M) \ .$$

If Q_1 and Q_2 are (q_1+q_2-m+1)-connected the map $E(*,*)=\Omega M \longrightarrow E(v_1,v_2)$ induces an isomorphism ([6,3.1])

$$\Omega^{fr}_{q_1+q_2-m}(E(*,*)) = \Omega^{fr}_{q_1+q_2-m}(\Omega M) \longrightarrow$$
$$\Omega^{fr}_{q_1+q_2-m}(E(v_1,v_2),\nu_{Q_1}\oplus\nu_{Q_2}\oplus\tau_M)$$

which is used as an identification.

Let $(f,b):M\longrightarrow X$ be an $(i-1)$-connected $2i$-dimensional normal map with a π_1-isomorphism reference map $X\longrightarrow B$, with the surgery obstruction $\sigma_*(f,b)=(K_i(M),\lambda,\mu)\in L_{2i}(\mathbb{Z}[\pi_1(B)])$ defined as in Chapter 5 of Wall [22]. Let $v_1,v_2,\ldots,v_k$ be a base of the kernel f.g. free $\mathbb{Z}[\pi_1(B)]$-module $K_i(M)=\pi_{i+1}(f)$. Represent each $v_j\in K_i(M)$ by a pointed framed immersion $v_j:S^i\longrightarrow M$ with a nullhomotopy in X. The values taken by the $(-)^i$-symmetric form $(K_i(M),\lambda)$ on the base elements are just the bordism intersections

$$\lambda(v_j,v_{j'}) \in \Omega_0^{fr}(E(v_j,v_{j'}),\nu_{S^i}\oplus\nu_{S^i}\oplus\tau_M)$$
$$= \Omega_0^{fr}(\Omega M) = H_0(\Omega M) = \mathbb{Z}[\pi_1(B)]$$
$$(1\leqslant j,j'\leqslant k) \ .$$

Now let $(g,c):N \longrightarrow Y$ be the $(i-1)$-connected $(n+2i)$-dimensional normal map with a π_1-isomorphism reference map $Y \longrightarrow E$ obtained from $(f,b):M \longrightarrow X$ by the pullback of the fibre bundle $F \longrightarrow E \xrightarrow{p} B$ along $X \longrightarrow B$. The pointed framed immersions $v_j:S^i \longrightarrow M$ $(1\leqslant j\leqslant k)$ with nullhomotopies in X lift to pointed framed immersions $w_j:S^i\times F \longrightarrow N$ with nullhomotopies in Y. On the chain level this corresponds to lifting the kernel $\mathbb{Z}[\pi_1(B)]$-module chain complex $C(f^!)=S^iK_i(M)=\bigoplus\limits_k S^i\mathbb{Z}[\pi_1(B)]$

to the kernel $\mathbb{Z}[\pi_1(E)]$-module chain complex

$C(g^!)=\bigoplus\limits_k S^iC(\widetilde{F})$. The bordism intersections

$$\lambda(w_j,w_{j'}) \in \Omega_n^{fr}(E(w_j,w_{j'}),\nu_{S^i\times F}\oplus\nu_{S^i\times F}\oplus\tau_N)$$
$$= \Omega_n^{fr}(\Omega M\times F,\nu_F) \quad (1\leqslant j,j'\leqslant k)$$

are the images of the bordism intersections $\lambda(v_j,v_{j'})$ under the geometric bordism transfer map

$$p^! = -\times F \ : \ \Omega_0^{fr}(\Omega M) \longrightarrow \Omega_n^{fr}(\Omega M\times F,\nu_F) \ ;$$
$$\lambda \longrightarrow \lambda\times F \ .$$

The Poincaré duality isomorphism of based f.g. free $\mathbb{Z}[\pi_1(B)]$-modules

$$(\lambda(v_j,v_{j'})) \ :$$
$$C(f^!)^{2i-*} = S^iK_i(M)^* \longrightarrow C(f^!) = S^iK_i(M)$$

is lifted to the Poincaré duality chain equivalence of

chain complexes of based f.g. free $\mathbb{Z}[\pi_1(E)]$-modules

$$(\lambda(w_j,w_{j'})) \ : \ C(g^!)^{n+2i-*} = \bigoplus_k S^i C(\tilde{F})^{n-*}$$

$$\longrightarrow C(g^!) = \bigoplus_k S^i C(\tilde{F}) \ .$$

Using the Poincaré duality $\mathbb{Z}[\pi_1(E)]$-module chain equivalence $[F]\cap - : C(\tilde{F})^{n-*} \longrightarrow C(\tilde{F})$, the action of ΩM on the $\pi_1(E)$-equivariant homotopy type of $\tilde{F}$ and Hurewicz maps there is defined a commutative diagram

$$\begin{tikzcd}
\Omega_0^{fr}(\Omega M) \arrow[r, "\cong"] \arrow[dd, "p^!"'] & H_0(\Omega M) \arrow[r, "\cong"] \arrow[dd, "-\otimes[F]"'] & \mathbb{Z}[\pi_1(B)] \arrow[d, "U"] \\
 & & H_0(\mathrm{Hom}_{\mathbb{Z}[\pi_1(E)]}(C(\tilde{F}),C(\tilde{F}))^{op} \arrow[d, "{[F]\cap -}"] \\
\Omega_n^{fr}(\Omega M\times F,\nu_F) \arrow[r] & H_n(\Omega M\times F) \arrow[r] & H_n(C(\tilde{F})\otimes_{\mathbb{Z}[\pi_1(E)]}C(\tilde{F}))
\end{tikzcd}$$

The anticlockwise composition gives the geometric surgery transfer $p^!_{geo}$ on the level of intersections, while the clockwise composition gives the algebraic surgery transfer $p^!_{alg}$.

