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# The total surgery obstruction

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Let $n \geqslant 5$.

According to the Browder-Novikov-Sullivan-Wall theory of surgery ([B1],[B2],[N],[Su1],[W1]) a finite n-dimensional Poincaré complex X is homotopy equivalent to a compact topological manifold if and only if

i) the Spivak normal fibration $\nu_X : X \longrightarrow BG(k)$ $(k \gg n)$ admits a topological reduction $\widetilde{\nu}_X : X \longrightarrow BTOP(k)$, in which case topological transversality applied to a degree 1 map $\rho_X : S^{n+k} \longrightarrow T(\nu_X)$ gives a topological manifold $M^n = \rho_X^{-1}(X) \subset S^{n+k}$ and a map of topological bundles $b : \widetilde{\nu}_M \longrightarrow \widetilde{\nu}_X$ covering the degree 1 map $f = \rho_X| : M \longrightarrow X$, and hence a surgery obstruction $\theta(f,b) \in L_n(\pi_1(X))$

ii) there exists a topological reduction $\widetilde{\nu}_X$ such that $\theta(f,b) = 0$, in which case the normal map $(f,b) : M \longrightarrow X$ is normal bordant to a homotopy equivalence. The theory was initially developed in the smooth and PL categories; the extension to the topological category is due to Kirby and Siebenmann ([KS]).

We present here the preliminary account of a theory which replaces the two-stage obstruction with a single invariant, 'the total surgery obstruction'.

We shall only consider the oriented case, but in principle there exists an unoriented version involving twisted coefficients. For the sake of the s-cobordism theorem we shall be working with simple homotopy types and the Wall $L^s$-groups, but there is also an ordinary homotopy version which we discuss briefly at the end. Thus Poincaré complexes will be finite, simple and oriented; manifolds will be compact, topological and oriented.

The invariant lies in one of the groups $\mathcal{S}_*(X)$ (defined for any space X) appearing in an exact sequence of abelian groups

$$\ldots \longrightarrow H_n(X;\underline{\mathbb{L}}_0) \xrightarrow{\sigma_*} L_n(\pi_1(X)) \longrightarrow \mathcal{S}_n(X) \longrightarrow H_{n-1}(X;\underline{\mathbb{L}}_0) \longrightarrow \ldots ,$$

where $\underline{\mathbb{L}}_0$ is a 1-connective $\Omega$-spectrum with 0th space homotopy equivalent to G/TOP and $\sigma_*$ is a universal assembly map. Both $\underline{\mathbb{L}}_0$ and $\sigma_*$ were originally constructed by

Quinn ([Q1],[Q2]) using geometric methods. Here, $\underline{\mathbb{L}}_0$ and $\sigma_*$ are constructed using algebraic methods, and the groups $\mathcal{S}_*(X)$ are the relative homotopy groups of a map of simplicial $\Omega$-spectra $\sigma_*:X_+\wedge\underline{\mathbb{L}}_0 \longrightarrow \underline{\mathbb{L}}_0(\pi_1(X))$ inducing the assembly maps $\sigma_*:H_*(X;\underline{\mathbb{L}}_0) = \pi_*(X_+\wedge\underline{\mathbb{L}}_0) \longrightarrow \pi_*(\underline{\mathbb{L}}_0(\pi_1(X))) = L_*(\pi_1(X))$ $(X_+ = X\cup\{pt.\})$.
There are also defined relative groups $\mathcal{S}_*(X,Y)$ for pairs $(X,Y)$, to fit into an exact sequence of abelian groups

$$\cdots \longrightarrow H_n(X,Y;\underline{\mathbb{L}}_0) \xrightarrow{\sigma_*} L_n(\pi_1(Y)\longrightarrow\pi_1(X)) \longrightarrow \mathcal{S}_n(X,Y) \longrightarrow H_{n-1}(X,Y;\underline{\mathbb{L}}_0) \longrightarrow \cdots .$$

The functor $\mathcal{S}_*$ satisfies the first five of the seven Eilenberg-Steenrod axioms for a homology theory, failing excision and dimension:

$$\mathcal{S}_*(\text{pushout square}) = \text{Cappell's } \mathrm{Unil}_* \ , \ \mathcal{S}_*(\text{pt.}) = 0 .$$

<u>Theorem 1</u> An n-dimensional Poincaré complex $X$ determines an element $s(X)\in\mathcal{S}_n(X)$, the <u>total surgery obstruction</u> of $X$, such that $s(X) = 0$ if and only if $X$ is simple homotopy equivalent to a closed topological manifold. The image of $s(X)$ in $H_{n-1}(X;\underline{\mathbb{L}}_0)$ is the obstruction to a topological reduction of the Spivak normal fibration $\nu_X:X \longrightarrow BSG$.

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There are also relative versions (and even n-ad versions) of Theorem 1:

<u>Theorem 1 (rel)</u> An n-dimensional Poincaré pair $(X,Y)$ determines an element $s(X,Y)\in\mathcal{S}_n(X,Y)$ such that $s(X,Y) = 0$ if and only if $(X,Y)$ is simple homotopy equivalent to a manifold with boundary.

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<u>Theorem 1 (rel $\partial$)</u> An n-dimensional Poincaré pair $(X,Y)$ with manifold boundary $Y$ determines an element $s_\partial(X,Y)\in\mathcal{S}_n(X)$ such that $s_\partial(X,Y) = 0$ if and only if $(X,Y)$ is simple homotopy equivalent to a manifold with boundary by an equivalence which restricts to a homeomorphism of the boundaries.

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The obstruction theory of Sullivan [Su1] for the problem of deforming a homotopy equivalence of manifolds to a homeomorphism has a natural expression as a total surgery obstruction:

<u>Corollary 1</u> A simple homotopy equivalence of closed n-dimensional manifolds $f:M \longrightarrow X$ determines an element $s(f) \in \mathcal{S}_{n+1}(X)$ such that $s(f) = 0$ if and only if f is homotopic to a homeomorphism.

<u>Proof</u>: Let W be the mapping cylinder of f, so that $(W, M \cup -X)$ is an (n+1)-dimensional Poincaré pair with manifold boundary. Define

$$s(f) = s_\partial(W, M \cup -X) \in \mathcal{S}_{n+1}(W) \ (= \mathcal{S}_{n+1}(X) \text{ by the homotopy invariance of } \mathcal{S}_*) \ .$$

By Theorem 1 (rel $\partial$) $s(f) = 0$ if and only if there exists a topological s-cobordism $(W';M',X')$ simple homotopy equivalent to $(W;M,X)$ by an equivalence which restricts to homeomorphisms of the boundary components. Now apply the topological s-cobordism theorem (in dimension $n+1 \geqslant 6$).

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There are also relative versions, Corollary 1 (rel) and Corollary 1 (rel $\partial$).

Given an n-dimensional Poincaré complex X let $\mathcal{S}^{TOP}(X)$ be the topological manifold structure set of X, defined as usual to be the set of equivalence classes of pairs

(closed n-dimensional topological manifold M,

orientation preserving simple homotopy equivalence $f:M \longrightarrow X$)

under the relation

$(M,f) \sim (M',f')$ if there exist a homeomorphism $h: M \longrightarrow M'$ and a homotopy $f'h \simeq f : M \longrightarrow X$ .

Define similarly structure sets $\mathcal{S}^{TOP}(X,Y)$ for Poincaré pairs $(X,Y)$, and also $\mathcal{S}^{TOP}_\partial(X,Y)$ for Poincaré pairs $(X,Y)$ with manifold boundary Y.

<u>Corollary 2</u> If X is a closed n-dimensional manifold the function

$$s : \mathcal{S}^{TOP}(X) \longrightarrow \mathcal{S}_{n+1}(X) \ ; \ (f:M \longrightarrow X) \longmapsto s(f)$$

is a bijection, and there is a natural identification of the Sullivan-Wall surgery exact sequence

$$\ldots \longrightarrow \mathcal{S}_\partial^{TOP}(X\times\Delta^1,\partial(X\times\Delta^1)) \longrightarrow [X\times\Delta^1,\partial(X\times\Delta^1);G/TOP,*] \xrightarrow{\theta} L_{n+1}(\pi_1(X))$$
$$\longrightarrow \mathcal{S}^{TOP}(X) \longrightarrow [X,G/TOP] \xrightarrow{\theta} L_n(\pi_1(X))$$

with the exact sequence

$$\ldots \longrightarrow \mathcal{S}_{n+2}(X) \longrightarrow H_{n+1}(X;\underline{\mathbb{L}}_0) \xrightarrow{\sigma_*} L_{n+1}(\pi_1(X))$$
$$\longrightarrow \mathcal{S}_{n+1}(X) \longrightarrow H_n(X;\underline{\mathbb{L}}_0) \xrightarrow{\sigma_*} L_n(\pi_1(X)) \quad .$$

In particular,

$$\mathcal{S}_\partial^{TOP}(X\times\Delta^k,\partial(X\times\Delta^k)) = \mathcal{S}_{n+k+1}(X) \ , \ [X\times\Delta^k,\partial(X\times\Delta^k);G/TOP,*] = H_{n+k}(X;\underline{\mathbb{L}}_0) \quad (k \geqslant 0) \ .$$

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Again, there are relative versions, Corollary 2 (rel) and Corollary 2 (rel$\partial$). If $(X,\partial X)$ is an n-dimensional manifold with boundary there are natural identifications

$$\begin{aligned} \mathcal{S}^{TOP}(X\times\Delta^k,\partial(X\times\Delta^k)) &= \mathcal{S}_{n+k+1}(X,\partial X) \\ \mathcal{S}_\partial^{TOP}(X\times\Delta^k,\partial(X\times\Delta^k)) &= \mathcal{S}_{n+k+1}(X) \end{aligned} \quad (k \geqslant 0) \quad .$$

We shall only sketch a proof of Theorem 1 here. There are 4 main ingredients:

i) the Browder-Novikov-Sullivan-Wall theory in the topological category

ii) the isomorphisms $\theta:\pi_*(G/TOP) \longrightarrow L_*(1)$ defined by the surgery obstruction

iii) transversality in Quinn's category of normal spaces and spherical fibrations

iv) the algebraic theory of surgery.

We start with a brief account of iv) - the first two instalments of a full account are due to appear shortly ([R2]).

Given a group $\pi$ and a (left) $\mathbb{Z}[\pi]$-module chain complex $C$ let $T\in\mathbb{Z}_2$ act on $C\otimes_{\mathbb{Z}[\pi]}C = C\otimes_{\mathbb{Z}}C/\{x\otimes gy - g^{-1}x\otimes y \mid x,y\in C, g\in\pi\}$ by $T(x\otimes y) = (-)^{|x||y|}y\otimes x$, and define the $\begin{cases}\mathbb{Z}_2\text{-hypercohomology}\\ \mathbb{Z}_2\text{-hyperhomology}\end{cases}$ groups $\begin{cases}Q^n(C) = H_n(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C\otimes_{\mathbb{Z}[\pi]}C))\\ Q_n(C) = H_n(W\otimes_{\mathbb{Z}[\mathbb{Z}_2]}(C\otimes_{\mathbb{Z}[\pi]}C))\end{cases}$ with $W$ the free $\mathbb{Z}[\mathbb{Z}_2]$-module resolution of $\mathbb{Z}$ $W : \ldots \longrightarrow \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1+T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}_2] \longrightarrow 0$.

An element $\begin{cases}\varphi\in Q^n(C)\\ \psi\in Q_n(C)\end{cases}$ is an equivalence class of collections $\begin{cases}\{\varphi_s\in (C\otimes_{\mathbb{Z}[\pi]}C)_{n+s} \mid s\geqslant 0\}\\ \{\psi_s\in (C\otimes_{\mathbb{Z}[\pi]}C)_{n-s} \mid s\geqslant 0\}\end{cases}$ such that

$$\begin{cases} d(\varphi_s) + (-)^{n+s-1}(\varphi_{s-1} + (-)^s T\varphi_{s-1}) = 0 \in (C\otimes_{\mathbb{Z}[\pi]}C)_{n+s-1} & (s\geqslant 0, \varphi_{-1}=0)\\ d(\psi_s) + (-)^{n-s-1}(\psi_{s+1} + (-)^{s+1} T\psi_{s+1}) = 0 \in (C\otimes_{\mathbb{Z}[\pi]}C)_{n-s-1} & (s\geqslant 0). \end{cases}$$

The $\begin{cases}\text{symmetric}\\ \text{quadratic}\end{cases}$ L-groups $\begin{cases}L^n(\pi)\\ L_n(\pi)\end{cases}$ $(n\geqslant 0)$ are defined to be the algebraic Poincaré cobordism groups of n-dimensional $\begin{cases}\text{symmetric}\\ \text{quadratic}\end{cases}$ Poincaré complexes over $\mathbb{Z}[\pi]$ $\begin{cases}(C, \varphi\in Q^n(C))\\ (C, \psi\in Q_n(C))\end{cases}$, with $C: C_n \xrightarrow{d} C_{n-1} \longrightarrow \ldots \longrightarrow C_1 \xrightarrow{d} C_0$ a based f.g. free $\mathbb{Z}[\pi]$-module chain complex such that slant product with the cycle $\begin{cases}\varphi_0\in (C\otimes_{\mathbb{Z}[\pi]}C)_n\\ (1+T)\psi_0\in (C\otimes_{\mathbb{Z}[\pi]}C)_n\end{cases}$ defines a simple chain equivalence $C^{n-*} = \mathrm{Hom}_{\mathbb{Z}[\pi]}(C, \mathbb{Z}[\pi])_{n-*} \longrightarrow C$. The quadratic L-groups are 4-periodic, $L_n(\pi) = L_{n+4}(\pi)$, being just the Wall surgery obstruction groups. The symmetric L-groups were introduced by Mishchenko [Mi]; they are not in general 4-periodic, $L^n(\pi) \neq L^{n+4}(\pi)$. There are defined symmetrization maps

$$1+T : L_n(\pi) \longrightarrow L^n(\pi) \ ; \ (C,\psi) \longmapsto (C, (1+T)\psi_0)$$

which are isomorphisms modulo 8-torsion. The cobordism classes of (n-1)-dimensional quadratic Poincaré complexes with an n-dimensional symmetric Poincaré null-cobordism define hyperquadratic L-groups $\hat{L}^n(\pi)$ $(n\geqslant 1)$ of exponent 8 which fit into a long exact sequence of abelian groups

$$\ldots \longrightarrow L_n(\pi) \xrightarrow{1+T} L^n(\pi) \xrightarrow{J} \hat{L}^n(\pi) \xrightarrow{H} L_{n-1}(\pi) \xrightarrow{1+T} L^{n-1}(\pi) \longrightarrow \ldots .$$

For example, $\begin{cases}L^0(\pi)\\ L_0(\pi)\end{cases}$ is the Witt group of non-singular $\begin{cases}\text{symmetric}\\ \text{quadratic}\end{cases}$ forms over $\mathbb{Z}[\pi]$.

The L-groups of the trivial group $\pi = \{1\}$ are given by

$$L^n(1) = \begin{cases} \mathbb{Z} & \text{(signature)} \\ \mathbb{Z}_2 & \text{(deRham)} \\ 0 & \\ 0 & \end{cases} \quad , \quad L_n(1) = \begin{cases} \mathbb{Z} & (\tfrac{1}{8}\text{(signature)}) \\ 0 & \\ \mathbb{Z}_2 & \text{(Arf)} \\ 0 & \end{cases}$$

$$\hat{L}^n(1) = \begin{cases} \mathbb{Z}_8 & \text{(signature (mod 8))} \\ \mathbb{Z}_2 & \text{(deRham)} \\ 0 & \\ \mathbb{Z}_2 & \text{(Arf)} \end{cases} \qquad \text{if } n \equiv \begin{cases} 0 \\ 1 \\ 2 \\ 3 \end{cases} \pmod 4$$

An n-dimensional geometric Poincaré complex $X$ is an n-dimensional finite CW complex together with a fundamental homology class $[X] \in H_n(X)$ such that cap product defines a simple chain equivalence of based f.g. free $\mathbb{Z}[\pi_1(X)]$-module chain complexes

$$[X] \cap - \; : \; C(\widetilde{X})^{n-*} \longrightarrow C(\widetilde{X}) \; ,$$

with $C(\widetilde{X})$ the cellular chain complex of the universal cover $\widetilde{X}$. Applying $H_*(\mathbb{Z} \otimes_{\mathbb{Z}[\pi_1(X)]} -)$ to a diagonal approximation $\Delta : C(\widetilde{X}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C(\widetilde{X}) \otimes_{\mathbb{Z}} C(\widetilde{X}))$ and evaluating $\Delta : H_n(X) \longrightarrow Q^n(C(\widetilde{X}))$ defines an n-dimensional symmetric Poincaré complex over $\mathbb{Z}[\pi_1(X)]$ $(C,\varphi) = (C(\widetilde{X}), \Delta[X])$, and hence a symmetric signature geometric Poincaré bordism invariant

$$\sigma^*(X) = (C(\widetilde{X}), \Delta[X]) \in L^n(\pi_1(X))$$

(which was introduced by Mishchenko [Mi]). Given a group morphism $\pi_1(X) \longrightarrow \pi$ we shall denote the image of $\sigma^*(X) \in L^n(\pi_1(X))$ in $L^n(\pi)$ also by $\sigma^*(X)$. For example, if $n = 4k$ $\sigma^*(X) \in L^{4k}(1) = \mathbb{Z}$ is just the ordinary signature of $X$.

An n-dimensional geometric Poincaré complex $X$ carries a stable equivalence class of Spivak normal structures

$$(\nu_X : X \longrightarrow BSG(k), \rho_X : S^{n+k} \longrightarrow T(\nu_X)) \quad ,$$

such as arise from an embedding $X \subset S^{n+k}$ $(k \gg n)$ by taking a closed regular neighbourhood $W$ of $X$ in $S^{n+k}$ and setting

$$S^{k-1} \longrightarrow \partial W \xrightarrow{\nu_X} W$$

$$\rho_X : S^{n+k} \xrightarrow{\text{collapse}} S^{n+k}/\overline{S^{n+k}-W} = W/\partial W = T(\nu_X) \ .$$

As usual, $[X] = h(\rho_X) \cap U_{\nu_X} \in H_n(X)$, with $h$ = Hurewicz map : $\pi_{n+k}(T(\nu_X)) \longrightarrow \dot{H}_{n+k}(T(\nu_X))$, $T(\nu_X)$ = Thom space of $\nu_X$, $U_{\nu_X}$ = Thom class $\in \dot{H}^k(T(\nu_X))$, $\dot{H}$ = reduced (co)homology.

