SOME SIMPLE EXAMPLES OF SYMPLECTIC MANIFOLDS

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ABSTRACT. This is a construction of closed symplectic manifolds with no Kaehler structure.

A symplectic manifold is a manifold of dimension $2k$ with a closed 2-form $\alpha$ such that $\alpha^k$ is nonsingular. If $M^{2k}$ is a closed symplectic manifold, then the cohomology class of $\alpha$ is nontrivial, and all its powers through $k$ are nontrivial. $M$ also has an almost complex structure associated with $\alpha$, up to homotopy.

It has been asked whether every closed symplectic manifold has also a Kaehler structure (the converse is immediate). A Kaehler manifold has the property that its odd dimensional Betti numbers are even. H. Guggenheimer claimed [1], [2] that a symplectic manifold also has even odd Betti numbers. In the review [3] of [1], Liberman noted that the proof was incomplete. We produce elementary examples of symplectic manifolds which are not Kaehler by constructing counterexamples to Guggenheimer's assertion.

There is a representation $\rho$ of $\mathbb{Z} \oplus \mathbb{Z}$ in the group of diffeomorphisms of $T^2$ defined by

$$(1,0) \xrightarrow{\rho} \text{id}, \quad (0,1) \xrightarrow{\rho} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

where "$[1 0]$" denotes the transformation of $T^2$ covered by the linear transformation of $\mathbb{R}^2$. This representation determines a bundle $M^4$ over $T^2$, with fiber $T^2$: $M^4 = T^2 \times_{\mathbb{Z} \oplus \mathbb{Z}} T^2$, where $\mathbb{Z} \oplus \mathbb{Z}$ acts on $T^2$ by covering transformations, and on $T^2$ by $\rho$ ($M^4$ can also be seen as $\mathbb{R}^4$ modulo a group of affine transformations). Let $\Omega_1$ be the standard volume form for $T^2$. Since $\rho$ preserves $\Omega_1$, this defines a closed 2-form $\Omega_1$ on $M^4$ which is nonsingular on each fiber. Let $\rho$ be projection to the base: then it can be checked that $\Omega_1 + \rho^* \Omega_1$ is a symplectic form. (It is, in general, true that $\Omega_1 + K \rho^* \Omega_1$ is a symplectic form, for any closed $\Omega_1$ which is a volume form for each fiber, and $K$ sufficiently large.) But $H_1(M^4) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$, so $M^4$ is not a Kaehler manifold.

Many more examples can be constructed. In the same vein, if $M^{2k}$ is a closed symplectic manifold, and if $N^{2k+2}$ fibers over $M^{2k}$ with the fundamental class of the fiber not homologous to zero in $N$, then $N$ is also a symplectic manifold. If, for instance, the Euler characteristic of the fiber is not zero, this
hypothesis is satisfied. To do this, one must see that if there is a closed 2-form \( \alpha \) whose integral on a fiber is nonzero, then \( \alpha \) is cohomologous to a 2-form \( \alpha' \) which is nonsingular on each fiber. To find \( \alpha' \), first find a 2-form \( \beta \), not necessarily closed, which is nonsingular on each fiber, and whose integral on each fiber agrees with that of \( \alpha \); this exists by convexity considerations. On each fiber, \( F \), there is a form \( \gamma_F \) such that \( \beta_F - (\alpha_1)_F = d(\gamma_F) \). This equation can also be solved differentiably in a small neighborhood of the base, so, by convexity considerations, there is a global 1-form \( \gamma \) such that on each fiber, \( \beta_F - (\alpha_1)_F = d(\gamma_F) \). Let \( \alpha = \alpha_1 + d(\gamma) \). If \( \Omega_1 \) is a symplectic form for \( M^{2k} \), then \( \Omega = \alpha + K(p^*\Omega_1) \) is a symplectic form for \( N^{2k+2} \), \( K \) is sufficiently large.

This construction, although it applies only to a narrow range of examples, nonetheless has a certain amount of flexibility. This leads me to make the

**Conjecture.** Every closed \( 2k \)-manifold which has an almost complex structure \( \tau \) and a real cohomology class \( \alpha \) such that \( \alpha^k \neq 0 \) has a symplectic structure realizing \( \tau \) and \( \alpha \).

I would like to thank Alan Weinstein for pointing out this question and for helpful discussions.

**References**

3. P. Liberman, review of [1], Zentralblatt für Mathematik 54 (1956), 68.

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