Appendix 2. A counterexample in symmetric L-theory

An n-dimensional Poincaré fibration $F\longrightarrow E\xrightarrow{p} B$ does not in general induce transfer maps in the symmetric L-groups $p^!:L^m(\mathbb{Z}[\pi_1(B)])\longrightarrow L^{m+n}(\mathbb{Z}[\pi_1(E)])$, either algebraically or geometrically. It is not possible to define $p^!$ geometrically since the symmetric L-groups are not geometrically realizable (Ranicki [16,7.6.8]). There are two obstructions to an algebraic definition of $p^!$, which requires the lifting of an m-dimensional symmetric Poincaré complex (C,ϕ) over $\mathbb{Z}[\pi_1(B)]$ representing an element $(C,\phi)\in L^m(\mathbb{Z}[\pi_1(B)])$ to an $(m+n)$-dimensional symmetric Poincaré complex $(C^!,\phi^!)$ over $\mathbb{Z}[\pi_1(E)]$ representing the putative transfer $p^!(C,\phi)=(C^!,\phi^!)\in L^{m+n}(\mathbb{Z}[\pi_1(E)])$. The symmetric L-groups are not 4-periodic, so it cannot be assumed that (C,ϕ)

is highly-connected as in the quadratic case. In the following discussion we assume that the fibre F is finite, and that the chain complex C consists of based f.g. free $\mathbb{Z}[\pi_1(B)]$-modules. The two obstructions to lifting (C,ϕ) to $(C^!,\phi^!)$ are given by:

i) it may not be possible to lift C to a based f.g. free $\mathbb{Z}[\pi_1(E)]$-module chain complex $C^!$ with a filtration $F_0C^! \subseteq F_1C^! \subseteq \ldots \subseteq F_mC^! = C^!$ such that the connecting chain maps between successive filtration quotients are given up to chain homotopy by

$$\partial = p^{\#}(d_C) : F_rC^!/F_{r-1}C^! = S^r p^{\#}(C_r) \longrightarrow$$

$$S(F_{r-1}C^!/F_{r-2}C^!) = S^r p^{\#}(C_{r-1}) \quad (1 \leqslant r \leqslant m)$$

where S^r denotes the r-fold dimension shift and $p^{\#}$ is the functor of §1

$$p^{\#} = -\otimes(C(\tilde{F}),U) : \mathbb{B}(\mathbb{Z}[\pi_1(B)]) \longrightarrow \mathbb{D}_n(\mathbb{Z}[\pi_1(E)]) \ ,$$

ii) even if $C^!$ exists, it may not be possible to lift the m-dimensional symmetric Poincaré structure ϕ on C to an (m+n)-dimensional symmetric Poincaré structure $\phi^!$ on $C^!$.

If C can be assembled over B in the sense of Ranicki and Weiss [20] then it can be lifted to $C^!$, but in general it is not possible to assemble $\mathbb{Z}[\pi_1(B)]$-module chain complexes, so already i) presents a non-trivial obstruction to the existence of transfer in symmetric L-theory. Even if the obstruction of i) vanishes (e.g. if B is an Eilenberg-MacLane space $K(\pi_1(B),1)$) then ii) may present a non-trivial obstruction. This is illustrated by the following example, which exhibits the failure of a projection of rings with involution $p:S \longrightarrow R = S/(1-t)$ (t = central unit $\in S$, $\bar{t} = t^{-1} \in S$) to induce an S^1-bundle symmetric L-theory transfer map $p^!:L^0(R) \longrightarrow L^1(S)$ analogous to the S^1-bundle quadratic L-theory transfer map $p^!:L_0(R) \longrightarrow L_1(S)$ (cf. 4.7). The

transfer $p^!(C,\phi)=(C^!,\phi^!)$ of a 0-dimensional symmetric Poincaré complex (= nonsingular symmetric form) (C,ϕ) over R with $C_0=R^k$ is defined if the symmetric $k\times k$ matrix

$$\phi_0 = (\phi_0)^* \in M_k(R)$$

can be lifted to a $k\times k$ matrix $\phi_0^! \in M_k(S)$ such that $p(\phi_0^!)=\phi_0 \in M_k(R)$ and

$$t\phi_0^! - (\phi_0^!)^* = (1-t)\phi_1^! \in M_k(S)$$

for some symmetric $k\times k$ matrix $\phi_1^!=(\phi_1^!)^* \in M_k(S)$, so that $(C^!,\phi^!)$ is a 1-dimensional symmetric Poincaré complex over S with $C^!=C(1-t:S^k \longrightarrow S^k)$. In particular, for

$$S = \mathbb{Z}_2[\mathbb{Z}_2\times\mathbb{Z}_2] = \mathbb{Z}_2[t,u]/(t^2-1,u^2-1) \ ,$$

$$\bar{t} = t \ , \ \bar{u} = t+u+1 \ ,$$

$$p : S \longrightarrow R = \mathbb{Z}_2[\mathbb{Z}_2] = \mathbb{Z}_2[u]/(u^2-1) \ ;$$

$$t \longrightarrow 1 \ , \ u \longrightarrow u$$

the transfer is not defined for the 0-dimensional symmetric Poincaré complex $(C,\phi)=(R,u)$ over R, for although C can be lifted to $C^!$ and ϕ_0 can be lifted to $\phi_0^!$ there does not exist a symmetric $\phi_1^!$. Both the obstructions to i) and ii) vanish for the visible symmetric L-groups $VL^*(\mathbb{Z}[\pi])$ of Weiss [23] provided that B is an Eilenberg-MacLane space $K(\pi_1(B),1)$, in which case there are defined transfer maps $p^!:VL^m(\mathbb{Z}[\pi_1(B)]) \longrightarrow VL^{m+n}(\mathbb{Z}[\pi_1(E)])$.

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