A normal map of n-dimensional geometric Poincaré complexes

$$(f,b) \ : \ (M,\nu_M,\rho_M) \longrightarrow (X,\nu_X,\rho_X)$$

is a map $f:M \longrightarrow X$ of degree 1, $f_*[M] = [X] \in H_n(X)$, together with specified Spivak normal structures $(\nu_M,\rho_M)$, $(\nu_X,\rho_X)$ and a stable map of spherical fibrations $b:\nu_M \longrightarrow \nu_X$ covering f such that $T(b)_*(\rho_M) = \rho_X \in \pi^S_{n+k}(T(\nu_X))$. Such a normal map determines an n-dimensional quadratic Poincaré complex over $\mathbb{Z}[\pi_1(X)]$ $(C,\psi)$, and there is defined a quadratic signature normal map bordism invariant

$$\sigma_*(f,b) = (C,\psi) \in L_n(\pi_1(X))$$

such that

$$(1+T)\sigma_*(f,b) = \sigma^*(M) - \sigma^*(X) \in L^n(\pi_1(X)) \ .$$

Here, $C = C(f^!)$ is the algebraic mapping cone of the Umkehr $\mathbb{Z}[\pi_1(X)]$-module chain map

$$f^! \ : \ C(\widetilde{X}) \xrightarrow[\simeq]{([X]\cap -)^{-1}} C(\widetilde{X})^{n-*} \xrightarrow{\widetilde{f}^*} C(\widetilde{M})^{n-*} \xrightarrow[\simeq]{([M]\cap -)} C(\widetilde{M})$$

with $\widetilde{M}$ the cover of M induced from the universal cover $\widetilde{X}$ of X by f, and $\psi$ is defined as follows. Let $\nu_{\widetilde{M}}:\widetilde{M} \longrightarrow M \xrightarrow{\nu_M} BSG(k)$, $\nu_{\widetilde{X}}:\widetilde{X} \longrightarrow X \xrightarrow{\nu_X} BSG(k)$, so that b lifts to a stable map $\widetilde{b}:\nu_{\widetilde{M}} \longrightarrow \nu_{\widetilde{X}}$ covering $\widetilde{f}:\widetilde{M} \longrightarrow \widetilde{X}$. The induced map of Thom spaces $T(\widetilde{b}):T(\nu_{\widetilde{M}}) \longrightarrow T(\nu_{\widetilde{X}})$ has an equivariant S-dual stable $\pi_1(X)$-equivariant map $F:\Sigma^\infty \widetilde{X}_+ \longrightarrow \Sigma^\infty \widetilde{M}_+$ $(\widetilde{X}_+ = \widetilde{X} \cup \{pt.\})$ inducing $f^!:C(\widetilde{X}) \longrightarrow C(\widetilde{M})$ on the chain level, and such that $(\Sigma^\infty \widetilde{f})F \simeq 1:\Sigma^\infty \widetilde{X}_+ \longrightarrow \Sigma^\infty \widetilde{X}_+$. The evaluation of the composite

$$\psi_F \ : \ H_n(X) \xrightarrow{(\text{adjoint } F)_*} \dot{H}_n(\Omega^\infty \Sigma^\infty \widetilde{M}_+/\pi) \xrightarrow{\text{projection}} Q_n(C(\widetilde{M})) \xrightarrow{e_\%} Q_n(C(f^!))$$

on the fundamental class $[X] \in H_n(X)$ defines $\psi = \psi_F[X] \in Q_n(C(f^!))$, where $e_\%$ is induced by the natural projection $e:C(\widetilde{M}) \longrightarrow C(f^!)$ and $\pi = \pi_1(X)$. The standard map $\coprod_{k \geq 0} E\Sigma_k \times_{\Sigma_k} (\prod_k \widetilde{M})/\pi \longrightarrow \Omega^\infty \Sigma^\infty \widetilde{M}_+/\pi$ is a group completion in homology, so that $\dot{H}_n(\Omega^\infty \Sigma^\infty \widetilde{M}_+/\pi) = \mathbb{Z}[\mathbb{Z}] \otimes_{\mathbb{Z}[\mathbb{N}]} (\bigoplus_{k \geq 1} H_n(E\Sigma_k \times_{\Sigma_k} (\prod_k \widetilde{M})/\pi))$ contains $H_n(E\Sigma_2 \times_{\Sigma_2} (\widetilde{M} \times \widetilde{M})/\pi) = Q_n(C(\widetilde{M}))$ as a direct summand.

An n-dimensional normal map $(f,b):M \longrightarrow X$ in the sense of Browder and Wall is a degree 1 map $f:M \longrightarrow X$ from an n-dimensional manifold M to an n-dimensional geometric Poincaré complex X, together with a stable map $b:\nu_M \longrightarrow \nu_X$ of topological bundles covering f, where $\nu_M:M \longrightarrow BSTOP(k)$ is the normal bundle of an embedding $M \subset S^{n+k}$. The algebraic theory of surgery identifies the surgery obstruction of (f,b) with the quadratic signature of the underlying normal map of geometric Poincaré complexes $(f,Jb):(M,J\nu_M,\rho_M) \longrightarrow (X,J\nu_X,\rho_X)$

$$\theta(f,b) = \sigma_*(f,Jb) \in L_n(\pi_1(X)) .$$

An n-dimensional normal space is a triple

$$(X,\nu_X:X \longrightarrow BSG(k),\rho_X:S^{n+k} \longrightarrow T(\nu_X))$$

consisting of an n-dimensional finite CW complex X, an oriented spherical fibration $\nu_X$ and a map $\rho_X$. There are evident notions of normal pair, normal bordism, normal space n-ad. Given a normal space $(X,\nu_X,\rho_X)$ it is possible to construct a stable $\pi_1(X)$-equivariant map $G:\Sigma^\infty Z \longrightarrow \Sigma^\infty \widetilde{X}_+$ inducing $[X]\cap -:C(\widetilde{X})^{n-*} \longrightarrow C(\widetilde{X})$ on the chain level, with Z an equivariant S-dual of $T(\nu_{\widetilde{X}})$ and $[X] = h(\rho_X) \cap U_{\nu_X} \in H_n(X)$. The quadratic construction now applies to define a hyperquadratic signature normal bordism invariant

$$\hat{\sigma}^*(X) \in \hat{L}^n(\pi_1(X))$$

(where $\hat{\sigma}^*(X)$ is short for $\hat{\sigma}^*(X,\nu_X,\rho_X)$) such that $H\hat{\sigma}^*(X) = (C,\psi) \in L_{n-1}(\pi_1(X))$, with C the algebraic mapping cone of $[X]\cap -:C(\widetilde{X})^{n-*} \longrightarrow C(\widetilde{X})$. An n-dimensional geometric Poincare complex X is essentially the same as an n-dimensional normal space $(X,\nu_X,\rho_X)$ such that $[X]\cap -:C(\widetilde{X})^{n-*} \longrightarrow C(\widetilde{X})$ is a chain equivalence, in which case $(\nu_X,\rho_X)$ is a Spivak normal structure, $Z = \widetilde{X}_+$, $G = 1$ and

$$\hat{\sigma}^*(X) = J\sigma^*(X) \in \hat{L}^n(\pi_1(X)) \quad , \quad H\hat{\sigma}^*(X) = 0 \in L_{n-1}(\pi_1(X)) \ ,$$

If (X,Y) is an (n+1)-dimensional normal pair with Poincaré boundary Y there is defined a quadratic signature (normal,Poincaré)-bordism invariant

$$\sigma_*(X,Y) = (C,\psi) \in L_n(\pi_1(X))$$

such that C is the algebraic mapping cone of $[X]\cap -:C(\widetilde{X})^{n+1-*} \longrightarrow C(\widetilde{X},\widetilde{Y})$ and

$$(1+T)\sigma_*(X,Y) = \sigma^*(Y) \in L^n(\pi_1(X)) .$$

The mapping cylinder W of a normal map of n-dimensional geometric Poincaré complexes $(f,b):(M,\nu_M,\rho_M) \longrightarrow (X,\nu_X,\rho_X)$ defines an (n+1)-dimensional normal pair $(W, M\cup -X)$ with Poincaré boundary $M\cup -X$, such that

$$\sigma_*(W,M\cup -X) = \sigma_*(f,b) \in L_n(\pi_1(X)) \ .$$

The various signature maps fit together to define a natural transformation of exact sequences of abelian groups (for any space K)

$$\begin{array}{ccccccccc}
\dots \longrightarrow & \Omega^N_{n+1}(K) & \longrightarrow & \Omega^{N,P}_{n+1}(K) & \longrightarrow & \Omega^P_n(K) & \longrightarrow & \Omega^N_n(K) & \longrightarrow \dots \\
 & \downarrow{\scriptstyle\hat{\sigma}^*} & & \downarrow{\scriptstyle\sigma_*} & & \downarrow{\scriptstyle\sigma^*} & & \downarrow{\scriptstyle\hat{\sigma}^*} & \\
\dots \longrightarrow & \hat{L}^{n+1}(\pi_1(K)) & \xrightarrow{H} & L_n(\pi_1(K)) & \xrightarrow{1+T} & L^n(\pi_1(K)) & \xrightarrow{J} & \hat{L}^n(\pi_1(K)) & \longrightarrow \dots \ ,
\end{array}$$

with $\begin{cases} \Omega^P_*(K) \\ \Omega^N_*(K) \\ \Omega^{N,P}_*(K) \end{cases}$ the bordism groups of $\begin{cases} \text{geometric Poincaré complexes} \\ \text{normal spaces} \\ \text{(normal, Poincaré) pairs} \end{cases}$ mapping to K.

I should like to thank Frank Quinn for inventing normal spaces ([Q3]), and for suggesting that they should have a hyperquadratic invariant. Unfortunately, the results and constructions of [Q1],[Q2],[Q3] have not yet been fully documented. The theory announced here is independent of Quinn's (although evidently influenced by its philosophy), with the following two exceptions:

i) Normal space transversality: given a spherical fibration $\nu:K \longrightarrow BSG(k)$ over a finite CW complex K and a map $\rho:S^{n+k} \longrightarrow T(\nu)$ to the Thom space $T(\nu)$ it is possible to deform $\rho$ by a homotopy to a map (also called $\rho$ ) for which $X=\rho^{-1}(K)\subset S^{n+k}$ has the structure of an n-dimensional normal space $(X,\nu_X,\rho_X)$ with

$$\nu_X : X \xrightarrow{\rho|} K \xrightarrow{\nu} BSG(k) \quad , \quad \rho_X : S^{n+k} \xrightarrow{\text{collapse}} S^{n+k}/\overline{S^{n+k}-W} = W/\partial W \longrightarrow T(\nu_X)$$

for some closed regular neighbourhood W of X in $S^{n+k}$, and with

$$\rho : S^{n+k} \xrightarrow{\rho_X} T(\nu_X) \longrightarrow T(\nu) \ .$$

Along with the relative normal transversality for maps of n-ads. It follows that the maps

$$\Omega^N_n(K) \longrightarrow H_n(K;\underline{MSG}) \ ; \ (X,\nu_X,\rho_X) \longmapsto (S^{n+k} \xrightarrow{\rho_X} T(\nu_X) \xrightarrow{\Delta} X_+\wedge T(\nu_X) \longrightarrow K_+\wedge MSG(k))$$

are isomorphisms, by analogy with the Pontrjagin-Thom isomorphisms for smooth bordism $\Omega^{SO}_n(K) \xrightarrow{\simeq} H_n(K;\underline{MSO})$ obtained by smooth transversality. (I am indebted to Norman Levitt for an elementary handle exchange argument establishing normal space transversality).

ii) Poincaré surgery: in the starred discussion surrounding Theorem 4 below (and Theorem 4 itself) we shall make use of the geometric Poincaré surgery theory initiated by Browder [B3], and developed further by Levitt [Le], Jones [J1] and Quinn [Q3]. Some details of the theory still remain obscure, especially in the non-simply-connected case. The main result of this theory is an exact sequence

$$\ldots \longrightarrow L_n(\pi_1(K)) \longrightarrow \Omega_n^P(K) \longrightarrow \Omega_n^N(K) \xrightarrow{H\hat{\partial}*} L_{n-1}(\pi_1(K)) \longrightarrow \ldots ,$$

or equivalently that the quadratic signature maps $\sigma_*:\Omega_{n+1}^{N,P}(K) \longrightarrow L_n(\pi_1(K))$ are isomorphisms, for any space K. It is immediate from the Wall realization theorem for surgery obstructions that the quadratic signature maps are split surjective, so that $\Omega_{n+1}^{N,P}(K) = L_n(\pi_1(K)) \oplus ?$, but it it is not so easy to see that ? = 0 (although almost certainly true). In particular, <u>the proof of Theorem 1 makes no use of geometric Poincaré surgery</u>, relying instead on the algebraic Poincaré surgery of [R2]. Assuming ? = 0 it is in fact possible to give an alternative proof of Theorem 1 which makes no use of algebraic Poincaré surgery, relying instead on geometric Poincaré surgery. (Follow the same steps as in the proof below, but with the $\left\{\begin{array}{l}\text{geometric Poincaré}\\ \text{normal space}\end{array}\right.$ bordism spectrum $\left\{\begin{array}{l}\underline{\Omega}^P(K(\pi,1))\\ \underline{\Omega}^N(K(\pi,1))\end{array}\right.$ in place of the $\left\{\begin{array}{l}\text{symmetric}\\ \text{hyperquadratic}\end{array}\right.$ $\mathbb{L}$-spectrum $\left\{\begin{array}{l}\underline{\mathbb{L}}^0(\pi)\\ \underline{\hat{\mathbb{L}}}^0(\pi)\end{array}\right.$. If ? = 0 the quadratic signature map $\sigma_*:\underline{\Omega}^{N,P}(K(\pi,1)) \longrightarrow \Sigma\underline{\mathbb{L}}_0(\pi)_\S$ to the suspension of the 1-connective quadratic $\mathbb{L}$-spectrum is a homotopy equivalence).

The original simply-connected surgery theory of Browder and Novikov was reformulated in terms of classifying spaces for normal maps (such as G/O, G/PL, G/TOP) by Sullivan [Su1] and Casson, and the non-simply-connected surgery theory of Wall was reformulated in terms of geometric classifying spaces by Quinn [Q1], see Rourke [Ro] and §17A of Wall [W1]. We shall now outline an algebraic construction of surgery classifying spaces, leading to an algebraic formulation of surgery.

Given an abelian group G let $\underline{K}(G)$ be the $\Omega$-spectrum with kth term the Eilenberg-MacLane space K(G,k). Given a connective spectrum $\underline{R}$ let $\underline{R}_\S$ denote the 1-connective covering of $\underline{R}$, i.e. the fibre of the evident map $\underline{R} \longrightarrow \underline{K}(\pi_0(\underline{R}))$.

Let $\pi$ be a group. A $\begin{cases}\text{symmetric}\\ \text{quadratic}\end{cases}$ Poincaré n-ad over $\mathbb{Z}[\pi]$ is an n-ad of chain complexes of based f.g. free $\mathbb{Z}[\pi]$-modules, together with a simple $\begin{cases}\text{symmetric}\\ \text{quadratic}\end{cases}$ Poincaré duality. (See §0 of Wall [W1] for the general properties of n-ads). For example, an algebraic Poincaré 1-ad (resp. 2-ad) is the same as an algebraic Poincaré complex (resp. pair). The $\begin{cases}\text{symmetric}\\ \text{quadratic}\end{cases}$ Poincaré n-ads over $\mathbb{Z}[\pi]$ are the simplexes of $-(k+1)$-connected Kan complexes $\begin{cases}\mathbb{L}^k(\pi)\\ \mathbb{L}_k(\pi)\end{cases}$ $(k\in\mathbb{Z})$ such that

$$\begin{cases}\Omega\mathbb{L}^k(\pi) = \mathbb{L}^{k+1}(\pi) \ , \ \pi_n(\mathbb{L}^k(\pi)) = L^{n+k}(\pi)\\ \Omega\mathbb{L}_k(\pi) = \mathbb{L}_{k+1}(\pi) \ , \ \pi_n(\mathbb{L}_k(\pi)) = L_{n+k}(\pi)\end{cases} \quad (k\in\mathbb{Z}, n+k\geqslant 0) \ .$$

Thus $\begin{cases}\underline{\mathbb{L}}^0(\pi) = \{\mathbb{L}^{-k}(\pi)\,|\,k\geqslant 0\}\\ \underline{\mathbb{L}}_0(\pi) = \{\mathbb{L}_{-k}(\pi)\,|\,k\geqslant 0\}\end{cases}$ is a connective $\Omega$-spectrum such that

$$\begin{cases}\pi_n(\underline{\mathbb{L}}^0(\pi)) = L^n(\pi)\\ \pi_n(\underline{\mathbb{L}}_0(\pi)) = L_n(\pi)\end{cases} \quad (n\geqslant 0) \ .$$

The cofibre of the symmetrization map $1+T:\underline{\mathbb{L}}_0(\pi)_{\S}\longrightarrow\underline{\mathbb{L}}^0(\pi)$ is a connective $\Omega$-spectrum $\underline{\hat{\mathbb{L}}}^0(\pi) = \{\hat{\mathbb{L}}^{-k}(\pi)\,|\,k\geqslant 0\}$ such that

$$\pi_n(\underline{\hat{\mathbb{L}}}^0(\pi)) = \begin{cases}\hat{L}^n(\pi) & (n\geqslant 1)\\ L^0(\pi) & (n=0)\end{cases} ,$$

which fits into a fibration sequence of spectra

$$\underline{\mathbb{L}}_0(\pi)_{\S} \xrightarrow{1+T} \underline{\mathbb{L}}^0(\pi) \xrightarrow{J} \underline{\hat{\mathbb{L}}}^0(\pi) \xrightarrow{H} \Sigma\underline{\mathbb{L}}_0(\pi)_{\S} \xrightarrow{1+T} \Sigma\underline{\mathbb{L}}^0(\pi) \ .$$

The tensor product of chain complex n-ads defines pairings of $\mathbb{L}$-spectra

$$\otimes : \underline{\mathbb{L}}^0(\pi)\wedge\underline{\mathbb{L}}^0(\rho) \longrightarrow \underline{\mathbb{L}}^0(\pi\times\rho) \ , \ \otimes : \underline{\mathbb{L}}^0(\pi)\wedge\underline{\mathbb{L}}_0(\rho) \longrightarrow \underline{\mathbb{L}}_0(\pi\times\rho)$$
$$\otimes : \underline{\hat{\mathbb{L}}}^0(\pi)\wedge\underline{\hat{\mathbb{L}}}^0(\rho) \longrightarrow \underline{\hat{\mathbb{L}}}^0(\pi\times\rho)$$

for any groups $\pi,\rho$ . On the L-group level the tensor product of chain complexes defines pairings

$$\otimes : L^m(\pi)\otimes_{\mathbb{Z}}L^n(\rho) \longrightarrow L^{m+n}(\pi\times\rho) \ , \ \otimes : L^m(\pi)\otimes_{\mathbb{Z}}L_n(\rho) \longrightarrow L_{m+n}(\pi\times\rho)$$
$$\otimes : \hat{L}^m(\pi)\otimes_{\mathbb{Z}}\hat{L}^n(\rho) \longrightarrow \hat{L}^{m+n}(\pi\times\rho) \ .$$

We shall write $\begin{cases} \underline{\mathbb{L}}^0(1) = \underline{\mathbb{L}}^0 \\ \underline{\mathbb{L}}_0(1)_{\S} = \underline{\mathbb{L}}_0 \\ \hat{\underline{\mathbb{L}}}^0(1) = \hat{\underline{\mathbb{L}}}^0 \end{cases}$. Both $\underline{\mathbb{L}}^0$ and $\hat{\underline{\mathbb{L}}}^0$ are ring spectra; every algebraic $\underline{\mathbb{L}}$-spectrum above is an $\underline{\mathbb{L}}^0$-module spectrum. There is defined a commutative braid of fibration sequences of spectra

$$\begin{tikzcd}[column sep=small]
\underline{\mathbb{L}}_0 \arrow[rr, bend left, "1+T"] \arrow[dr, "1+T"] & & \underline{\mathbb{L}}^0 \arrow[dr, "J"] \arrow[rr, bend left] & & \underline{K}(L^0(1)) \arrow[dr] \arrow[rr, bend left] & & \Sigma\hat{\underline{\mathbb{L}}}^0_{\S} \\
& \underline{\mathbb{L}}^0_{\S} \arrow[ur] \arrow[dr, "J"] & & \hat{\underline{\mathbb{L}}}^0 \arrow[ur] \arrow[dr, "H"] & & \Sigma\underline{\mathbb{L}}^0_{\S} \arrow[ur, "J"] \arrow[dr] & \\
\Sigma^{-1}\underline{K}(L^0(1)) \arrow[ur] \arrow[rr, bend right] & & \hat{\underline{\mathbb{L}}}^0_{\S} \arrow[ur] \arrow[rr, bend right, "H"] & & \Sigma\underline{\mathbb{L}}_0 \arrow[ur, "1+T"] \arrow[rr, bend right, "1+T"] & & \Sigma\underline{\mathbb{L}}^0
\end{tikzcd}$$

Given a space K let $\begin{cases} \underline{\Omega}^P(K) \\ \underline{\Omega}^N(K) \end{cases}$ be the connective $\Omega$-spectrum of Kan complexes of maps $f: X \longrightarrow K$ from $\begin{cases} \text{geometric Poincaré} \\ \text{normal space} \end{cases}$ n-ads X to K such that $\begin{cases} \pi_n(\underline{\Omega}^P(K)) = \Omega_n^P(K) \\ \pi_n(\underline{\Omega}^N(K)) = \Omega_n^N(K) \end{cases}$ $(n \geqslant 0)$ is the nth $\begin{cases} \text{geometric Poincaré} \\ \text{normal space} \end{cases}$ bordism group of K.
The cofibre of the forgetful map $\underline{\Omega}^P(K) \longrightarrow \underline{\Omega}^N(K)$ is denoted by $\underline{\Omega}^{N,P}(K)$, so that $\pi_n(\underline{\Omega}^{N,P}(K)) = \Omega_n^{N,P}(K)$ $(n \geqslant 0)$ is the nth (normal, Poincaré) pair bordism group of K.
The cartesian product of topological n-ads defines pairings of spectra

$$\begin{cases} \otimes : \underline{\Omega}^P(K) \wedge \underline{\Omega}^P(L) \longrightarrow \underline{\Omega}^P(K \times L) \\ \otimes : \underline{\Omega}^N(K) \wedge \underline{\Omega}^N(L) \longrightarrow \underline{\Omega}^N(K \times L) \\ \otimes : \underline{\Omega}^P(K) \wedge \underline{\Omega}^{N,P}(L) \longrightarrow \underline{\Omega}^{N,P}(K \times L) \end{cases}$$

for any spaces K,L. We shall write $\underline{\Omega}^Q(pt.) = \underline{\Omega}^Q$ $(Q = P, N, (N,P))$. Let $\underline{K}_+$ be the suspension spectrum of $K_+ = K \cup \{pt.\}$, with kth term $\Sigma^k K_+ = S^k \wedge K_+$. A singular simplex in K is a particular example of a $\begin{cases} \text{geometric Poincaré} \\ \text{normal space} \end{cases}$ n-ad mapping to K, so there

is defined a forgetful map $\begin{cases}\sigma^*:\underline{K}_+ \longrightarrow \underline{\Omega}^P(K)\\ \hat{\sigma}^*:\underline{K}_+ \longrightarrow \underline{\Omega}^N(K)\end{cases}$. The composites

$$\begin{cases}\sigma^* : \underline{K}_+\wedge\underline{\Omega}^P \xrightarrow{\sigma^*\wedge 1} \underline{\Omega}^P(K)\wedge\underline{\Omega}^P \xrightarrow{\quad\otimes\quad} \underline{\Omega}^P(K)\\ \hat{\sigma}^* : \underline{K}_+\wedge\underline{\Omega}^N \xrightarrow{\hat{\sigma}^*\wedge 1} \underline{\Omega}^N(K)\wedge\underline{\Omega}^N \xrightarrow{\quad\otimes\quad} \underline{\Omega}^N(K)\\ \sigma_* : \underline{K}_+\wedge\underline{\Omega}^{N,P} \xrightarrow{\sigma^*\wedge 1} \underline{\Omega}^P(K)\wedge\underline{\Omega}^{N,P} \xrightarrow{\quad\otimes\quad} \underline{\Omega}^{N,P}(K)\end{cases}$$

induce the assembly maps appearing in the natural transformation of exact sequences

$$\begin{tikzcd}
\cdots \arrow[r] & H_{n+1}(K;\underline{\Omega}^{N,P}) \arrow[r] \arrow[d, "\sigma_*"] & H_n(K;\underline{\Omega}^P) \arrow[r] \arrow[d, "\sigma^*"] & H_n(K;\underline{\Omega}^N) \arrow[r] \arrow[d, "\hat{\sigma}^*"] & H_n(K;\underline{\Omega}^{N,P}) \arrow[r] \arrow[d, "\sigma_*"] & \cdots \\
\cdots \arrow[r] & \Omega^{N,P}_{n+1}(K) \arrow[r] & \Omega^P_n(K) \arrow[r] & \Omega^N_n(K) \arrow[r] & \Omega^{N,P}_n(K) \arrow[r] & \cdots
\end{tikzcd}$$

The assembly maps $\hat{\sigma}^*:H_n(K;\underline{\Omega}^N)\longrightarrow\Omega^N_n(K)$ are isomorphisms inverse to the natural maps $\Omega^N_n(K) \longrightarrow H_n(K;\underline{MSG}) = H_n(K;\underline{\Omega}^N)$, identifying $\underline{MSG} = \underline{\Omega}^N$ by normal transversality. (The Pontrjagin-Thom isomorphisms $H_n(K;\underline{MSO}) \xrightarrow{\sim} \Omega^{SO}_n(K)$ have a similar expression as assembly maps).

The chain complex of the universal cover $\widetilde{X}$ of a geometric Poincaré n-ad X defines a symmetric Poincaré n-ad over $\mathbb{Z}[\pi_1(|X|)]$ $(C(\widetilde{X}),\Delta[X])$, so there is defined a map of $\Omega$-spectra

$$\sigma^* : \underline{\Omega}^P(K) \longrightarrow \underline{\mathbb{L}}^0(\pi_1(K))$$

inducing the symmetric signature $\sigma^*:\Omega^P_n(K)\longrightarrow L^n(\pi_1(K))$ in the homotopy groups.

Similarly for $\begin{cases}\text{normal space}\\ \text{(normal,Poincaré) pair}\end{cases}$ n-ads, with a map of $\Omega$-spectra

$$\begin{cases}\hat{\sigma}^* : \underline{\Omega}^N(K) \longrightarrow \underline{\hat{\mathbb{L}}}^0(\pi_1(K))\\ \sigma_* : \underline{\Omega}^{N,P}(K) \longrightarrow \Sigma\underline{\mathbb{L}}_0(\pi_1(K))\end{cases}$$

inducing the $\begin{cases}\text{hyperquadratic}\\ \text{quadratic}\end{cases}$ signature $\begin{cases}\hat{\sigma}^*:\Omega^N_n(K)\longrightarrow L^n(\pi_1(K))\\ \sigma_*:\Omega^{N,P}_{n+1}(K)\longrightarrow L_n(\pi_1(K))\end{cases}$. The pairings $\otimes$ defined for the $\Omega$-spectra correspond to $\otimes$ for the $\mathbb{L}$-spectra. In particular, $\begin{cases}\sigma^*:\underline{\Omega}^P \longrightarrow \underline{\mathbb{L}}^0\\ \hat{\sigma}^*:\underline{\Omega}^N \longrightarrow \underline{\hat{\mathbb{L}}}^0\end{cases}$ is a morphism of ring spectra.

For an Eilenberg-MacLane space $K = K(\pi,1)$ the composite

$$\sigma^* : \underline{K}_+ \xrightarrow{\sigma^*} \underline{\Omega}^P(K) \xrightarrow{\sigma^*} \underline{\mathbb{L}}^0(\pi)$$

can be defined algebraically, using the standard simplicial model for $K(\pi,1)$. On the 1-skeleton $K(\pi,1)^{(1)} = \pi$ $\sigma^*$ sends $g \in \pi$ to the 1-dimensional symmetric Poincaré complex over $\mathbb{Z}[\pi]$ $\sigma^*(g) = (C, \varphi \in Q^1(C))$ defined by

$$\begin{array}{ccccc} C^{1-*} : & C^0 = \mathbb{Z}[\pi] & \xrightarrow{d^* = 1-g^{-1}} & C^1 = \mathbb{Z}[\pi] & \\ \varphi_0 \big\downarrow & \varphi_0 = 1 \big\downarrow & \varphi_1 = 1 \swarrow & \big\downarrow \varphi_0 = -g & \\ C : & C_1 = \mathbb{Z}[\pi] & \xrightarrow{d = 1-g} & C_0 = \mathbb{Z}[\pi] & . \end{array}$$

This is the symmetric Poincaré complex corresponding to the simple automorphism $g : (\mathbb{Z}[\pi],1) \longrightarrow (\mathbb{Z}[\pi],1)$ of the non-singular symmetric form over $\mathbb{Z}[\pi]$ $(\mathbb{Z}[\pi],1)$. For the generator $g \in \pi = \mathbb{Z}$ $\sigma^*(g)$ is just the symmetric Poincaré complex $\sigma^*(S^1)$ of $K(\mathbb{Z},1) = S^1$.

Given a space $X$ use the composite

$$\sigma^* : \underline{X}_+ \longrightarrow \underline{K}(\pi_1(X,1)_+ \xrightarrow{\sigma^*} \underline{\mathbb{L}}^0(\pi_1(X))$$

(which is also the composite $\sigma^* : \underline{X}_+ \xrightarrow{\sigma^*} \underline{\Omega}^P(X) \xrightarrow{\sigma^*} \underline{\mathbb{L}}^0(\pi_1(X))$) to define assembly maps of spectra

$$\begin{aligned} \sigma^* &: \underline{X}_+ \wedge \underline{\mathbb{L}}^0 \xrightarrow{\sigma^* \wedge 1} \underline{\mathbb{L}}^0(\pi_1(X)) \wedge \underline{\mathbb{L}}^0 \xrightarrow{\otimes} \underline{\mathbb{L}}^0(\pi_1(X)) \\ \sigma_* &: \underline{X}_+ \wedge \underline{\mathbb{L}}_0 \xrightarrow{\sigma^* \wedge 1} \underline{\mathbb{L}}^0(\pi_1(X)) \wedge \underline{\mathbb{L}}_0 \xrightarrow{\otimes} \underline{\mathbb{L}}_0(\pi_1(X))_{\S} \\ \hat{\sigma}^* &: \underline{X}_+ \wedge \underline{\hat{\mathbb{L}}}^0 \xrightarrow{\sigma^* \wedge 1} \underline{\mathbb{L}}^0(\pi_1(X)) \wedge \underline{\hat{\mathbb{L}}}^0 \xrightarrow{\otimes} \underline{\hat{\mathbb{L}}}^0(\pi_1(X)) \quad , \end{aligned}$$

and hence a natural transformation of exact sequences of abelian groups

$$\begin{array}{ccccccccccc} \cdots \longrightarrow & H_n(X;\underline{\mathbb{L}}_0) & \xrightarrow{1+T} & H_n(X;\underline{\mathbb{L}}^0) & \xrightarrow{J} & H_n(X;\underline{\hat{\mathbb{L}}}^0) & \xrightarrow{H} & H_{n-1}(X;\underline{\mathbb{L}}_0) & \longrightarrow \cdots \\ & \big\downarrow \sigma_* & & \big\downarrow \sigma^* & & \big\downarrow \hat{\sigma}^* & & \big\downarrow \sigma_* & \\ \cdots \longrightarrow & L_n(\pi_1(X)) & \xrightarrow{1+T} & L^n(\pi_1(X)) & \xrightarrow{J} & \hat{L}^n(\pi_1(X)) & \xrightarrow{H} & L_{n-1}(\pi_1(X)) & \longrightarrow \cdots . \end{array}$$

Define the quadratic $\mathcal{S}$-groups $\mathcal{S}_*(X)$ of a space $X$ by

$$\mathcal{S}_n(X) = \pi_n(\sigma_* : X_+ \wedge \underline{\mathbb{L}}_0 \longrightarrow \underline{\mathbb{L}}_0(\pi_1(X))_{\S}) \quad ,$$

to fit into an exact sequence of abelian groups

$$\cdots \longrightarrow H_n(X;\underline{\mathbb{L}}_0) \xrightarrow{\sigma_*} L_n(\pi_1(X)) \longrightarrow \mathcal{S}_n(X) \longrightarrow H_{n-1}(X;\underline{\mathbb{L}}_0) \longrightarrow \cdots .$$

The construction of the algebraic assembly maps $\sigma_*$ and of the groups $\mathcal{S}_*(X)$ was motivated by Quinn's analysis of the surgery exact sequence in terms of geometric assembly maps ([Q1],[Q2]), and by the higher Whitehead groups $Wh_*(X)$ of Waldhausen [Wa]. Loday [Lo] has obtained similar maps in the context of Karoubi's hermitian K-theory, and also in algebraic K-theory. The maps $\sigma_*$ are L-theoretic analogues of the maps $H_*(X;\underline{\underline{K}}(\mathbb{Z})) \longrightarrow K_*(\mathbb{Z}[\pi_1(X)])$ used to define $Wh_*(X)$ to fit into an exact sequence

$$\ldots \longrightarrow H_n(X;\underline{\underline{K}}(\mathbb{Z})) \longrightarrow K_n(\mathbb{Z}[\pi_1(X)]) \longrightarrow Wh_n(X) \longrightarrow H_{n-1}(X;\underline{\underline{K}}(\mathbb{Z})) \longrightarrow \ldots ,$$

with $\underline{\underline{K}}(\mathbb{Z})$ the spectrum of the algebraic K-theory of $\mathbb{Z}$, $\pi_*(\underline{\underline{K}}(\mathbb{Z})) = K_*(\mathbb{Z})$.
For example, $Wh_1(K(\pi,1)) = Wh(\pi)$, $Wh_0(K(\pi,1)) = \tilde{K}_0(\mathbb{Z}[\pi])$. The groups $\mathcal{S}_*(X)$ are thus L-theoretic analogues of $Wh_*(X)$.

Transversality in the $\begin{Bmatrix}\text{topological}\\ \text{normal}\end{Bmatrix}$ category allows us to replace the Thom spectrum $\begin{Bmatrix}\underline{\mathrm{MSTOP}}\\ \underline{\mathrm{MSG}}\end{Bmatrix}$ by the homotopy equivalent $\Omega$-spectrum $\begin{Bmatrix}\underline{\Omega}^{STOP}\\ \underline{\Omega}^{N}\end{Bmatrix}$ of Kan complexes of $\begin{Bmatrix}\text{topological manifold}\\ \text{normal space}\end{Bmatrix}$ n-ads. (It may be objected that we have ignored the absence of topological transversality in dimension 4, but there is at least enough of it to define a forgetful map $\underline{\mathrm{MSTOP}} \longrightarrow \underline{\Omega}^P$, which is all we need. See Scharlemann [Sch]).
Let $\underline{\mathrm{MS}(G/TOP)}$ be the fibre of the forgetful map $\underline{\mathrm{MSTOP}} \longrightarrow \underline{\mathrm{MSG}}$, the spectrum with kth space $\mathrm{MS}(G(k)/TOP(k))$, the homotopy-theoretic fibre of $\mathrm{MSTOP}(k) \longrightarrow \mathrm{MSG}(k)$.
Then $\Sigma\underline{\mathrm{MS}(G/TOP)}$ is homotopy equivalent to $\underline{\Omega}^{N,STOP}$, the cofibre of $\underline{\Omega}^{STOP} \longrightarrow \underline{\Omega}^N$.

The $\begin{Bmatrix}\text{symmetric}\\ \text{hyperquadratic}\\ \text{quadratic}\end{Bmatrix}$ signature map $\begin{Bmatrix}\sigma^*: \Omega_n^{STOP}(K) \longrightarrow L^n(\pi_1(K))\\ \hat{\sigma}^*: \Omega_n^N(K) \longrightarrow \hat{L}^n(\pi_1(K))\\ \sigma_*: \Omega_{n+1}^{N,STOP}(K) \longrightarrow L_n(\pi_1(K))\end{Bmatrix}$ factors through the algebraic assembly map

$$\begin{cases}\sigma^* : \Omega_n^{STOP}(K) = H_n(K;\underline{\mathrm{MSTOP}}) \xrightarrow{\sigma^*} H_n(K;\underline{\underline{L}}^0) \xrightarrow{\sigma^*} L^n(\pi_1(K))\\ \hat{\sigma}^* : \Omega_n^N(K) = H_n(K;\underline{\mathrm{MSG}}) \xrightarrow{\hat{\sigma}^*} H_n(K;\hat{\underline{\underline{L}}}^0) \xrightarrow{\hat{\sigma}^*} \hat{L}^n(\pi_1(K))\\ \sigma_* : \Omega_{n+1}^{N,STOP}(K) = H_n(K;\underline{\mathrm{MS}(G/TOP)}) \xrightarrow{\sigma_*} H_n(K;\underline{\underline{L}}_0) \xrightarrow{\sigma_*} L_n(\pi_1(K))\end{cases} .$$

(These factorizations can be interpreted in terms of characteristic numbers, in particular for the surgery obstructions of normal maps of manifolds, which can then be used to determine the homotopy types of the $\mathbb{L}$-spaces, following the work of Sullivan [Su1] and Morgan and Sullivan [MS] in the simply-connected case. See Wall [W3], Jones [J2], Taylor and Williams [TaW] for generalizations to the non-simply-connected case. In [TaW] it is shown that the algebraic $\mathbb{L}$-spectra become generalized Eilenberg-MacLane spectra localized at 2, and wedges of $\underline{bo}$-coefficient spectra localized away from 2).

Given a ring $\Omega$-spectrum $\underline{R} = \{R_k = \Omega R_{k+1}, \otimes : R_j \wedge R_k \longrightarrow R_{j+k}, 1_k : S^k \longrightarrow R_k\}$ let $B\underline{R}G$ be the classifying space for stable $\underline{R}$-oriented spherical fibrations over finite CW complexes, and let $R_\otimes$ be the component of $1 \in \pi_0(\underline{R})$ in $R_0$. If $\pi_0(\underline{R}) = \mathbb{Z}$ the morphism $\underline{R} \longrightarrow \underline{K}(\mathbb{Z})$ induces a forgetful map $B\underline{R}G \longrightarrow B\underline{K}(\mathbb{Z})G = BSG$, and there is defined a fibration sequence of spaces

$$R_\otimes \longrightarrow B\underline{R}G \longrightarrow BSG \ .$$

In particular, we have defined a commutative braid of fibration sequences

$$\begin{tikzcd}[column sep=small]
\mathbb{L}_0 \arrow[dr, "1+T"] \arrow[rr, bend left] & & B\underline{\mathbb{L}}^0G \arrow[dr, "J"] \arrow[rr, bend left] & & BSG \\
& \mathbb{L}^0_\otimes \arrow[ur] \arrow[dr, "J"] & & B\underline{\hat{\mathbb{L}}}^0G \arrow[ur] & \\
& & \hat{\mathbb{L}}^0_\otimes \arrow[ur] & &
\end{tikzcd}$$

with $\mathbb{L}_0$ the 0th term of $\underline{\mathbb{L}}_0 = \underline{\mathbb{L}}_0(1)_S$, i.e. the connected Kan complex of quadratic Poincaré n-ads over $\mathbb{Z}$ such that $\pi_n(\mathbb{L}_0) = L_n(1)$ $(n \geqslant 1)$.

We have defined a commutative square of ring spectra

$$\begin{tikzcd}
\underline{MSTOP} \arrow[r, "\sigma^*"] \arrow[d, "J"'] & \underline{\mathbb{L}}^0 \arrow[d, "J"] \\
\underline{MSG} \arrow[r, "\hat{\sigma}^*"] & \underline{\hat{\mathbb{L}}}^0
\end{tikzcd} \ .$$

An oriented $\left\{\begin{array}{l}\text{topological bundle } \alpha : K \longrightarrow BSTOP(k) \\ \text{spherical fibration } \beta : K \longrightarrow BSG(k)\end{array}\right.$ over a finite CW complex K has a canonical $\left\{\begin{array}{l}\underline{MSTOP}\text{-} \\ \underline{MSG}\text{-}\end{array}\right.$ orientation $\left\{\begin{array}{l}U(\alpha) \in \dot{H}^k(T(\alpha);\underline{MSTOP}) \\ \hat{U}(\beta) \in \dot{H}^k(T(\beta);\underline{MSG})\end{array}\right.$, and hence also a canonical

$\left\{\begin{matrix}\underline{\mathbb{L}}^0\text{-} \\ \hat{\underline{\mathbb{L}}}^0\text{-}\end{matrix}\right.$ orientation $\left\{\begin{matrix}\sigma^*U(\alpha)\in \dot{H}^k(T(\alpha);\underline{\mathbb{L}}^0) \\ \hat{\sigma}^*\hat{U}(\beta)\in \dot{H}^k(T(\beta);\hat{\underline{\mathbb{L}}}^0)\end{matrix}\right.$. There is induced a morphism of fibrations

$$\begin{tikzcd} G/TOP \arrow[r] \arrow[d, "\sigma_*"'] & BSTOP \arrow[r, "J"] \arrow[d, "\sigma^*U"'] & BSG \arrow[d, "\hat{\sigma}^*\hat{U}"] \\ \mathbb{L}_0 \arrow[r] & B\underline{\mathbb{L}}^0G \arrow[r, "J"] & B\hat{\underline{\mathbb{L}}}^0G \end{tikzcd}$$

with $\sigma_*:G/TOP \longrightarrow \mathbb{L}_0$ the map associating to each singular simplex $\Delta \longrightarrow G/TOP$ the quadratic Poincaré n-ad $\sigma_*(f,b)$ over $\mathbb{Z}$ of the normal map of manifold n-ads $(f,b):M \longrightarrow \Delta$ that it classifies. Now $\sigma_*:G/TOP \longrightarrow \mathbb{L}_0$ induces the surgery obstruction isomorphisms

$$\sigma_* = \theta : \pi_*(G/TOP) \longrightarrow \pi_*(\mathbb{L}_0) = L_*(1) \quad ,$$

so that it is a homotopy equivalence by J.H.C.Whitehead's theorem. The right hand square is thus a homotopy-theoretic pullback, and for any spherical fibration $\beta:K \longrightarrow BSG(k)$ there is an identification of sets of equivalence classes

$\{$stable topological reductions $\tilde{\beta}:K \longrightarrow BSTOP$ of $\beta:K \longrightarrow BSG(k)\}$

$= \{$pairs $(V,h)$ consisting of a map $V:T(\beta) \longrightarrow \mathbb{L}^{-k}$ and a homotopy

$h : JV \simeq \hat{V} : T(\beta) \longrightarrow \hat{\mathbb{L}}^{-k}\}$

for some fixed map $\hat{V}:T(\beta) \longrightarrow \hat{\mathbb{L}}^{-k}$ representing the canonical $\underline{\mathbb{L}}^0$- orientation $\hat{\sigma}^*\hat{U}(\beta)\in \dot{H}^k(T(\beta);\hat{\underline{\mathbb{L}}}^0) = [T(\beta),\hat{\mathbb{L}}^{-k}]$. We thus have an equivalence of categories

$\{$stable oriented topological bundles (over finite CW complexes)$\}$

$\approx \{$stable spherical fibrations with an $\underline{\mathbb{L}}^0$- orientation lifting the canonical $\hat{\underline{\mathbb{L}}}^0$- orientation$\}$.

Localizing away from 2 we have the Sullivan [Su2] characterization of stable topological bundles as $KO[\frac{1}{2}]$-oriented spherical fibrations, with

$$\mathbb{L}_0[\tfrac{1}{2}] = BO[\tfrac{1}{2}] \quad , \quad \underline{\mathbb{L}}^0[\tfrac{1}{2}] = \underline{bo}[\tfrac{1}{2}] \quad , \quad \hat{\underline{\mathbb{L}}}^0[\tfrac{1}{2}] = \underline{K}(\mathbb{Z})[\tfrac{1}{2}] \quad .$$

I should like to thank Graeme Segal and Frank Quinn for discussions pertaining to the L-theoretic characterization of topological bundles. (It is in fact equivalent to the Levitt-Morgan-Brumfiel characterization of stable topological bundles as spherical fibrations with geoemtric Poincaré transversality [LeM],[BM]. Unstably, the result $G(k)/\widetilde{TOP(k)} = G/TOP$ $(k\geqslant 3)$ of Rourke and Sanderson [RS] applies to show that there is an equivalence of categories

$\{$oriented topological k-block bundles (over finite CW complexes)$\}$

$\approx \{$(k-1)-spherical fibrations with an $\underline{\mathbb{L}}^0$-orientation lifting the canonical $\hat{\underline{\mathbb{L}}}^0$- orientation$\}$ .

The homotopy equivalence $\sigma_*: G/TOP \longrightarrow \mathbb{L}_0$ is not an H-map from the H-space structure on G/TOP defined by the Whitney sum of bundles to the H-space structure on $\mathbb{L}_0$ defined by the direct sum of quadratic Poincaré n-ads. The latter is equivalent to the Quinn disjoint union of surgery problems addition, and also to the Sullivan characteristic variety addition in G/TOP. The former is expressed in terms of the latter by $(a,b) \longmapsto a \oplus b \oplus (a \otimes b)$. Madsen and Milgram [MM] show that there exists no (2-local) homotopy equivalence $B(G/TOP) \longrightarrow \mathbb{L}_{-1}$ extending the above diagram to the right by a commutative square

$$\begin{array}{ccc} BSG & \longrightarrow & B(G/TOP) \\ {\scriptstyle \hat{\sigma}*\hat{U}} \downarrow & & \downarrow \\ B\hat{\underline{\mathbb{L}}}^0 G & \longrightarrow & \mathbb{L}_{-1} \end{array} .$$

Here, $\mathbb{L}_{-1}$ is the 1st term of $\underline{\mathbb{L}}_0$, the delooping of $\mathbb{L}_0$ defined by the universal cover of the connected Kan complex $\mathbb{L}_{-1}(1)$ of quadratic Poincaré n-ads over $\mathbb{Z}$ such that $\pi_n(\mathbb{L}_{-1}(1)) = L_{n-1}(1)$ $(n \geqslant 1)$. Localizing at 2 we have

$$\mathbb{L}_0(1)_{(2)} = \prod_{i=0}^{\infty}(K(\mathbb{Z}_{(2)},4i) \times K(\mathbb{Z}_2,4i+2)) \ , \ \mathbb{L}_{-1}(1)_{(2)} = \prod_{i=0}^{\infty}(K(\mathbb{Z}_{(2)},4i+1) \times K(\mathbb{Z}_2,4i+3))$$

$$\underline{\mathbb{L}}^0{}_{(2)} = \prod_{i=0}^{\infty}(\Sigma^{4i}\underline{K}(\mathbb{Z}_{(2)}) \times \Sigma^{4i+1}\underline{K}(\mathbb{Z}_2)) \ ,$$

$$\hat{\underline{\mathbb{L}}}^0{}_{(2)} = \underline{K}(\mathbb{Z}_{(2)}) \times \prod_{i=0}^{\infty}(\Sigma^{4i+1}\underline{K}(\mathbb{Z}_2) \times \Sigma^{4i+3}\underline{K}(\mathbb{Z}_2) \times \Sigma^{4i+4}\underline{K}(\mathbb{Z}_8)) \ ).$$

Given an oriented spherical fibration $\beta: K \longrightarrow BSG(k)$ over a finite CW complex K define

$$t(\beta) = H\hat{\sigma}*\hat{U}(\beta) \in \dot{H}^{k+1}(T(\beta);\underline{\mathbb{L}}_0) \ ,$$

the image of the canonical $\hat{\underline{\mathbb{L}}}^0$- orientation $\hat{\sigma}*\hat{U}(\beta) \in \dot{H}^k(T(\beta);\hat{\underline{\mathbb{L}}}^0)$ under the map H appearing in the exact sequence

$$\ldots \longrightarrow \dot{H}^k(T(\beta);\underline{\mathbb{L}}_0) \xrightarrow{1+T} \dot{H}^k(T(\beta);\underline{\mathbb{L}}^0) \xrightarrow{J} \dot{H}^k(T(\beta);\hat{\underline{\mathbb{L}}}^0) \xrightarrow{H} \dot{H}^{k+1}(T(\beta);\underline{\mathbb{L}}_0) \rightarrow \ldots \ .$$

By the above, $\beta$ admits a stable topological reduction $\tilde{\beta}: K \longrightarrow BSTOP$ if and

only if $t(\beta) = 0$. (We have that $t(\beta)$ is a torsion element, and that

$$\underline{\mathbb{L}}_0[\tfrac{1}{2}] = \underline{\mathrm{bso}}[\tfrac{1}{2}] \quad , \quad \underline{\mathbb{L}}_{0(2)} = \prod_{i=0}^{\infty}(\Sigma^{4i+2}\underline{K}(\mathbb{Z}_2)\times\Sigma^{4i+4}\underline{K}(\mathbb{Z}_{(2)})) \ .$$

Localized at 2 $t(\beta)$ can be expressed as a stable characteristic class

$$t(\beta)_{(2)} \in \prod_{i=1}^{\infty} H^{4i-1}(K;\mathbb{Z}_2)\oplus \mathrm{im}(H^{4i}(K;\mathbb{Z}_8)\longrightarrow H^{4i+1}(K;\mathbb{Z}_{(2)})) \ .$$

Away from 2 $t(\beta)$ is the obstruction to a $KO[\frac{1}{2}]$-orientation of $\beta$

$$t(\beta)[\tfrac{1}{2}] = \widetilde{KSO}^{k+1}(T(\beta))[\tfrac{1}{2}] \quad ).$$

Given an n-dimensional geometric Poincaré complex X let $\mathcal{T}^{TOP}(X)$ be the topological normal map bordism set of X, defined as usual to be the set of equivalence classes of normal maps $(f,b):M\longrightarrow X$ in the sense of Browder and Wall, under the relation

$(f,b)\sim(f',b')$ if there exists a normal map

$$((g;f,f'),(c;b,b')) \ : \ (N;M,M')\longrightarrow (X\times I;X\times 0,X\times 1) \ .$$

The surgery obstruction function

$$\theta \ : \mathcal{T}^{TOP}(X)\longrightarrow L_n(\pi_1(X)) \ ; \ (f,b)\longmapsto \sigma_*(f,Jb)$$

fits into the Sullivan-Wall surgery exact sequence of sets

$$L_{n+1}(\pi_1(X))\longrightarrow \mathcal{S}^{TOP}(X)\longrightarrow \mathcal{T}^{TOP}(X)\xrightarrow{\ \theta\ } L_n(\pi_1(X)) \ .$$

In the case $\mathcal{T}^{TOP}(X)\neq\emptyset$ (i.e. if the Spivak normal fibration $\nu_X:X\longrightarrow BSG$ admits a topological reduction) we shall express $\theta$ in terms of the assembly map $\sigma_*:H_n(X;\underline{\mathbb{L}}_0)\longrightarrow L_n(\pi_1(X))$.

Let $G(k)/TOP(k)$ denote the homotopy-theoretic fibre of the forgetful map $J:BSTOP(k)\longrightarrow BSG(k)$, as usual, and let $MS(G(k)/TOP(k))$ be the homotopy-theoretic fibre of the forgetful map of Thom spaces $J:MSTOP(k)\longrightarrow MSG(k)$ $(k\geqslant 0)$. The canonical topological bundle $\eta_k:G(k)/TOP(k)\longrightarrow BSTOP(k)$ has a canonical fibre homotopy trivialization $h_k:J\eta_k\simeq J\varepsilon^k:G(k)/TOP(k)\longrightarrow BSG(k)$. The canonical $\underline{MSTOP}$-orientation $U(\eta_k)\in \dot{H}^k(T(\eta_k);\underline{MSTOP})$ is represented by the induced map of Thom spaces

$$U(\eta_k) \ : \ T(\eta_k) = \Sigma^k(G(k)/TOP(k))_+\longrightarrow MSTOP(k) \ ,$$

using $h_k$ to identify $T(\eta_k) = T(\varepsilon^k) = \Sigma^k(G(k)/TOP(k))_+$. The canonical

$\underline{\mathrm{MSTOP}}$-orientation $U(\varepsilon^k) \in \dot{H}^k(T(\varepsilon^k);\underline{\mathrm{MSTOP}})$ of the trivial topological bundle $\varepsilon^k : G(k)/TOP(k) \longrightarrow BSTOP(k)$ is represented by the composite

$$U(\varepsilon^k) \ : \ T(\varepsilon^k) = \Sigma^k(G(k)/TOP(k))_+ \xrightarrow{\text{collapse}} \Sigma^k(S^0) = S^k \xrightarrow{1_k} MSTOP(k) \ .$$

The fibre homotopy $h_k : J\eta_k \simeq J\varepsilon^k : G(k)/TOP(k) \longrightarrow BSG(k)$ determines a homotopy

$$T(h_k) \ : \ JU(\eta_k) \simeq JU(\varepsilon^k) \ : \ \Sigma^k(G(k)/TOP(k))_+ \longrightarrow MSG(k) \ ,$$

and hence a map

$$\Gamma_k \ : \ G(k)/TOP(k) \longrightarrow \Omega^k MS(G(k)/TOP(k))$$

such that

$$\text{adjoint}\, U(\eta_k) - \text{adjoint}\, U(\varepsilon^k) \ : \ G(k)/TOP(k) \xrightarrow{\Gamma_k} \Omega^k MS(G(k)/TOP(k)) \longrightarrow \Omega^k MSTOP(k)$$

(up to homotopy). The maps $\Gamma_k$ $(k \geqslant 0)$ fit together to define a map

$$\Gamma = \varinjlim_k \Gamma_k : G/TOP = \varinjlim_k G(k)/TOP(k) \longrightarrow \Omega^\infty MS(G/TOP) = \varinjlim_k \Omega^k MS(G(k)/TOP(k)) \ .$$

Now $\Omega^\infty MS(G/TOP)$ is the infinite loop space corresponding to the (normal,manifold) bordism spectrum with a dimension shift, $\underline{MS(G/TOP)} = \Sigma^{-1}\underline{\Omega}^{N,STOP}$, and so can be regarded as a Kan complex of (normal,manifold)-pair n-ads. The quadratic signature of such n-ads defines a map

$$\sigma_* \ : \Omega^\infty MS(G/TOP) \longrightarrow \mathbb{L}_0 \ .$$

The map $\Gamma : G/TOP \longrightarrow \Omega^\infty MS(G/TOP)$ sends a singular simplex in G/TOP to the mapping cylinder of the normal map of manifold n-ads that it classifies. The composite

$$\sigma_* \ : \ G/TOP \xrightarrow{\Gamma} \Omega^\infty MS(G/TOP) \xrightarrow{\sigma_*} \mathbb{L}_0$$

is the homotopy equivalence defined previously.

Let X be an n-dimensional geometric Poincaré complex, and let

$$(\nu_X : X \longrightarrow BSG(k), \rho_X : S^{n+k} \longrightarrow T(\nu_X))$$

be a Spivak normal structure. The composite

$$\alpha_X \ : \ S^{n+k} \xrightarrow{\rho_X} T(\nu_X) \xrightarrow{\Delta} X_+ \wedge T(\nu_X)$$

is an S-duality map between $X_+$ and $T(\nu_X)$, so that for any spectrum $\underline{R} = \{R_k, \Sigma R_k \longrightarrow R_{k+1}\}$ there are defined isomorphisms

$$\alpha_X : \dot{H}^*(T(\nu_X);\underline{R}) = \varinjlim_j [\Sigma^j T(\nu_X), R_{j+*}] \xrightarrow{\sim} H_{n+k-*}(X;\underline{R}) = \varinjlim_j \pi_{n+j+k-*}(X_+ \wedge R_j) \; ;$$
$$\{g_j : \Sigma^j T(\nu_X) \longrightarrow R_{j+*}\} \longmapsto \{S^{n+j+k} \xrightarrow{\Sigma^j \alpha_X} X_+ \wedge \Sigma^j T(\nu_X) \xrightarrow{1\wedge g_j} X_+ \wedge R_{j+*}\} \; .$$

Any two Spivak normal structures on X $(\nu_X,\rho_X)$, $(\nu'_X,\rho'_X)$ are related by a stable fibre homotopy equivalence $c:\nu_X \longrightarrow \nu'_X$ over $1:X \longrightarrow X$ such that $T(c)_*(\rho_X) = \rho'_X \in \pi^S_{n+k'}(T(\nu'_X))$, and any two such fibre homotopy equivalences are related by a stable fibre homotopy. The Browder-Novikov transversality construction of normal maps identifies

$\mathcal{T}^{TOP}(X)$ = the set of equivalence classes of topological normal structures

$$(\nu_X : X \longrightarrow BSTOP(k), \rho_X : S^{n+k} \longrightarrow T(\nu_X)) \; .$$

Thus if $\mathcal{T}^{TOP}(X) \neq \emptyset$ and $x_0 = ((f_0,b_0):M_0 \longrightarrow X) \in \mathcal{T}^{TOP}(X)$ is the normal map bordism class associated to some topological normal structure $(\nu_0 : X \longrightarrow BSTOP(k_0)$, $\rho_0 : S^{n+k_0} \longrightarrow T(\nu_0))$ we have the usual bijections (depending on $x_0$)

$\mathcal{T}^{TOP}(X) \approx$ the set of equivalence classes of stable topological reductions

$$\widetilde{\nu}_0 : X \longrightarrow BSTOP \text{ of } J\nu_0 : X \longrightarrow BSG(k_0) \; ,$$

and

$$x_0 : \mathcal{T}^{TOP}(X) \xrightarrow{\sim} [X, G/TOP] \; ; \; ((f_1,b_1):M_1 \longrightarrow X) \longmapsto (\nu_1 - \nu_0, c) \; ,$$

with $(\nu_1 : X \longrightarrow BSTOP(k_1), \rho_1 : S^{n+k_1} \longrightarrow T(\nu_1))$ a topological normal structure associated to $(f_1,b_1) \in \mathcal{T}^{TOP}(X)$. Let $\alpha_0 : S^{n+k_0} \xrightarrow{\rho_0} T(\nu_0) \xrightarrow{\Delta} X_+ \wedge T(\nu_0)$ be the S-duality map determined by $(\nu_0,\rho_0)$. The image of the canonical $\underline{MSTOP}$-orientation $U(\nu_0) \in \dot{H}^k(T(\nu_0);\underline{MSTOP})$ under the S-duality isomorphism

$$\alpha_0 : \dot{H}^{k_0}(T(\nu_0);\underline{MSTOP}) \xrightarrow{\sim} H_n(X;\underline{MSTOP}) = \Omega_n^{STOP}(X)$$

is the $\underline{MSTOP}$-orientation $[X]_0 = (M_0,f_0) \in \Omega_n^{STOP}(X)$ of X determined by $(f_0,b_0) \in \mathcal{T}^{TOP}(X)$. For any $\underline{MSTOP}$-module spectrum $\underline{R} = \{R_j, \Sigma R_j \longrightarrow R_{j+1}, \otimes : MSTOP(j) \wedge R_k \longrightarrow R_{j+k}\}$ there is defined an $\underline{R}$-coefficient Thom isomorphism

$$- \cup U(\nu_0) : H^0(X;\underline{R}) \xrightarrow{\sim} \dot{H}^{k_0}(T(\nu_0);\underline{R}) \; ;$$
$$\{g_j : \Sigma^j X_+ \longrightarrow R_j\} \longmapsto \{\Sigma^j T(\nu_0) \xrightarrow{\Delta} T(\nu_0) \wedge \Sigma^j X_+ \xrightarrow{U(\nu_0)\wedge g_j} MSTOP(k_0) \wedge R_j \xrightarrow{\otimes} R_{j+k_0}\} \; ,$$

so that the composite

$$[X]_0 \cap - : H^0(X;\underline{R}) \xrightarrow[\sim]{U(\nu_0)\cup -} \dot{H}^{k_0}(T(\nu_0);\underline{R}) \xrightarrow[\sim]{\alpha_0} H_n(X;\underline{R})$$

is an $\underline{R}$-coefficient Poincaré duality isomorphism. (This point of view derives from G.W.Whitehead's treatment of orientability with respect to extraordinary (co)homology theories, and from Atiyah's reformulation of Thom's smooth cobordism theory in terms of $\underline{MSO}$-orientations). In particular, $\underline{MSTOP}$ and $\underline{MS(G/TOP)}$ are $\underline{MSTOP}$-module spectra. Let $\Phi : G/TOP \longrightarrow \Omega^{\infty}MSTOP = \varinjlim_{k} \Omega^{k}MSTOP(k)$ be the map which restricts to the adjoints $(G(k)/TOP(k))_{+} \longrightarrow \Omega^{k}MSTOP(k)$ of the canonical $\underline{MSTOP}$-orientations $U(\gamma_k) : \Sigma^{k}(G(k)/TOP(k))_{+} \longrightarrow MSTOP(k)$, so that

$$\Phi - 1 \ : \ G/TOP \xrightarrow{\ \Gamma\ } \Omega^{\infty}MS(G/TOP) \longrightarrow \Omega^{\infty}MSTOP \quad .$$

Given a topological bundle $\eta : X \longrightarrow BSTOP(j)$ and a fibre homotopy trivialization $h : J\eta \simeq \varepsilon^{j} : X \longrightarrow BSG(j)$ there is defined a topological normal structure

$$(\nu_1 = \eta \oplus \nu_0 : X \longrightarrow BSTOP(k_1), \rho_1 : S^{n+k_1} \xrightarrow{\ \Sigma^{j}\rho_0\ } \Sigma^{j}T(\nu_0) = T(\varepsilon^{j} \oplus \nu_0) \xrightarrow[\simeq]{\ T(h\oplus 1)^{-1}\ } T(\nu_1)) \ ,$$

where $k_1 = j + k_0$. The image of the classifying map $(\eta, h) : X \longrightarrow G/TOP$ under the bijection $x_0^{-1} : [X, G/TOP] \xrightarrow{\ \simeq\ } \mathcal{T}^{TOP}(X)$ is the bordism class of the normal map $(f_1, b_1) : M_1 \longrightarrow X$ associated to $(\nu_1, \rho_1)$. The composite

$$\begin{array}{l} [X, G/TOP] \xrightarrow{\ \Phi\ } [X_{+}, \Omega^{\infty}MSTOP] = H^{0}(X;\underline{MSTOP}) \xrightarrow[\simeq]{\ -\cup U(\nu_0)\ } \dot{H}^{k_0}(T(\nu_0);\underline{MSTOP}) \\ (=[X_{+}, G/TOP]) \\ \qquad \xrightarrow[\simeq]{\ \Sigma^{j}\ } \dot{H}^{k_1}(T(\varepsilon^{j} \oplus \nu_0);\underline{MSTOP}) \xrightarrow[\simeq]{\ T(h\oplus 1)^{*}\ } \dot{H}^{k_1}(T(\nu_1);\underline{MSTOP}) \end{array}$$

sends $(\eta, h) \in [X, G/TOP]$ to the canonical MSTOP-orientation $U(\nu_1) \in \dot{H}^{k_1}(T(\nu_1);\underline{MSTOP})$. The composite

$$\begin{array}{l} \alpha_1 \ : \ \dot{H}^{k_1}(T(\nu_1);\underline{MSTOP}) \xrightarrow[\simeq]{\ T(h\oplus 1)^{*-1}\ } \dot{H}^{k_1}(T(\varepsilon^{j} \oplus \nu_0);\underline{MSTOP}) \\ \qquad \xrightarrow[\simeq]{\ \Sigma^{-j}\ } \dot{H}^{k_0}(T(\nu_0);\underline{MSTOP}) \xrightarrow[\simeq]{\ \alpha_0\ } H_{n}(X;\underline{MSTOP}) \end{array}$$

is the S-duality isomorphism determined by $(\nu_1, \rho_1)$. The composite

$$\begin{array}{l} [X, G/TOP] \xrightarrow{\ \Gamma\ } [X, \Omega^{\infty}MS(G/TOP)] = H^{0}(X;\underline{MS(G/TOP)}) \xrightarrow[\simeq]{\ [X]_0 \cap -\ } H_{n}(X;\underline{MS(G/TOP)}) \\ \qquad = \Omega_{n+1}^{N,STOP}(X) \end{array}$$

sends $(\eta, h) \in [X, G/TOP]$ to $(W_1 \cup_X -W_0, M_1 \cup -M_0) \in \Omega_{n+1}^{N,STOP}(X)$, where $W_i$ is the mapping cylinder of $f_i : M_i \longrightarrow X$ $(i = 0,1)$. Let $\sigma^{*}[X]_0 \in H_n(X;\underline{\mathbb{L}}^{0})$ be the $\underline{\mathbb{L}}^{0}$-orientation of X determined by $[X]_0 \in H_n(X;\underline{MSTOP})$, so that there is defined a commutative diagram

$$\begin{array}{ccccc}
[X,G/TOP] & \xrightarrow{\Gamma} & [X,\Omega^{\infty}MS(G/TOP)] = H^0(X;\underline{MS(G/TOP)}) & \xrightarrow[\sim]{[X]_0\cap -} & H_n(X;\underline{MS(G/TOP)}) = \Omega^{N,STOP}_{n+1}(X) \\
\sigma_* \downarrow \wr & & \downarrow \sigma_* & & \downarrow \sigma_* \\
[X,\mathbb{L}_0] & \xrightarrow[\sim]{1} & H^0(X;\underline{\mathbb{L}}_0) & \xrightarrow[\sim]{\sigma^*[X]_0\cap -} & H_n(X;\underline{\mathbb{L}}_0)
\end{array} .$$

Furthermore, there is defined a commutative diagram

$$\begin{array}{ccc}
\Omega^{N,STOP}_{n+1}(X) & \longrightarrow & \Omega^{N,P}_{n+1}(X) \\
\sigma_* \downarrow & & \downarrow \sigma_* \\
H_n(X;\underline{\mathbb{L}}_0) & \xrightarrow{\sigma_*} & L_n(\pi_1(X))
\end{array} ,$$

and

$$(W_1 \cup_X -W_0, M_1 \cup -M_0) = (W_1, M_1 \cup -X) - (W_0, M_0 \cup -X) \in \Omega^{N,P}_{n+1}(X) .$$

Thus the surgery obstruction $\theta(f_1,b_1) = \sigma_*(W_1,M_1\cup -X) \in L_n(\pi_1(X))$ of $(f_1,b_1) \in \mathcal{J}^{TOP}(X)$ is given by

$$\begin{aligned}
\theta(f_1,b_1) &= \sigma_*(W_1\cup_X -W_0, M_1\cup -M_0) + \sigma_*(W_0, M_0\cup -X) \\
&= \sigma_*(x_1) + \theta(f_0,b_0) \in L_n(\pi_1(X)) ,
\end{aligned}$$

where $\sigma_*(x_1) \in L_n(\pi_1(X))$ is the image of $(f_1,b_1)$ under the composite

$$\mathcal{J}^{TOP}(X) \xrightarrow[\sim]{x_0} [X,G/TOP] \xrightarrow[\sim]{\sigma^*[X]_0\cap -} H_n(X;\underline{\mathbb{L}}_0) \xrightarrow{\sigma_*} L_n(\pi_1(X)) .$$

We now define the total surgery obstruction $s(X) \in \mathcal{S}_n(X)$ of an n-dimensional geometric Poincaré complex X, as follows. Let $(\nu_X : X \longrightarrow BSG(k), \rho_X : S^{n+k} \longrightarrow T(\nu_X))$ be a Spivak normal structure of X, and let $\alpha_X : S^{n+k} \xrightarrow{\rho_X} T(\nu_X) \xrightarrow{\Delta} X_+ \wedge T(\nu_X)$ be the corresponding S-duality map. Consider the commutative diagram

$$\begin{array}{ccccc}
\dot{H}^k(T(\nu_X);\underline{\hat{\mathbb{L}}}^0) & \xrightarrow[\sim]{\alpha_X} & H_n(X;\underline{\hat{\mathbb{L}}}^0) & \xrightarrow{\hat{\sigma}^*} & \hat{L}^n(\pi_1(X)) \\
H \downarrow & & H \downarrow & & H \downarrow \\
\dot{H}^{k+1}(T(\nu_X);\underline{\mathbb{L}}_0) & \xrightarrow[\sim]{\alpha_X} & H_{n-1}(X;\underline{\mathbb{L}}_0) & \xrightarrow{\sigma_*} & L_{n-1}(\pi_1(X))
\end{array} .$$

The canonical $\underline{\hat{\mathbb{L}}}^0$-orientation $\hat{V} = \hat{\sigma}^*\hat{U}(\nu_X) \in \dot{H}^k(T(\nu_X);\underline{\hat{\mathbb{L}}}^0)$ is such that

i) $H(\hat{V}) = t(\nu_X) \in \dot{H}^{k+1}(T(\nu_X);\underline{\mathbb{L}}_0)$ is the obstruction to a stable topological reduction of $\nu_X$

ii) $\hat{\sigma}^*\alpha_X(\hat{V}) = \hat{\sigma}^*(X) = J\sigma^*(X) \in \hat{L}^n(\pi_1(X))$ is the hyperquadratic signature of X, with $\sigma^*(X) \in L^n(\pi_1(X))$ the symmetric signature of X.

Thus $\sigma_*(\alpha_X H(\hat{V})) = HJ\sigma^*(X) = 0 \in L_{n-1}(\pi_1(X))$, and working on the $\mathbb{L}_0(\pi_1(X))$-space level we can use the $\mathbb{Z}[\pi_1(X)]$-coefficient Poincaré duality on the chain level to obtain an explicit null-homotopy of a simplex representing $\sigma_*(\alpha_X H(\hat{V})) \in L_{n-1}(\pi_1(X))$, and hence an element $s(X) \in \pi_n(\sigma_* : X_+ \wedge \underline{\mathbb{L}}_0 \longrightarrow \underline{\mathbb{L}}_0(\pi_1(X))_{\S}) = \mathcal{S}_n(X)$. The image of $s(X)$ in $H_{n-1}(X;\underline{\mathbb{L}}_0)$ is the S-dual of $t(\nu_X) \in \dot{H}^{k+1}(T(\nu_X);\underline{\mathbb{L}}_0)$. If $t(\nu_X) = 0$ choose a stable topological reduction $\tilde{\nu}_0 : X \longrightarrow BSTOP$ of $\nu_X$, let $x_0 = (f_0,b_0) \in \mathcal{T}^{TOP}(X)$ be the corresponding normal map, and let $[X]_0 = \alpha_X(\sigma^* U(\nu_0)) \in H_n(X;\underline{\mathbb{L}}^0)$ denote the $\underline{\mathbb{L}}^0$-orientation of X determined by the canonical $\underline{\mathbb{L}}^0$-orientation of $\nu_0$ $\sigma^* U(\nu_0) \in \dot{H}^k(T(\nu_X);\underline{\mathbb{L}}^0)$. By the above, the surgery obstruction function is given by

$$\theta : \mathcal{T}^{TOP}(X) \longrightarrow L_n(\pi_1(X)) \ ; \ x_1 \longmapsto \sigma_*(x_1) + \theta(x_0) \ ,$$

where $\sigma_*(x_1)$ is the evaluation of the composite

$$\mathcal{T}^{TOP}(X) \xrightarrow[\simeq]{x_0} [X, G/TOP] \xrightarrow[\simeq]{\sigma_*} [X, \mathbb{L}_0] = H^0(X;\underline{\mathbb{L}}_0) \xrightarrow[\simeq]{[X]_0 \cap -} H_n(X;\underline{\mathbb{L}}_0) \xrightarrow{\sigma_*} L_n(\pi_1(X)) \ .$$

The composite $\mathcal{T}^{TOP}(X) \xrightarrow{\theta} L_n(\pi_1(X)) \longrightarrow \mathcal{S}_n(X)$ sends every element $x_1 \in \mathcal{T}^{TOP}(X)$ to $s(X) \in \mathcal{S}_n(X)$, and the inverse image of $s(X)$ in $L_n(\pi_1(X))$ is precisely the coset of the subgroup $\mathrm{im}(\sigma_* : H_n(X;\underline{\mathbb{L}}_0) \longrightarrow L_n(\pi_1(X)))$ consisting of the surgery obstructions $\theta(x_1) \in L_n(\pi_1(X))$ of all the elements $x_1 \in \mathcal{T}^{TOP}(X)$. The surgery exact sequence has been extended to the right

$$L_{n+1}(\pi_1(X)) \longrightarrow \mathcal{S}^{TOP}(X) \longrightarrow \mathcal{T}^{TOP}(X) \xrightarrow{\theta} L_n(\pi_1(X)) \longrightarrow \mathcal{S}_n(X) \longrightarrow H_{n-1}(X;\underline{\mathbb{L}}_0) \longrightarrow \ldots \ ,$$

with $s(X) = 0 \in \mathcal{S}_n(X)$ if and only if there exists a normal map $x_1 = (f_1,b_1) \in \mathcal{T}^{TOP}(X)$ with surgery obstruction $\theta(f_1,b_1) = 0 \in L_n(\pi_1(X))$, i.e. if and only if X is simple homotopy equivalent to a closed topological manifold.

This completes the sketch of the proof of Theorem 1.

In order to identify $\mathcal{S}^{TOP}(X) = \mathcal{S}_{n+1}(X)$ for an n-dimensional manifold X note that an element $x\in\mathcal{S}_{n+1}(X)$ is defined by a pair (y,z) consisting of a normal map bordism class $y \in H_n(X;\underline{\mathbb{L}}_0) = \mathcal{T}^{TOP}(X)$ such that $\sigma_*(y) = \theta(y) = 0 \in L_n(\pi_1(X))$, together with a particular solution z of the associated surgery problem. Such a pair (y,z) is essentially the same as a homotopy triangulation $(f:M\longrightarrow X)\in\mathcal{S}^{TOP}(X)$. The function $\mathcal{S}_{n+1}(X)\longrightarrow\mathcal{S}^{TOP}(X)$ ; $x = (y,z)\longmapsto (f:M\longrightarrow X)$ is an inverse for the total surgery obstruction function $s:\mathcal{S}^{TOP}(X)\longrightarrow\mathcal{S}_{n+1}(X)$.

The identification of the structure sets $\mathcal{S}_{\partial}^{TOP}(X\times\Delta^k,\partial(X\times\Delta^k))$ $(k\geqslant 0)$ for an n-dimensional manifold with boundary $(X,\partial X)$ with a sequence of universally defined abelian groups $\mathcal{S}_{n+k+1}(X)$ is implicit in Quinn's identification ([Q2]) of the surgery obstruction function

$$\theta : \mathcal{T}_{\partial}^{TOP}(X\times\Delta^k,\partial(X\times\Delta^k)) = [X\times\Delta^k,\partial(X\times\Delta^k);G/TOP,*] \longrightarrow L_{n+k}(\pi_1(X))$$

with the restrictions of universally defined abelian group morphisms

$$A : H_{n+k}(X;\underline{\mathcal{L}}) \longrightarrow L_{n+k}(\pi_1(X))$$

to $\mathrm{im}(H_{n+k}(X;\underline{\mathcal{L}}_{\S})\hookrightarrow H_{n+k}(X;\underline{\mathcal{L}}))$. See the forthcoming Princeton Ph.D. thesis of Andrew Nicas for induction theorems for the structure sets which exploit this group structure. (I am indebted to Larry Siebenmann for the following description of the assembly map A. Given a finite CW complex X let W be the closed regular neighbourhood of X for some embedding $X\subset S^q$ $(q\gg \dim X)$. Then $(W,\partial W)$ is a framed q-dimensional manifold with boundary, enjoying universal Poincaré duality. Let $\underline{\mathcal{L}} = \{\mathcal{L}_{-k} = \Omega\mathcal{L}_{-k-1}\,|\,k\in\mathbb{Z}\}$ be the connective $\Omega$-spectrum with kth space $\mathcal{L}_{-k}$ the Kan complex of normal maps of manifold n-ads such that $\pi_{n+k}(\mathcal{L}_{-k}) = L_n(1)$ $(n,n+k\geqslant 0)$ i.e. Quinn's surgery spectrum, with $\mathcal{L}_0 \simeq L_0(1)\times G/TOP$ [Q1]. Define

$$A : H_n(X;\underline{\mathcal{L}}) = H_n(W;\underline{\mathcal{L}}) = H^{q-n}(W,\partial W;\underline{\mathcal{L}}) = [W,\partial W;\mathcal{L}_{n-q},*] \longrightarrow L_n(\pi_1(X))$$

by sending a simplicial map $(W,\partial W)\longrightarrow(\mathcal{L}_{n-q},*)$ to the surgery obstruction $\sigma_*(f,b)\in L_n(\pi_1(X))$ of the n-dimensional normal map $(f,b):M\longrightarrow N$ obtained by glueing together ("assembling") the normal maps classified by the composites $\Delta^q\hookrightarrow W\longrightarrow\mathcal{L}_{n-q}$, which comes equipped with a reference map $N\longrightarrow W\simeq X$. The quadratic signature map $\sigma_*:\underline{\mathcal{L}}\longrightarrow\underline{\mathbb{L}}_0(1)$ is a homotopy equivalence, and

$$\sigma_* : H_n(X;\underline{\mathbb{L}}_0) = H_n(X;\underline{\mathcal{L}}_{\S}) \longrightarrow H_n(X;\underline{\mathbb{L}}_0(1)) = H_n(X;\underline{\mathcal{L}}) \xrightarrow{\;A\;} L_n(\pi_1(X)) \;).$$

Any simple homotopy invariant of an n-dimensional geometric Poincaré complex X which vanishes if X has the simple homotopy type of a manifold can now be expressed in terms of the total surgery obstruction $s(X) \in \mathcal{S}_n(X)$. We have already dealt with the obstruction to a topological reduction of the Spivak normal fibration, the image of $s(X)$ in $H_{n-1}(X;\underline{\mathbb{L}}_0)$. Examples of geometric Poincaré complexes without topological reduction were first obtained by Gitler and Stasheff [GS], and Wall - of course, at the time it was only clear there was no PL reduction, but the subsequent computation $TOP/PL \simeq K(\mathbb{Z}_2,3)$ implied that there was also no topological reduction. (The Hambleton-Milgram [HM] geometric Poincaré splitting obstruction for a double cover of a 2m-dimensional geometric Poincaré complex X (which need not be oriented) is a part of the topological reducibility obstruction, being the image of $s(X) \in \mathcal{S}_{2m}(X^w)$ under the composite

$$\mathcal{S}_{2m}(X^w) \longrightarrow H^w_{2m-1}(X;\underline{\mathbb{L}}_0) \xrightarrow{p_*} H^w_{2m-1}(B\Sigma_2;\underline{\mathbb{L}}_0) \xrightarrow{c} L_{2m-2}(\mathbb{Z}_2^w) = \mathbb{Z}_2 \quad ,$$

where w refers to homology and L-theory with orientation-twisted coefficients, $p: X \longrightarrow B\Sigma_2$ is the classifying map of the covering, and c is the codimension 1 Arf invariant). The symmetric signature $\sigma^*(X) \in L^n(\pi_1(X))$ is a simple homotopy invariant of X such that $\sigma^*(X) \in \operatorname{coker}(\sigma^*: H_n(X;\underline{\mathbb{L}}^0) \longrightarrow L^n(\pi_1(X)))$ vanishes if X has the simple homotopy type of a manifold. We shall express this invariant in terms of $s(X)$ in Theorem 2 below. For example, if $n = 2m$ and $\pi_1(X) \longrightarrow \pi$ is a morphism to a finite group $\pi$, the image of this invariant in $\operatorname{coker}(\sigma^*: H_n(K(\pi,1);\underline{\mathbb{L}}^0) \longrightarrow L^n(\pi)) \otimes \mathbb{Z}[\frac{1}{2}]$ is the corresponding multisignature of X reduced modulo the multisignatures of closed manifolds, i.e. those with equal components (cf. p.175 of Wall [W1]). The 4-dimensional geometric Poincaré complexes X of Wall [W2] such that $\pi_1(X) = \mathbb{Z}_p$, $\sigma^*(\widetilde{X}) \neq p\sigma^*(X) \in L^4(1) = \mathbb{Z}$ are thus detected by this invariant. (There is no problem in defining the total surgery obstruction $s(X) \in \mathcal{S}_n(X)$ for $n \leqslant 4$, or in showing that $s(X) = 0$ if X has the simple homotopy type of a manifold. However, the usual difficulties with low-dimensional geometric surgery prevent us from deducing the converse).

The construction of the assembly map $\sigma_*: X_+\wedge\underline{\mathbb{L}}_0 \longrightarrow \underline{\mathbb{L}}_0(\pi_1(X))_{\S}$ generalizes to a natural transformation of commutative braids of fibration sequences of spectra

$$\sigma : X_+\wedge\underline{\mathbb{L}} \longrightarrow \underline{\mathbb{L}}(\pi_1(X))$$

(for any space X), from

$$\begin{tikzcd}[column sep=small]
& X_+\wedge\underline{\mathbb{L}}_0 \arrow[rr, bend left, "1\wedge(1+T)"] \arrow[dr, "1\wedge(1+T)"] & & X_+\wedge\underline{\mathbb{L}}^0 \arrow[rr, bend left] \arrow[dr, "1\wedge J"] & & X_+\wedge\underline{K}(L^0(1)) \\
X_+\wedge\underline{\mathbb{L}} : & & X_+\wedge\underline{\mathbb{L}}^0_{\S} \arrow[ur] \arrow[dr, "1\wedge J"] & & X_+\wedge\hat{\underline{\mathbb{L}}}^0 \arrow[ur] \arrow[dr, "H"] & \\
& X_+\wedge\Sigma^{-1}\underline{K}(L^0(1)) \arrow[ur] \arrow[rr, bend right] & & X_+\wedge\hat{\underline{\mathbb{L}}}^0_{\S} \arrow[ur] \arrow[rr, bend right, "H"'] & & X_+\wedge\Sigma\underline{\mathbb{L}}_0
\end{tikzcd}$$

to

$$\begin{tikzcd}[column sep=small]
& \underline{\mathbb{L}}_0(\pi_1(X))_{\S} \arrow[rr, bend left, "1+T"] \arrow[dr, "1+T"] & & \underline{\mathbb{L}}^0(\pi_1(X)) \arrow[rr, bend left] \arrow[dr, "J"] & & \underline{K}(L^0(\pi_1(X)) \\
\underline{\mathbb{L}}(\pi_1(X)) : & & \underline{\mathbb{L}}^0(\pi_1(X))_{\S} \arrow[ur] \arrow[dr, "J"] & & \hat{\underline{\mathbb{L}}}^0(\pi_1(X)) \arrow[ur] \arrow[dr, "H"] & \\
& \Sigma^{-1}\underline{K}(L^0(\pi_1(X))) \arrow[ur] \arrow[rr, bend right] & & \hat{\underline{\mathbb{L}}}^0(\pi_1(X))_{\S} \arrow[ur] \arrow[rr, bend right, "H"'] & & \Sigma\underline{\mathbb{L}}_0(\pi_1(X))_{\S}
\end{tikzcd}$$

The relative homotopy groups of all the maps appearing in $\sigma: X_+\wedge\underline{\mathbb{L}} \longrightarrow \underline{\mathbb{L}}(\pi_1(X))$ define a commutative braid of exact sequences of abelian groups

$$
\mathcal{S}(X):\quad
\begin{tikzcd}
\mathcal{S}_n(X) \arrow[rr, bend left, "1+T"] \arrow[dr, "1+T"] & & \mathcal{S}^n(X) \arrow[rr, bend left] \arrow[dr, "J"] & & H_{n-1}(X;L^0(1)) \\
& \mathcal{S}^n(X)_\S \arrow[ur] \arrow[dr, "J"] & & \hat{\mathcal{S}}^n(X) \arrow[ur] \arrow[dr, "H"] & \\
H_n(X;L^0(1)) \arrow[ur] \arrow[rr, bend right] & & \hat{\mathcal{S}}^n(X)_\S \arrow[ur] \arrow[rr, bend right, "H"] & & \mathcal{S}_{n-1}(X)
\end{tikzcd}
,
$$

and there are defined a commutative diagram with exact rows and columns

$$
\begin{tikzcd}
& \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & H_{n+1}(X;\hat{\underline{\mathbb{L}}}^0) \arrow[r, "\hat{\sigma}^*"] \arrow[d, "H"] & \hat{L}^{n+1}(\pi_1(X)) \arrow[r] \arrow[d, "H"] & \hat{\mathcal{S}}^{n+1}(X) \arrow[r] \arrow[d, "H"] & H_n(X;\hat{\underline{\mathbb{L}}}^0) \arrow[r] \arrow[d, "H"] & \cdots \\
\cdots \arrow[r] & H_n(X;\underline{\mathbb{L}}_0) \arrow[r, "\sigma_*"] \arrow[d, "1+T"] & L_n(\pi_1(X)) \arrow[r] \arrow[d, "1+T"] & \mathcal{S}_n(X) \arrow[r] \arrow[d, "1+T"] & H_{n-1}(X;\underline{\mathbb{L}}_0) \arrow[r] \arrow[d, "1+T"] & \cdots \\
\cdots \arrow[r] & H_n(X;\underline{\mathbb{L}}^0) \arrow[r, "\sigma^*"] \arrow[d, "J"] & L^n(\pi_1(X)) \arrow[r] \arrow[d, "J"] & \mathcal{S}^n(X) \arrow[r] \arrow[d, "J"] & H_{n-1}(X;\underline{\mathbb{L}}^0) \arrow[r] \arrow[d, "J"] & \cdots \\
\cdots \arrow[r] & H_n(X;\hat{\underline{\mathbb{L}}}^0) \arrow[r, "\hat{\sigma}^*"] \arrow[d] & \hat{L}^n(\pi_1(X)) \arrow[r] \arrow[d] & \hat{\mathcal{S}}^n(X) \arrow[r] \arrow[d] & H_{n-1}(X;\hat{\underline{\mathbb{L}}}^0) \arrow[r] \arrow[d] & \cdots \\
& \vdots & \vdots & \vdots & \vdots &
\end{tikzcd}
$$

and the corresponding diagram with $\begin{cases} \underline{\mathbb{L}}^0{}_\S, \mathcal{S}^*(X)_\S \\ \hat{\underline{\mathbb{L}}}^0{}_\S, \hat{\mathcal{S}}^*(X)_\S \end{cases}$ in place of $\begin{cases} \underline{\mathbb{L}}^0, \mathcal{S}^*(X) \\ \hat{\underline{\mathbb{L}}}^0, \hat{\mathcal{S}}^*(X) \end{cases}$.

If X is an n-dimensional geometric Poincaré complex the image of the total surgery obstruction $s(X) \in \mathcal{S}_n(X)$ in $H_{n-1}(X;\underline{\mathbb{L}}_0)$ is the image under H of the canonical $\hat{\underline{\mathbb{L}}}^0$- orientation $[\hat{X}] \in H_n(X;\hat{\underline{\mathbb{L}}}^0)$.

For any space X there is defined a commutative exact braid

$$\begin{tikzcd}[column sep=small]
H_n(X;\underline{\mathbb{L}}^0_\S) \arrow[dr] \arrow[rr, bend left] & & L^n(\pi_1(X)) \arrow[dr] \arrow[rr, bend left] & & \mathcal{S}^n(X) \\
& H_n(X;\underline{\mathbb{L}}^0) \arrow[ur, "\sigma^*"] \arrow[dr] & & \mathcal{S}^n(X)_\S \arrow[ur] \arrow[dr] & \\
\mathcal{S}^{n+1}(X) \arrow[ur] \arrow[rr, bend right] & & H_n(X;L^0(1)) \arrow[ur] \arrow[rr, bend right] & & H_{n-1}(X;\underline{\mathbb{L}}^0_\S)
\end{tikzcd},$$

giving rise to the exact sequence

$$\ldots \longrightarrow H_n(X;\underline{\mathbb{L}}^0) \longrightarrow L^n(\pi_1(X)) \oplus H_n(X;L^0(1)) \longrightarrow \mathcal{S}^n(X)_\S \longrightarrow H_{n-1}(X;\underline{\mathbb{L}}^0) \longrightarrow \ldots .$$

Theorem 2 Let X be an n-dimensional geometric Poincaré complex, with total surgery obstruction $s(X) \in \mathcal{S}_n(X)$.

i) The symmetrization $(1+T)s(X)_\S \in \mathcal{S}^n(X)_\S$ is the image of

(symmetric signature $\sigma^*(X)$, fundamental class $[X]) \in L^n(\pi_1(X)) \oplus H_n(X;L^0(1))$ ,

so that $(1+T)s(X)_\S = 0$ if and only if X has an $\underline{\mathbb{L}}^0$-orientation $[X] \in H_n(X;\underline{\mathbb{L}}^0)$ which assembles to $\sigma^*([X]) = \sigma^*(X) \in L^n(\pi_1(X))$.

ii) The image of $(1+T)s(X)_\S \in \mathcal{S}^n(X)_\S$ in $H_{n-1}(X;\underline{\mathbb{L}}^0_\S)$ is the obstruction to an $\underline{\mathbb{L}}^0$- orientation of X, or equivalently of the Spivak normal fibration $\nu_X : X \longrightarrow BSG$.

iii) The symmetrization $(1+T)s(X) \in \mathcal{S}^n(X)$ is the image of $\sigma^*(X) \in L^n(\pi_1(X))$, so that $(1+T)s(X) = 0$ if and only if $\sigma^*(X) \in \mathrm{im}(\sigma^* : H_n(X;\underline{\mathbb{L}}^0) \longrightarrow L^n(\pi_1(X)))$.

[]

It should be noted that the symmetrization maps

$$1+T : \mathcal{S}_n(X) \longrightarrow \mathcal{S}^n(X)_\S$$

are isomorphisms modulo 8-torsion (for any space X), since the hyperquadratic L-groups $\hat{L}^*(\pi_1(X))$ are of exponent 8, and hence so are $\pi_*(\hat{\underline{\mathbb{L}}}^0_\S) = \hat{L}^*(1)$, $\hat{\mathcal{S}}^*(X)_\S$. Thus if X is an n-dimensional geometric Poincaré complex $s(X)[\frac{1}{2}] = 0 \in \mathcal{S}_n(X)[\frac{1}{2}]$ if and only if X has a $KO[\frac{1}{2}]$-orientation $[X] \in KO_n(X)[\frac{1}{2}]$ which assembles to the symmetric signature away from 2 $\sigma^*[X] = \sigma^*(X)[\frac{1}{2}] \in L^n(\pi_1(X))[\frac{1}{2}]$. Here, we can identify the assembly map $\sigma^* : H_n(X;\underline{\mathbb{L}}^0) \longrightarrow L^n(\pi_1(X))$ localized away from 2 with the composite $KO_n(X)[\frac{1}{2}] \longrightarrow KO_n(K(\pi_1(X),1))[\frac{1}{2}] \xrightarrow{1_\pi^!} L_n(\pi_1(X))[\frac{1}{2}] = L^n(\pi_1(X))[\frac{1}{2}]$, where $1_\pi^!$ is as defined on p.265 of Wall [W1], and $\underline{\mathbb{L}}^0[\frac{1}{2}] = \underline{bo}[\frac{1}{2}]$ as before.

An n-dimensional geometric Poincaré complex X carries an equivalence class of triples $(\sigma^*(X),[\hat{X}],j)$ consisting of a map $\sigma^*(X):\underline{S}^n \longrightarrow \underline{\mathbb{L}}^0(\pi_1(X))$ representing the symmetric signature $\sigma^*(X) \in [\underline{S}^n,\underline{\mathbb{L}}^0(\pi_1(X))] = L^n(\pi_1(X))$, a map $[\hat{X}]:\underline{S}^n \longrightarrow X_+ \wedge \underline{\hat{\mathbb{L}}}^0$ representing the canonical $\underline{\hat{\mathbb{L}}}^0$- orientation $[\hat{X}] \in [\underline{S}^n, X_+ \wedge \underline{\hat{\mathbb{L}}}^0] = H_n(X;\underline{\hat{\mathbb{L}}}^0)$, and a homotopy

$$j \ : \ J\sigma^*(X) \simeq \hat{\sigma}^*[\hat{X}] \ : \ \underline{S}^n \longrightarrow \underline{\hat{\mathbb{L}}}^0(\pi_1(X)) \ .$$

Fixing one such triple $(\sigma^*(X),[\hat{X}],j)$ we can express the original two-stage obstruction theory for X to be simple homotopy equivalent to a manifold entirely in terms of the algebraic $\mathbb{L}$-spectra: $\mathcal{S}^{TOP}(X) \neq \emptyset$ if and only if

i) $[\hat{X}] \in \mathrm{im}(J:H_n(X;\underline{\mathbb{L}}^0) \longrightarrow H_n(X;\underline{\hat{\mathbb{L}}}^0))$, in which case a choice of map $[X]:\underline{S}^n \longrightarrow X_+ \wedge \underline{\mathbb{L}}^0$ and homotopy $g:J[X] \simeq [\hat{X}]:\underline{S}^n \longrightarrow X_+ \wedge \underline{\hat{\mathbb{L}}}^0$ together with j determine an element $\theta([X],g) \in L_n(\pi_1(X))$ with images $s(X) \in \mathcal{S}_n(X)$, $\sigma^*([X]) - \sigma^*(X) \in L^n(\pi_1(X))$

ii) there exists a pair $([X],g)$ such that $\theta([X],g) = 0$.

(In geometric terms $([X],g)$ corresponds to a topological reduction $\tilde{\nu}_X:X \longrightarrow BSTOP$ of the Spivak normal fibration $\nu_X:X \longrightarrow BSG$, and if $(f,b):M \longrightarrow X$ is the associated normal map then $\theta([X],g) = \theta(f,b) \in L_n(\pi_1(X))$ is the surgery obstruction, and $[X] = f_*[M] \in H_n(X;\underline{\mathbb{L}}^0)$ is the image of the canonical $\underline{\mathbb{L}}^0$-orientation $[M] \in H_n(M;\underline{\mathbb{L}}^0)$ of the manifold M, so that $\sigma^*([X]) = \sigma^*(M) \in L^n(\pi_1(X))$).

The invariant $(1+T)s(X)_{\S} \in \mathcal{S}^n(X)_{\S}$ is the primary obstruction of a distinct two-stage theory: $\mathcal{S}^{TOP}(X) \neq \emptyset$ if and only if

i)' there exists an $\underline{\mathbb{L}}^0$- orientation $[X] \in H_n(X;\underline{\mathbb{L}}^0)$ such that $\sigma^*([X]) = \sigma^*(X) \in L^n(\pi_1(X))$, in which case a choice of representative map $[X]:\underline{S}^n \longrightarrow X_+ \wedge \underline{\mathbb{L}}^0$ and of a homotopy $h:\sigma^*(X) \simeq \sigma^*[X]:\underline{S}^n \longrightarrow \underline{\mathbb{L}}^0(\pi_1(X))$ together with j determine an element $\hat{s}([X],h)_{\S} \in \hat{\mathcal{S}}^{n+1}(X)_{\S}$ with images $s(X) \in \mathcal{S}_n(X)$, $J[X] - [\hat{X}] \in H_n(X;\underline{\hat{\mathbb{L}}}^0{}_{\S})$

ii)' there exists a pair $([X],h)$ such that $\hat{s}([X],h)_{\S} = 0$.

(In the previous theory the primary obstruction $t(\nu_X) \in \dot{H}^{k+1}(T(\nu_X);\underline{\mathbb{L}}_0) = H_{n-1}(X;\underline{\mathbb{L}}_0)$ is a torsion element, with the 2-primary torsion of exponent 8. In this theory

the secondary obstruction $\hat{s}([X],h)_{\S} \in \hat{\mathcal{S}}^{n+1}(X)_{\S}$ is 2-primary torsion of exponent 8). Combining the two approaches we have that $\mathcal{S}^{TOP}(X) \neq \emptyset$ if and only if there exists a quadruple $([X],g,h,i)$ consisting of a map $[X]:\underline{S}^n \longrightarrow X_+ \wedge \underline{\mathbb{L}}^0$, homotopies $g:J[X] \simeq [\hat{X}]:\underline{S}^n \longrightarrow X_+ \wedge \hat{\underline{\mathbb{L}}}^0$, $h:\sigma^*(X) \simeq \sigma^*[X]:\underline{S}^n \longrightarrow \underline{\mathbb{L}}^0(\pi_1(X))$, and a homotopy of homotopies $i:(\hat{\sigma}^*g)(Jh) \simeq j : J\sigma^*(X) \simeq \hat{\sigma}^*[\hat{X}] : \underline{S}^n \longrightarrow \hat{\underline{\mathbb{L}}}^0(\pi_1(X))$.
An n-dimensional manifold X carries an equivalence class of such quadruples $([X],g,h,i)$, with $[X] \in H_n(X;\underline{\mathbb{L}}^0)$ the canonical $\underline{\mathbb{L}}^0$-orientation, $J[X] = [\hat{X}] \in H_n(X;\hat{\underline{\mathbb{L}}}^0)$ the canonical $\hat{\underline{\mathbb{L}}}^0$- orientation, and $\sigma^*([X]) = \sigma^*(X) \in L^n(\pi_1(X))$ the symmetric signature. Conversely, an n-dimensional geometric Poincaré complex X is simple homotopy equivalent to a manifold if and only if it admits such a quadruple $([X],g,h,i)$. (In geometric terms $([X],g)$ corresponds to a particular topological reduction of the Spivak normal fibration $\nu_X$, and $(h,i)$ to a particular solution of the associated surgery problem). We can thus identify:

$$\mathcal{S}^{TOP}(X) = \text{the set of equivalence classes of quadruples } ([X],g,h,i) \ ,$$

and if $\mathcal{S}^{TOP}(X) \neq \emptyset$ (i.e. if $s(X) = 0 \in \mathcal{S}_n(X)$) then choosing one manifold structure on X as a base point of $\mathcal{S}^{TOP}(X)$ we have the bijection of Corollary 2 to Theorem 1

$$s : \mathcal{S}^{TOP}(X) \longrightarrow \mathcal{S}_{n+1}(X) \ ; \ (f:M \longrightarrow X) \longmapsto s(f) \ .$$

This defines an equivalence of categories

{compact n-dimensional topological manifolds,
homotopy classes of homeomorphisms}
$\approx$ {n-dimensional geometric Poincaré complexes with extra structure $([X],g,h,i)$,
homotopy classes of simple homotopy equivalences preserving the extra structure} .

By the above, an n-dimensional geometric Poincaré complex X is simple homotopy equivalent to a closed topological manifold if and only if there exists an element $[X] \in H_n(X;\underline{\mathbb{L}}^0)$ such that

i) $J[X] = [\hat{X}] \in H_n(X;\hat{\underline{\mathbb{L}}}^0)$ is the canonical $\hat{\underline{\mathbb{L}}}^0$-orientation of X, in which case $[X] \in H_n(X;\underline{\mathbb{L}}^0)$ is an $\underline{\mathbb{L}}^0$-orientation (since $\pi_0(\underline{\mathbb{L}}^0) = \pi_0(\hat{\underline{\mathbb{L}}}^0) = L^0(1)$)

ii) $\sigma^*([X]) = \sigma^*(X) \in L^n(\pi_1(X))$ is the symmetric signature of X

iii) the relations i) and ii) are compatible on the $\mathbb{L}$-space level, i.e. can be realized by a quadruple $([X],g,h,i)$.

In certain cases we can ensure that condition iii) is redundant:

Theorem 3 Let X be an n-dimensional geometric Poincaré complex such that the hyperquadratic signature map $\hat{\sigma}^*: H_{n+1}(X;\hat{\underline{\mathbb{L}}}^0) \longrightarrow \hat{L}^{n+1}(\pi_1(X))$ is onto. Then X is simple homotopy equivalent to a closed topological manifold if and only if there exists an $\underline{\mathbb{L}}^0$-orientation $[X]\in H_n(X;\underline{\mathbb{L}}^0)$ such that $J[X] = [\hat{X}]\in H_n(X;\hat{\underline{\mathbb{L}}}^0)$ and $\sigma^*([X]) = \sigma^*(X)\in L^n(\pi_1(X))$.

Proof: Given such an $\underline{\mathbb{L}}^0$-orientation $[X]$ there are defined homotopies $g: J[X]\simeq[\hat{X}]: \underline{S}^n \longrightarrow X_+\wedge\hat{\underline{\mathbb{L}}}^0$, $h:\sigma^*(X)\simeq\sigma^*([X]) : \underline{S}^n \longrightarrow \underline{\mathbb{L}}^0(\pi_1(X))$. These determine an element $\hat{\sigma}([X],g,h)\in\hat{L}^{n+1}(\pi_1(X))$, the obstruction to the existence of a homotopy of homotopies $i : (\hat{\sigma}^*g)(Jh)\simeq j : J\sigma^*(X)\simeq\hat{\sigma}^*[\hat{X}] : \underline{S}^n \longrightarrow \hat{\underline{\mathbb{L}}}^0(\pi_1(X))$. Now $H\hat{\sigma}^*([X],g,h) = \theta([X],g) = \theta(f,b)\in L_n(\pi_1(X))$ is the surgery obstruction of the normal map $(f,b):M\longrightarrow X$ associated to the topological reduction of $\nu_X$ determined by $([X],g)$. By assumption $\hat{\sigma}([X],g,h)\in \mathrm{im}(\hat{\sigma}^*: H_{n+1}(X;\hat{\underline{\mathbb{L}}}^0) \longrightarrow \hat{L}^{n+1}(\pi_1(X)))$, so that $\theta(f,b)\in\mathrm{im}(\sigma_*: H_n(X;\underline{\mathbb{L}}_0)\longrightarrow L_n(\pi_1(X)))$ and there exists a topological reduction with 0 surgery obstruction.

[]

In particular, suppose that $\pi$ is a group such that $K(\pi,1)$ is an n-dimensional geometric Poincaré complex for which $\sigma^*: H_n(K(\pi,1);\underline{\mathbb{L}}^0)\longrightarrow L^n(\pi)$ is an isomorphism and $\hat{\sigma}^*: H_{n+1}(K(\pi,1);\hat{\underline{\mathbb{L}}}^0)\longrightarrow \hat{L}^{n+1}(\pi)$ is onto. Then $K(\pi,1)$ is simple homotopy equivalent to a closed topological manifold if and only if the composite $L^n(\pi) \xrightarrow[\simeq]{\sigma^{*-1}} H_n(K(\pi,1);\underline{\mathbb{L}}^0) \xrightarrow{J} H_n(K(\pi,1);\hat{\underline{\mathbb{L}}}^0)$ sends the symmetric signature $\sigma^*(K(\pi,1))\in L^n(\pi)$ to the canonical $\hat{\underline{\mathbb{L}}}^0$-orientation $[\widehat{K(\pi,1)}]\in H_n(K(\pi,1);\hat{\underline{\mathbb{L}}}^0)$.

(The hypothesis of Theorem 3 is not satisfied in general: the infinitely generated subgroup $\mathbb{Z}_2^\infty\subseteq \mathrm{Unil}_{4k+2}(1;\mathbb{Z},\mathbb{Z}_2) = \mathrm{coker}(L_{4k+2}(\mathbb{Z})\oplus L_{4k+2}(\mathbb{Z}_2)\longrightarrow L_{4k+2}(\mathbb{Z}*\mathbb{Z}_2))$ constructed by Cappell [C] can be used to detect an infinitely generated subgroup $\mathbb{Z}_2^\infty\subseteq\mathrm{coker}(\hat{\sigma}^*: H_{4k+3}(K(\mathbb{Z}*\mathbb{Z}_2,1);\hat{\underline{\mathbb{L}}}^0)\longrightarrow\hat{L}^{4k+3}(\mathbb{Z}*\mathbb{Z}_2))$. This also shows that the hyperquadratic signature map $\hat{\sigma}^*:\Omega_n^N(K)\longrightarrow\hat{L}^n(\pi_1(K))$ is not onto in general).

For any space K there is defined a natural transformation of exact sequences

$$\begin{array}{ccccccccc}
\cdots \longrightarrow \Omega^N_{n+1}(K) & \longrightarrow & \Omega^{N,P}_{n+1}(K) & \longrightarrow & \Omega^P_n(K) & \longrightarrow & \Omega^N_n(K) & \longrightarrow & \cdots \\
\downarrow{\scriptstyle H\hat{\sigma}^*} & & \downarrow{\scriptstyle \sigma_*} & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle H\hat{\sigma}^*} & & \\
\cdots \longrightarrow H_n(K;\underline{\mathbb{L}}_0) & \xrightarrow{\sigma_*} & L_n(\pi_1(K)) & \longrightarrow & \mathcal{S}_n(K) & \longrightarrow & H_{n-1}(K;\underline{\mathbb{L}}_0) & \longrightarrow & \cdots
\end{array}$$

with $\sigma_*:\Omega^{N,P}_{n+1}(K) \longrightarrow L_n(\pi_1(K))$ the quadratic signature map and

$$H\hat{\sigma}^* : \Omega^N_{n+1}(K) = H_{n+1}(K;\underline{\Omega}^N) \xrightarrow{\hat{\sigma}^*} H_{n+1}(K;\underline{\hat{\mathbb{L}}}^0) \xrightarrow{H} H_n(K;\underline{\mathbb{L}}_0)$$

$$s : \Omega^P_n(K) \longrightarrow \mathcal{S}_n(K) \ ; \ (f:X \longrightarrow K) \longmapsto f_*s(X) \ .$$

In particular, the quadratic signature $\sigma_*(f,b) = \sigma_*(W,M\cup -X) \in L_n(\pi_1(X))$ of a normal map of n-dimensional geometric Poincaré complexes

$$(f,b) : (M,\nu_M,\rho_M) \longrightarrow (X,\nu_X,\rho_X)$$

has image

$$[\sigma_*(f,b)] = f_*s(M) - s(X) \in \mathcal{S}_n(X) \ ,$$

where W is the mapping cylinder of f, $(W,M\cup -X) \in \Omega^{N,P}_{n+1}(X)$.

For any space K define a morphism of abelian groups

$$L_n(\pi_1(K)) \longrightarrow \Omega^P_n(K) \ ; \ x \longmapsto (f:X \longrightarrow K)$$

as follows. Let Y be an (n-1)-dimensional manifold (possibly with boundary) equipped with a map $Y \longrightarrow K$ inducing an isomorphism $\pi_1(Y) \xrightarrow{\sim} \pi_1(K)$.
By Wall's realization theorem every element $x \in L_n(\pi_1(K))$ is the surgery obstruction $x = \sigma_*(F,B)$ of a normal map of manifold triads

$$(F,B) : (Z;Y,Y') \longrightarrow (Y\times I;Y\times 0,Y\times 1)$$

such that $F| = 1 : Y \longrightarrow Y\times 0$ and $F| = h : Y' \longrightarrow Y\times 1$ is a simple homotopy equivalence. Define $X = Z/Y \underset{h}{\simeq} Y'$ to be the n-dimensional geometric Poincaré complex obtained from Z by glueing Y to Y' by h, let $g:X \longrightarrow Y\times S^1$ be the degree 1 map obtained from F, and define $f:X \longrightarrow K$ to be the composite

$$f : X \xrightarrow{g} Y\times S^1 \xrightarrow{\text{projection}} Y \longrightarrow K \ .$$

Now g is covered by a bundle map of topological reductions of the Spivak normal fibrations such that the quadratic signature $\sigma_*(g,c) \in L_n(\pi_1(Y\times S^1))$ of the

corresponding normal map of geometric Poincaré complexes

$$(g,c)\ :\ (X,\nu_X,\rho_X)\longrightarrow(Y\times S^1,\nu_{Y\times S^1},\rho_{Y\times S^1})$$

has image

$$\sigma_*(g,c) = x\in L_n(\pi_1(K))\ .$$

By the above

$$[\sigma_*(g,c)] = g_*s(X) - s(Y\times S^1)\in\mathcal{S}_n(Y\times S^1)\ ,$$

and $s(Y\times S^1) = 0$, so that

$$[x] = f_*s(X)\in\mathcal{S}_n(K)\ .$$

(Incidentally, the image $[x]\in\mathcal{S}_n(Y)$ is the obstruction to deforming the simple homotopy equivalence $h:Y'\longrightarrow Y$ to a homeomorphism, $[x] = s(h)\in\mathcal{S}_n(Y)$, cf. Corollary 1 to Theorem 1 above). The composite

$$L_n(\pi_1(K))\longrightarrow\Omega_n^P(K)\xrightarrow{\ s\ }\mathcal{S}_n(K)$$

is thus the canonical map $L_n(\pi_1(K))\longrightarrow\mathcal{S}_n(K)$.

********

We have the following extension of the Levitt-Jones-Quinn geometric Poincaré surgery exact sequence [Le],[J1],[Q3]

$$\ldots\longrightarrow\Omega_{n+1}^N(K)\longrightarrow L_n(\pi_1(K))\longrightarrow\Omega_n^P(K)\longrightarrow\Omega_n^N(K)\longrightarrow\ \ldots\ .$$

<u>Theorem 4</u> For any space K there is defined a commutative braid of exact sequences of abelian groups

$$\begin{tikzcd}[column sep=small]
\mathcal{S}_{n+1}(K) \arrow[dr] \arrow[rr, bend left] & & H_n(K;\underline{\Omega}^P) \arrow[dr, "\sigma^*"] \arrow[rr, bend left] & & \Omega_n^N(K) \arrow[dr] \arrow[rr, bend left, "H\hat{\sigma}^*"] & & L_{n-1}(\pi_1(K)) \\
& H_n(K;\underline{\mathbb{L}}_0) \arrow[ur] \arrow[dr, "\sigma_*"] & & \Omega_n^P(K) \arrow[ur] \arrow[dr, "s"] & & H_{n-1}(K;\underline{\mathbb{L}}_0) \arrow[ur, "\sigma_*"] \arrow[dr] & \\
\Omega_{n+1}^N(K) \arrow[ur] \arrow[rr, bend right, "H\hat{\sigma}^*"'] & & L_n(\pi_1(K)) \arrow[ur] \arrow[rr, bend right] & & \mathcal{S}_n(K) \arrow[ur] \arrow[rr, bend right] & & H_{n-1}(K;\underline{\Omega}^P)
\end{tikzcd}$$

[]

For example, $\Omega_n^P(T^n) = H_n(T^n;\underline{\Omega}^P)\oplus\mathcal{S}_n(T^n)\ ,\ \mathcal{S}_n(T^n) = L_0(1)$

(since $\sigma_*:H_n(T^n;\underline{\mathbb{L}}_0) = \bigoplus_{i=1}^{n}\binom{n}{i}L_i(1)\hookrightarrow L_n(\pi_1(T^n)) = \bigoplus_{i=0}^{n}\binom{n}{i}L_i(1)$).

From the point of view of geometric Poincaré surgery theory there are defined equivalences of categories

$\{$stable oriented topological bundles (over finite CW complexes)$\}$

$\approx$ $\{$stable spherical fibrations with an $\underline{\Omega}^P$-orientation lifting the canonical $\underline{\Omega}^N$-orientation$\}$,

$\{$compact oriented n-dimensional topological manifolds$\}$

$\approx$ $\{$n-dimensional geometric Poincaré complexes X with an $\underline{\Omega}^P$-orientation $[X]\in H_n(X;\underline{\Omega}^P)$ which assembles to $\sigma^*([X]) = (1:X\longrightarrow X)\in\Omega^P_n(X)\}$.

********

Product with the symmetric signature $\sigma^*(\mathbb{C}P^2)\in L^4(1)$ $(=1\in\mathbb{Z})$ of the complex projective plane $\mathbb{C}P^2$ defines the periodicity isomorphisms in the quadratic L-groups

$$\sigma^*(\mathbb{C}P^2)\otimes - \; : \; L_n(\pi)\longrightarrow L_{n+4}(\pi) \quad (n\geqslant 0)$$

for any group $\pi$. For any space K there is defined a commutative braid of exact sequences of abelian groups

$$\begin{tikzcd}[column sep=tiny]
H_{n+1}(K;L_0(1))\oplus H_{n+3}(K;L_2(1)) \arrow[dr] \arrow[rr, bend left] & & H_n(K;\underline{\mathbb{L}}_0) \arrow[dr, "\sigma^*(\mathbb{C}P^2)\otimes-"] \arrow[rr, bend left, "\sigma_*"] & & L_n(\pi_1(K)) \arrow[dr] \arrow[rr, bend left] & & \mathcal{S}_{n+4}(K) \\
& \mathcal{S}_{n+1}(K) \arrow[ur] \arrow[dr, "\sigma^*(\mathbb{C}P^2)\otimes-"'] & & H_{n+4}(K;\underline{\mathbb{L}}_0) \arrow[ur, "\sigma_*"'] \arrow[dr] & & \mathcal{S}_n(K) \arrow[ur, "\sigma^*(\mathbb{C}P^2)\otimes-"'] & \\
L_{n+1}(\pi_1(K)) \arrow[ur] \arrow[rr, bend right] & & \mathcal{S}_{n+5}(K) \arrow[ur] \arrow[rr, bend right] & & H_n(K;L_0(1))\oplus H_{n+2}(K;L_2(1)) \arrow[ur] & &
\end{tikzcd} ,$$

involving the products $\sigma^*(\mathbb{C}P^2)\otimes - : \Sigma^4\underline{\mathbb{L}}_0\longrightarrow\underline{\mathbb{L}}_0$ and the homotopy-theoretic analysis

$$\underline{\mathbb{L}}_0[\tfrac{1}{2}] = \underline{\mathrm{bso}}[\tfrac{1}{2}] \quad , \quad \underline{\mathbb{L}}_{0(2)} = \prod_{i=1}^{\infty}\Sigma^{4i}\underline{K}(L_0(1)_{(2)})\times\Sigma^{4i-2}\underline{K}(L_2(1)) \; .$$

The maps $H_{n+4}(K;\underline{\mathbb{L}}_0)\longrightarrow H_n(K;L_0(1))\oplus H_{n+2}(K;L_2(1))$ have odd torsion cokernel.

(More generally, we have that the $\left\{\begin{array}{l}\text{symmetric}\\ \text{quadratic}\end{array}\right.$ signature of a product is given by

$\left\{\begin{array}{l}\sigma^*(M^m\times N^n) = \sigma^*(M)\otimes\sigma^*(N)\in L^{m+n}(\pi_1(M\times N))\\ \sigma_*(1\times(f,b):M^m\times N^n\longrightarrow M\times X) = \sigma^*(M)\otimes\sigma_*(f,b)\in L_{m+n}(\pi_1(M\times X))\end{array}\right.$. Product with the canonical $\underline{\mathbb{L}}^0$-orientation $[M]\in H_m(M;\underline{\mathbb{L}}^0)$ of an m-dimensional manifold M defines a map $[M]\otimes - : \mathcal{S}_n(X)\longrightarrow\mathcal{S}_{m+n}(M\times X)$ (for any space X) compatible with the product map $\sigma^*(M)\otimes - : L_n(\pi_1(X))\longrightarrow L_{m+n}(\pi_1(M\times X))$ $(\sigma^*(M) = \sigma^*([M])\in L^m(\pi_1(M)))$. If X is an n-dimensional geometric Poincaré complex $s(M\times X) = [M]\otimes s(X)\in\mathcal{S}_{m+n}(M\times X)$. The maps appearing above are $\sigma^*(\mathbb{C}P^2)\otimes - : \mathcal{S}_n(K)\xrightarrow{[\mathbb{C}P^2]\otimes -}\mathcal{S}_{n+4}(\mathbb{C}P^2\times K)\xrightarrow{\text{proj.}_*}\mathcal{S}_{n+4}(K)$).

<u>Theorem 5</u> i) If X is a connected n-dimensional geometric Poincaré complex there are defined periodicity isomorphisms

$$\sigma^*(\mathbb{C}P^2)\otimes - : \mathcal{S}_{n+k}(X) \longrightarrow \mathcal{S}_{n+k+4}(X) \quad (k \geqslant 2)$$

and an exact sequence

$$0 \longrightarrow \mathcal{S}_{n+1}(X) \xrightarrow{\sigma^*(\mathbb{C}P^2)\otimes -} \mathcal{S}_{n+5}(X) \longrightarrow L_0(1) \longrightarrow \mathcal{S}_n(X) \xrightarrow{\sigma^*(\mathbb{C}P^2)\otimes -} \mathcal{S}_{n+4}(X) \longrightarrow \dots .$$

ii) If (X,Y) is an n-dimensional geometric Poincaré pair with X connected and Y non-empty there are defined periodicity isomorphisms

$$\sigma^*(\mathbb{C}P^2)\otimes - : \mathcal{S}_{n+k}(X) \longrightarrow \mathcal{S}_{n+k+4}(X) \quad (k \geqslant 1)$$

and an exact sequence

$$\begin{aligned} 0 \longrightarrow \mathcal{S}_n(X) \xrightarrow{\sigma^*(\mathbb{C}P^2)\otimes -} \mathcal{S}_{n+4}(X) &\longrightarrow H_{n-1}(X;L_0(1)) \\ &\longrightarrow \mathcal{S}_{n-1}(X) \xrightarrow{\sigma^*(\mathbb{C}P^2)\otimes -} \mathcal{S}_{n+3}(X) \longrightarrow \dots . \end{aligned}$$

[]

In particular, if (X,Y) is an n-dimensional manifold with boundary we have the structure set 4-periodicity of Appendix C of Essay V of Kirby and Siebenmann [KS] (which is due to Siebenmann)

$$\mathcal{S}_\partial^{TOP}(X\times\Delta^k,\partial(X\times\Delta^k)) = \mathcal{S}_\partial^{TOP}(X\times\Delta^{k+4},\partial(X\times\Delta^{k+4})) \ (= \mathcal{S}_{n+k+1}(X)) \quad (n \geqslant 5)$$

for $k \geqslant 1$, and if X is connected and Y is non-empty also for k = 0. In the closed case $\mathcal{S}^{TOP}(X) \neq \mathcal{S}_\partial^{TOP}(X\times\Delta^4,\partial(X\times\Delta^4))$ in general, contradicting Siebenmann's claim for periodicity in this case also. (This discrepancy was pointed out to me by Andrew Nicas). For example,

$$\mathcal{S}_{n+k}(S^n) = \begin{cases} L_{k-1}(1) & \text{if } k \geqslant 2 \\ 0 & \text{if } k = 0,1 \end{cases} \quad (n \geqslant 2)$$

so that

$$\mathcal{S}_\partial^{TOP}(S^n\times\Delta^4,\partial(S^n\times\Delta^4)) = \mathcal{S}_{n+5}(S^n) = L_4(1) \neq \mathcal{S}^{TOP}(S^n) = \mathcal{S}_{n+1}(S^n) = 0 \quad (n \geqslant 5).$$

On the other hand,

$$\mathcal{S}_{n+k}(T^n) = \begin{cases} 0 & \text{if } k \geqslant 1 \\ L_0(1) & \text{if } k = 0 \end{cases} \quad (n \geqslant 1)$$

so that

$$\mathcal{S}_\partial^{TOP}(T^n\times\Delta^k,\partial(T^n\times\Delta^k)) = \mathcal{S}_\partial^{TOP}(T^n\times\Delta^{k+4},\partial(T^n\times\Delta^{k+4})) = \mathcal{S}_{n+k+1}(T^n) = 0 \quad (k \geqslant 0, n \geqslant 5).$$

In conclusion, we note that it is also possible to define quadratic $\mathcal{S}$-groups $\left\{\begin{matrix}\mathcal{S}^h_*(X)\\ \mathcal{S}^p_*(X)\end{matrix}\right.$ appropriate to $\left\{\begin{matrix}\text{finite}\\ \text{infinite}\end{matrix}\right.$ homotopy types and the $\left\{\begin{matrix}\text{free}\\ \text{projective}\end{matrix}\right.$ L-groups $\left\{\begin{matrix}L^h_*(\pi)\\ L^p_*(\pi)\end{matrix}\right.$, which fit into a commutative braid of exact sequences of abelian groups

$$\begin{tikzcd}[column sep=small]
\hat{H}^{n+1}(\mathbb{Z}_2;\widetilde{K}_0(\mathbb{Z}[\pi_1(X)])) \arrow[dr] \arrow[rr, bend left] & & \mathcal{S}^h_n(X) \arrow[dr] \arrow[rr, bend left] & & H_{n-1}(X;\underline{\mathbb{L}}_0) \arrow[dr, "\sigma_*"] \arrow[rr, bend left, "\sigma_*"] & & L^p_{n-1}(\pi_1(X)) \\
& L^h_n(\pi_1(X)) \arrow[ur] \arrow[dr] & & \mathcal{S}^p_n(X) \arrow[ur] \arrow[dr] & & L^h_{n-1}(\pi_1(X)) \arrow[ur] \arrow[dr] & \\
H_n(X;\underline{\mathbb{L}}_0) \arrow[ur, "\sigma_*"] \arrow[rr, bend right, "\sigma_*"'] & & L^p_n(\pi_1(X)) \arrow[ur] \arrow[rr, bend right] & & \hat{H}^n(\mathbb{Z}_2;\widetilde{K}_0(\mathbb{Z}[\pi_1(X)])) \arrow[ur] \arrow[rr, bend right] & & \mathcal{S}^h_{n-1}(X)
\end{tikzcd}$$

involving the Tate $\mathbb{Z}_2$-cohomology groups of the duality involution $[P] \longmapsto [P^*]$ ($P^* = \mathrm{Hom}_A(P,A)$, $A = \mathbb{Z}[\pi_1(X)]$) on the reduced projective class group $\widetilde{K}_0(\mathbb{Z}[\pi_1(X)])$. There is a similar braid relating $\mathcal{S}^h_*(X)$ and $\mathcal{S}^s_*(X) = \mathcal{S}_*(X)$, involving the duality involution $\tau(f:P \longrightarrow Q) \longmapsto \tau(f^*:Q^* \longrightarrow P^*)$ on the Whitehead group $Wh(\pi_1(X))$.
The free symmetric L-groups $L^*_h(\pi)$ are related to the projective symmetric L-groups $L^*_p(\pi)$ by an exact sequence

$$\ldots \longrightarrow \hat{H}^{n+1}(\mathbb{Z}_2;\widetilde{K}_0(\mathbb{Z}[\pi])) \longrightarrow L^n_h(\pi) \longrightarrow L^n_p(\pi) \longrightarrow \hat{H}^n(\mathbb{Z}_2;\widetilde{K}_0(\mathbb{Z}[\pi])) \longrightarrow L^{n-1}_h(\pi) \longrightarrow \ldots$$

(which actually connects with the quadratic L-group sequence for $L^h_*(\pi), L^p_*(\pi)$ on setting $L^n(\pi) = L_{n+4k}(\pi)$ ($n \leqslant -3$, $n+4k \geqslant 0$), see [R2]) and similarly for $L^*_s(\pi) \equiv L^*(\pi)$, $L^*_h(\pi)$, $Wh(\pi)$. Thus it is also possible to define symmetric $\mathcal{S}$-groups $\left\{\begin{matrix}\mathcal{S}^*_h(X)\\ \mathcal{S}^*_p(X)\end{matrix}\right.$ with properties analogous to those of $\mathcal{S}^*_s(X) \equiv \mathcal{S}^*(X)$, $\left\{\begin{matrix}\mathcal{S}^h_*(X)\\ \mathcal{S}^p_*(X)\end{matrix}\right.$.
The hyperquadratic L-groups are such that

$$\hat{L}^*_p(\pi) = \hat{L}^*_h(\pi) = \hat{L}^*_s(\pi) \equiv \hat{L}^*(\pi) \quad ,$$

and accordingly we define

$$\hat{\mathcal{S}}^*_p(X) = \hat{\mathcal{S}}^*_h(X) = \hat{\mathcal{S}}^*_s(X) \equiv \hat{\mathcal{S}}^*(X) \ .$$

Similarly for $\mathcal{S}^*(X)_{\S}$, $\hat{\mathcal{S}}^*(X)_{\S}$.

<u>Theorem 1(h)</u> A finite n-dimensional geometric Poincaré complex X determines an element $s(X) \in \mathcal{S}_n^h(X)$ such that $s(X) = 0$ if and only if X is homotopy equivalent to a closed topological manifold. The image of $s(X)$ in $H_{n-1}(X;\underline{\mathbb{L}}_0)$ is the obstruction to a topological reduction of the Spivak normal fibration $\nu_X : X \longrightarrow BSG$.
The symmetrization $(1+T)s(X) \in \mathcal{S}_h^n(X)$ is the image of the symmetric signature $\sigma^*(X) \in L_h^n(\pi_1(X))$. The image of $s(X)$ in $\hat{H}^n(\mathbb{Z}_2;Wh(\pi_1(X)))$ is the class of the Whitehead torsion $\tau(X) \in Wh(\pi_1(X))$ of the chain equivalence $[X] \cap - : C(\widetilde{X})^{n-*} \longrightarrow C(\widetilde{X})$.
[]

Furthermore, if X is an n-dimensional manifold then $\mathcal{S}_{n+1}^h(X)$ can be identified with the set of concordance classes of topological h-triangulations of X, i.e. pairs

(n-dimensional manifold M, homotopy equivalence $f:M \longrightarrow X$)

with $(M,f) \sim (M',f')$ if there exist an h-cobordism $(W;M,M')$ and a homotopy equivalence

$$(g;f,f') : (W;M,M') \longrightarrow (X \times I; X \times 0, X \times 1) .$$

<u>Theorem 1(p)</u> A finitely dominated n-dimensional geometric Poincaré complex X determines an element $s(X) \in \mathcal{S}_n^p(X)$ such that $s(X) = 0$ if and only if $X \times S^1$ is homotopy equivalent to a closed topological manifold. The image of $s(X)$ in $H_{n-1}(X;\underline{\mathbb{L}}_0)$ is the obstruction to a topological reduction of the Spivak normal fibration $\nu_X : X \longrightarrow BSG$. The symmetrization $(1+T)s(X) \in \mathcal{S}_p^n(X)$ is the image of the symmetric signature $\sigma^*(X) \in L_p^n(\pi_1(X))$. The image of $s(X)$ in $\hat{H}^n(\mathbb{Z}_2;\widetilde{K}_0(\mathbb{Z}[\pi_1(X)]))$ is the class of the Wall finiteness obstruction $[C(\widetilde{X})] \in \widetilde{K}_0(\mathbb{Z}[\pi_1(X)])$.
[]

Theorem 1(p) is the special case of Theorem 1(h) obtained by first noting that $X \times S^1$ has the homotopy type of a finite complex and then applying the algebraic aplitting theorem $L_{n+1}^h(\pi \times \mathbb{Z}) = L_{n+1}^h(\pi) \oplus L_n^p(\pi)$ ([R1]) to identify

$$s(X \times S^1) = (0, s(X)) \in \mathcal{S}_{n+1}^h(X \times S^1) = \mathcal{S}_{n+1}^h(X) \oplus \mathcal{S}_n^p(X) .$$

(The definitive version of the non-compact manifold surgery theories of Taylor [Ta] and Maumary [Ma] should interpret $s(X) \in \mathcal{S}_n^p(X)$ as the total obstruction to X being homotopy equivalent to a topological manifold allowed a certain degree of non-compactness, such as an end).

The invariant $s(X)\in\mathcal{S}^p_n(X)$ may be of interest in the classification of free actions of finite groups on spheres, the "topological spherical space form problem" (cf. Swan [Sw], Thomas and Wall [ThW], Madsen,Thomas and Wall [MTW]) since its definition does not presuppose a vanishing of the finiteness obstruction. If $\pi$ is a finite group with cohomology of period dividing $n+1$, to every generator $g\in H^{n+1}(K(\pi,1))$ there is associated a finitely dominated n-dimensional geometric Poincaré complex $X_g$ equipped with an isomorphism $\pi_1(X_g)\xrightarrow{\sim}\pi$, a homotopy equivalence $\tilde{X}_g\xrightarrow{\sim}S^n$, and first k-invariant $g\in H^{n+1}(K(\pi,1))$. Ultimately, it might be possible to give a direct description of $s(X_g)\in\mathcal{S}^p_n(X_g)$. In this connection, it should also be mentioned that the $\mathcal{S}$-groups (in each of the categories s,h,p) behave well with respect to finite covers $p:\overline{X}\longrightarrow X$, with transfer maps defining a natural transformation of exact sequences of abelian groups

$$\begin{array}{ccccccccc}
\cdots\longrightarrow & H_n(X;\underline{\mathbb{L}}_0) & \xrightarrow{\sigma_*} & L_n(\pi_1(X)) & \longrightarrow & \mathcal{S}_n(X) & \longrightarrow & H_{n-1}(X;\underline{\mathbb{L}}_0) & \longrightarrow\cdots \\
 & \downarrow p^! & & \downarrow p^! & & \downarrow p^! & & \downarrow p^! & \\
\cdots\longrightarrow & H_n(\overline{X};\underline{\mathbb{L}}_0) & \xrightarrow{\sigma_*} & L_n(\pi_1(\overline{X})) & \longrightarrow & \mathcal{S}_n(\overline{X}) & \longrightarrow & H_{n-1}(\overline{X};\underline{\mathbb{L}}_0) & \longrightarrow\cdots
\end{array},$$

using the canonical S-map $p^!:\Sigma^\infty X_+\longrightarrow\Sigma^\infty\overline{X}_+$ to define

$$p^!\ :\ H_n(X;\underline{\mathbb{L}}_0)\longrightarrow H_n(\overline{X};\underline{\mathbb{L}}_0)\quad ,$$

and the restriction of $\pi_1(X)$-action to $\pi_1(\overline{X})$-action to define

$$p^!\ :\ L_n(\pi_1(X))\longrightarrow L_n(\pi_1(\overline{X}))\ ;\ (C,\psi)\longmapsto(p^!C,p^!\psi)\ .$$

If $\begin{cases}X\\(f,b):M\longrightarrow X\end{cases}$ is an n-dimensional $\begin{cases}\text{geometric Poincaré complex}\\\text{normal map}\end{cases}$ then so is $\begin{cases}\overline{X}\\(\overline{f},\overline{b}):\overline{M}\longrightarrow\overline{X}\end{cases}$, and

$$\begin{cases}s(\overline{X})\ =\ p^!s(X)\in\mathcal{S}_n(\overline{X})\ ,\ \sigma^*(\overline{X})\ =\ p^!\sigma^*(X)\in L^n(\pi_1(\overline{X}))\\ \sigma_*(\overline{f},\overline{b})\ =\ p^!\sigma_*(f,b)\in L_n(\pi_1(\overline{X}))\end{cases}.$$

Similarly for the $\begin{cases}\text{symmetric}\\\text{hyperquadratic}\end{cases}$ $\mathcal{S}$-groups $\begin{cases}\mathcal{S}^*(X),\ \mathcal{S}^*(X)_{\S}\\\hat{\mathcal{S}}^*(X),\ \hat{\mathcal{S}}^*(X)_{\S}\end{cases}.$

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